Maxima is a computer algebra system, implemented in Lisp.

Maxima is derived from the Macsyma system, developed at MIT in the years 1968 through 1982 as part of Project MAC. MIT turned over a copy of the Macsyma source code to the Department of Energy in 1982; that version is now known as DOE Macsyma. A copy of DOE Macsyma was maintained by Professor William F. Schelter of the University of Texas from 1982 until his death in 2001. In 1998, Schelter obtained permission from the Department of Energy to release the DOE Macsyma source code under the GNU Public License, and in 2000 he initiated the Maxima project at SourceForge to maintain and develop DOE Macsyma, now called Maxima.
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1 Introduction to Maxima

Start Maxima with the command "maxima". Maxima will display version information and a prompt. End each Maxima command with a semicolon. End the session with the command "quit();". Here's a sample session:

```maxima
$ maxima
Maxima 5.45.1 https://maxima.sourceforge.io
using Lisp SBCL 2.0.1.debian
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
The function bug_report() provides bug reporting information.
(%i1) factor(10!);
8 4 2
(%o1) 2 3 5 7
(%i2) expand ((x + y)^6);
6 5 2 4 3 3 4 2 5 6
(%o2) y + 6 x y + 15 x y + 20 x y + 15 x y + 6 x y + x
(%i3) factor (x^6 - 1);
2 2
(%o3) (x - 1) (x + 1) (x - x + 1) (x + x + 1)
(%i4) quit();
$
```

Maxima can search the info pages. Use the `describe` command to show information about the command or all the commands and variables containing a string. The question mark? (exact search) and double question mark ?? (inexact search) are abbreviations for `describe`:

```maxima
(%i1) ?? integ
0: Functions and Variables for Elliptic Integrals
1: Functions and Variables for Integration
2: Introduction to Elliptic Functions and Integrals
3: Introduction to Integration
4: askinteger (Functions and Variables for Simplification)
5: integerp (Functions and Variables for Miscellaneous Options)
6: integer_partitions (Functions and Variables for Sets)
7: integrate (Functions and Variables for Integration)
8: integrate_use_rootsof (Functions and Variables for Integration)
9: integration_constant_counter (Functions and Variables for Integration)
10: nonnegintegerp (Functions and Variables for linearalgebra)
Enter space-separated numbers, 'all' or 'none': 5 4

-- Function: integerp (<expr>)
    Returns 'true' if <expr> is a literal numeric integer, otherwise 'false'.

    'integerp' returns false if its argument is a symbol, even if the
argument is declared integer.

Examples:

```
(%i1) integerp (0);
(%o1) true
(%i2) integerp (1);
(%o2) true
(%i3) integerp (-17);
(%o3) true
(%i4) integerp (0.0);
(%o4) false
(%i5) integerp (1.0);
(%o5) false
(%i6) integerp (%pi);
(%o6) false
(%i7) integerp (n);
(%o7) false
(%i8) declare (n, integer);
(%o8) done
(%i9) integerp (n);
(%o9) false
```

```
-- Function: askinteger (<expr>, integer)
-- Function: askinteger (<expr>)
-- Function: askinteger (<expr>, even)
-- Function: askinteger (<expr>, odd)

'askinteger (<expr>, integer)' attempts to determine from the 'assume' database whether <expr> is an integer. 'askinteger' prompts the user if it cannot tell otherwise, and attempt to install the information in the database if possible. 'askinteger (<expr>)' is equivalent to 'askinteger (<expr>, integer)'.

'askinteger (<expr>, even)' and 'askinteger (<expr>, odd)' likewise attempt to determine if <expr> is an even integer or odd integer, respectively.
```

(%i1) true

To use a result in later calculations, you can assign it to a variable or refer to it by its automatically supplied label. In addition, % refers to the most recent calculated result:

```
(%i1) u: expand ((x + y)^6);
   6 5 2 4 3 3 4 2 5 6
  y + 6 x y + 15 x y + 20 x y + 15 x y + 6 x y + x
(%o1)
(%i2) diff (u, x);
   5 4 2 3 3 2 4 5
  6 y + 30 x y + 60 x y + 60 x y + 30 x y + 6 x
(%o2)
```
Chapter 1: Introduction to Maxima

\(\text{(\%i3) factor (\%o2);}\)
\(\text{5}\)
\(\text{(\%o3) } 6 (y + x)\)

Maxima knows about complex numbers and numerical constants:

\(\text{(\%i1) cos(\%pi);}\)
\(\text{(-1)}\)
\(\text{(\%i2) exp(\%i*\%pi);}\)
\(\text{(-1)}\)

Maxima can do differential and integral calculus:

\(\text{(\%i1) u: expand ((x + y)^6);}\)
\(\text{6 5 4 3 2 1}\)
\(\text{(\%o1) } x^6 + 6 x^5 y + 15 x^4 y^2 + 20 x^3 y^3 + 15 x^2 y^4 + 6 x y^5 + y^6\)
\(\text{(\%i2) diff (\%, x);}\)
\(\text{5 4 3 2 1}\)
\(\text{(\%o2) } 6 y + 30 x y + 60 x^2 y + 60 x^3 y + 30 x^4 y + 6 x^5\)
\(\text{(\%i3) integrate (1/(1 + x^3), x);}\)
\(\text{2 x - 1}\)
\(\text{2 atan(--------)}\)
\(\text{log(x - x + 1)}\)
\(\text{log(x + 1)}\)
\(\text{6 sqrt(3)}\)
\(\text{(\%o3) } - \frac{2 x - 1}{6} + \frac{2 x - 1}{3} + \frac{2 x - 1}{3}\)

Maxima can solve linear systems and cubic equations:

\(\text{(\%i1) linsolve ([3*x + 4*y = 7, 2*x + a*y = 13], [x, y]);}\)
\(\text{7 a - 52 25}\)
\(\text{(\%o1) } [x = \frac{7 a - 52}{25}, y = \frac{7 a - 52}{25}]\)
\(\text{(\%i2) solve (x^3 - 3*x^2 + 5*x = 15, x);}\)
\(\text{3 a - 8 3 a - 8}\)
\(\text{(\%o2) } [x = -\sqrt{5} \cdot i, x = \sqrt{5} \cdot i, x = 3]\)

Maxima can solve nonlinear sets of equations. Note that if you don’t want a result printed, you can finish your command with $ instead of ;.

\(\text{(\%i1) eq_1: x^2 + 3*x*y + y^2 = 0$}\)
\(\text{(\%i2) eq_2: 3*x + y = 1$}\)
\(\text{(\%i3) solve ([eq_1, eq_2]);}\)
\(\text{3 sqrt(5) + 7 sqrt(5) + 3}\)
\(\text{(\%o3) } [[y = \frac{3 \sqrt{5} - 7}{2}, x = \frac{\sqrt{5} - 3}{2}],\}
\(\text{3 sqrt(5) - 7 sqrt(5) - 3}\)
\(\text{[y = \frac{\sqrt{5} - 3}{2}, x = \frac{3 \sqrt{5} - 7}{2}]]}\)

Maxima can generate plots of one or more functions:

\(\text{(\%i1) plot2d (sin(x)/x, [x, -20, 20]$}\)
(%i2) plot2d ([atan(x), erf(x), tanh(x)], [x, -5, 5], [y, -1.5, 2])$

(%i3) plot3d (sin(sqrt(x^2 + y^2))/sqrt(x^2 + y^2),
       [x, -12, 12], [y, -12, 12])$

(%i3) plot3d (sin(sqrt(x^2 + y^2))/sqrt(x^2 + y^2),
       [x, -12, 12], [y, -12, 12])$
\[
\frac{\sin(\sqrt{y^2+x^2})}{\sqrt{y^2+x^2}}
\]
2 Bug Detection and Reporting

2.1 Functions and Variables for Bug Detection and Reporting

run_testsuite ([options])

Run the Maxima test suite. Tests producing the desired answer are considered “passes,” as are tests that do not produce the desired answer, but are marked as known bugs.

run_testsuite takes the following optional keyword arguments:

- display_all
  Display all tests. Normally, the tests are not displayed, unless the test fails. (Defaults to false).

- display_known_bugs
  Displays tests that are marked as known bugs. (Default is false).

- tests
  This is a single test or a list of tests that should be run. Each test can be specified by either a string or a symbol. By default, all tests are run. The complete set of tests is specified by testsuite_files.

- time
  Display time information. If true, the time taken for each test file is displayed. If all, the time for each individual test is shown if display_all is true. The default is false, so no timing information is shown.

- share_tests
  Load additional tests for the share directory. If true, these additional tests are run as a part of the testsuite. If false, no tests from the share directory are run. If only, only the tests from the share directory are run. Of course, the actual set of test that are run can be controlled by the tests option. The default is false.

For example run_testsuite(display_known_bugs = true, tests=[rtest5]) runs just test rtest5 and displays the test that are marked as known bugs.

run_testsuite(display_all = true, tests=["rtest1", rtest1a]) will run tests rtest1 and rtest2, and displays each test.

run_testsuite changes the Maxima environment. Typically a test script executes kill to establish a known environment (namely one without user-defined functions and variables) and then defines functions and variables appropriate to the test.

run_testsuite returns done.

testsuite_files

[Option variable]

testsuite_files is the set of tests to be run by run_testsuite. It is a list of names of the files containing the tests to run. If some of the tests in a file are known to fail, then instead of listing the name of the file, a list containing the file name and the test numbers that fail is used.

For example, this is a part of the default set of tests:

["rtest13s", ["rtest14", 57, 63]]
This specifies the testsuite consists of the files "rtest13s" and "rtest14", but "rtest14" contains two tests that are known to fail: 57 and 63.

**share_testsuite_files**

[Option variable]

*share_testsuite_files* is the set of tests from the *share* directory that is run as a part of the test suite by *run_testsuite*.

**bug_report ()**

[Function]

Prints out Maxima and Lisp version numbers, and gives a link to the Maxima project bug report web page. The version information is the same as reported by *build_info*.

When a bug is reported, it is helpful to copy the Maxima and Lisp version information into the bug report.

*bug_report* returns an empty string "".

**build_info ()**

[Function]

Returns a summary of the parameters of the Maxima build, as a Maxima structure (defined by *defstruct*). The fields of the structure are: *version*, *timestamp*, *host*, *lisp_name*, and *lisp_version*. When the pretty-printer is enabled (via *display2d*), the structure is displayed as a short table.

See also *bug_report*.

Examples:

```
(%i1) build_info ();
(%o1)
Maxima version: "5.36.1"
Maxima build date: "2015-06-02 11:26:48"
Host type: "x86_64-unknown-linux-gnu"
Lisp implementation type: "GNU Common Lisp (GCL)"
Lisp implementation version: "GCL 2.6.12"
(%i2) x : build_info ()$
(%i3) x@version;
(%o3) 5.36.1
(%i4) x@timestamp;
(%o4) 2015-06-02 11:26:48
(%i5) x@host;
(%o5) x86_64-unknown-linux-gnu
(%i6) x@lisp_name;
(%o6) GNU Common Lisp (GCL)
(%i7) x@lisp_version;
(%o7) GCL 2.6.12
```

Maxima version: "5.36.1"
Maxima build date: "2015-06-02 11:26:48"
Host type: "x86_64-unknown-linux-gnu"
Lisp implementation type: "GNU Common Lisp (GCL)"
Lisp implementation version: "GCL 2.6.12"
The Maxima version string can (here 5.36.1) can look very different:

```
(%i1) build_info();
(%o1)
Maxima version: "branch_5_37_base_331_g8322940_dirty"
Maxima build date: "2016-01-01 15:37:35"
Host type: "x86_64-unknown-linux-gnu"
Lisp implementation type: "CLISP"
Lisp implementation version: "2.49 (2010-07-07) (built 3605577779) (memory 3660647857)"
```

In that case, Maxima was not build from a released sourcecode, but directly from the GIT-checkout of the sourcecode. In the example, the checkout is 331 commits after the latest GIT tag (usually a Maxima (major) release (5.37 in our example)) and the abbreviated commit hash of the last commit was "8322940".

Front-ends for maxima can add information about currently being used by setting the variables `maxima_frontend` and `maxima_frontend_version` accordingly.
3 Help

3.1 Documentation

The Maxima on-line user’s manual can be viewed in different forms. From the Maxima interactive prompt, the user’s manual is viewed as plain text by the `describe` function. The user’s manual is viewed as `info` hypertext by the `info` viewer program and as a web page by any ordinary web browser.

`example` displays examples for many Maxima functions. For example,

```
(%i1) example (integrate);
```

yields

```
(%i2) test(f):=block([u],u:integrate(f,x),ratsimp(f-diff(u,x)))
(%o2) test(f) := block([u], u : integrate(f, x),
                ratsimp(f - diff(u, x)))
(%i3) test(sin(x))
(%o3) 0
(%i4) test(1/(x+1))
(%o4) 0
(%i5) test(1/(x^2+1))
(%o5) 0
```

and additional output.

3.2 Functions and Variables for Help

`apropos (name)`

[Function]

Searches for Maxima names which have `name` appearing anywhere within them; `name` must be a string or symbol. Thus, `apropos (exp)` returns a list of all the flags and functions which have `exp` as part of their names, such as `expand`, `exp`, and `exponentialize`. So, if you can only remember part of the name of a Maxima command or variable, you can use this command to find the rest of the name. Similarly, you can type `apropos (tr_)` to find a list of many of the switches relating to the translator, most of which begin with `tr_`.

`apropos("")` returns a list with all Maxima names.

`apropos` returns the empty list `[]`, if no name is found.

Example:

Show all Maxima symbols which have `gamma` in the name:

```
(%i1) apropos("gamma");
(%o1) [%gamma, Gamma, gamma_expand, gammalim, makegamma, prefer_gamma_incomplete, gamma, gamma-incomplete, gamma_incomplete, gamma_incomplete_generalized, gamma_incomplete_generalized_regularized, gamma_incomplete_lower, gamma_incomplete_regularized, log_gamma]
```

The same example, using the symbol `gamma`, rather than the string:
The number of symbols in the current Maxima session. This will vary.

```lisp
(%i3) length(apropos(""));
(%o3) 2338
```

**demo** (*filename*)

Evaluates Maxima expressions in *filename* and displays the results. **demo** pauses after evaluating each expression and continues after the user enters a carriage return. (If running in Xmaxima, **demo** may need to see a semicolon ; followed by a carriage return.)

demo searches the list of directories **file_search_demo** to find *filename*. If the file has the suffix dem, the suffix may be omitted. See also **file_search**.

demo evaluates its argument. **demo** returns the name of the demonstration file.

Example:

```lisp
(%i1) demo ("disol");

batching /home/wfs/maxima/share/simplification/disol.dem
  At the _ prompt, type ';' followed by enter to get next demo
(%i2) load("disol")

-
(%i3) exp1 : a (e (g + f) + b (d + c))
(%o3) a (e (g + f) + b (d + c))

-
(%i4) disolate(exp1, a, b, e)
(%t4) d + c
(%t5) g + f
(%o5) a (%t5 e + %t4 b)
```

**describe**

describe (**string**)  
describe (**string**, exact)  
describe (**string**, inexact)

describe (**string**) is equivalent to describe (**string**, exact).

describe (**string**, exact) finds an item with title equal (case-insensitive) to **string**, if there is any such item.
describe(string, inexact) finds all documented items which contain string in their titles. If there is more than one such item, Maxima asks the user to select an item or items to display.

At the interactive prompt, ? foo (with a space between ? and foo) is equivalent to describe("foo", exact), and ?? foo is equivalent to describe("foo", inexact). describe("", inexact) yields a list of all topics documented in the on-line manual. describe quotes its argument. describe returns true if some documentation is found, otherwise false.

To display the topics using a browser see [output_format_for_help], page 14. Also see [browser], page 14, and [url_base], page 14, to configure how to display the HTML files.

See also Section 3.1 [Documentation], page 11.

Example:

(%i1) ?? integ
0: Functions and Variables for Elliptic Integrals
1: Functions and Variables for Integration
2: Introduction to Elliptic Functions and Integrals
3: Introduction to Integration
4: askinteger (Functions and Variables for Simplification)
5: integerp (Functions and Variables for Miscellaneous Options)
6: integer_partitions (Functions and Variables for Sets)
7: integrate (Functions and Variables for Integration)
8: integrate_use_rootsof (Functions and Variables for Integration)
9: integration_constant_counter (Functions and Variables for Integration)
10: nonnegintegerp (Functions and Variables for linearalgebra)
Enter space-separated numbers, 'all' or 'none': 7 8

-- Function: integrate (<expr>, <x>)
-- Function: integrate (<expr>, <x>, <a>, <b>)
Attempts to symbolically compute the integral of <expr> with respect to <x>. ‘integrate (<expr>, <x>)’ is an indefinite integral, while ‘integrate (<expr>, <x>, <a>, <b>)’ is a definite integral, [...]

-- Option variable: integrate_use_rootsof
Default value: ‘false'

When ‘integrate_use_rootsof' is ‘true' and the denominator of a rational function cannot be factored, ‘integrate' returns the integral in a form which is a sum over the roots (not yet known) of the denominator. [...]

In this example, items 7 and 8 were selected (output is shortened as indicated by [...]). All or none of the items could have been selected by entering all or none, which can be abbreviated a or n, respectively.

output_format_for_help

[Option variable]
Default value: text

output_format_for_help controls how describe displays help.
output_format_for_help can be set to one of the following values:

text Help is displayed as plain text sent to a terminal. This is the default.
html Help is displayed using a browser to display the HTML version of the manual.
frontend Help is displayed using the frontend’s help system. If no frontend is running then an error is signaled. For example, wxMaxima and xmaxima are some frontends for maxima.

Any other value is a error.
See also browser, and url_base.

browser

[Option variable]
This specifies the command to use to open an HTML file. This is a format string of the form <cmd> "A where "A is replaced by the URL of the HTML file and <cmd> is some program that takes an arg and opens up a browser to the given URL.

On windows, the default setting is "start "A", which uses the default browser to display the html file. You may replace it with e.g. start firefox "A, start chrome "A or start iexplore "A if you want to use Firefox, Chrome, or Internet Explorer instead of the default browser.

On other OSes, the user’s default browser should be used automatically (using xdg-open on Linux/Unix and open on MacOS). You can also set the browser variable to use a non default browser, e.g. browser:"firefox "A"; or browser:"chromium "A";.

See also output_format_for_help, and url_base.

url_base

[Option variable]
When displaying help using a browser, url_base defines the URL to use. It defaults to a file:// path pointing to the directory containing the html files for documentation. However, you could use http://localhost:8080/ or some other URL that has the HTML help files. But this requires those URLs to have exactly the same HTML files in the info directory because a table is needed to translate a topic to the appropriate location in an html file.

See also [output_format_for_help], page 14, and browser.

example

[Function]

eexample (topic)
eexample ()
eexample (topic) displays some examples of topic, which is a symbol or a string. To get examples for operators like if, do, or lambda the argument must be a string, e.g. example ("do"). example is not case sensitive. Most topics are function names.
example ( ) returns the list of all recognized topics.

The name of the file containing the examples is given by the global option variable manual_demo, which defaults to "manual.demo".

example quotes its argument. example returns done unless no examples are found or there is no argument, in which case example returns the list of all recognized topics.

Examples:

```
(%i1) example(append);
(%i2) append([y+x,0,-3.2],[2.5e+20,x])
(%o2) [y + x, 0, -3.2, 2.5e+20, x]
(%o2) done
(%i3) example("lambda");
(%i4) lambda([x,y,z],x^2+y^2+z^2)
(%o4) lambda([x, y, z], x^2 + y^2 + z^2)
(%i5) %(1,2,a)
(%o5) a + 5
(%i6) 1+2+a
(%o6) a + 3
(%o6) done
```

manual_demo [Option variable]

Default value: "manual.demo"

manual_demo specifies the name of the file containing the examples for the function example. See example.
4 Command Line

4.1 Introduction to Command Line

4.2 Functions and Variables for Command Line

__

__ is the input expression currently being evaluated. That is, while an input expression expr is being evaluated, __ is expr.

__ is assigned the input expression before the input is simplified or evaluated. However, the value of __ is simplified (but not evaluated) when it is displayed.

__ is recognized by batch and load. In a file processed by batch, __ has the same meaning as at the interactive prompt. In a file processed by load, __ is bound to the input expression most recently entered at the interactive prompt or in a batch file; __ is not bound to the input expressions in the file being processed. In particular, when load (filename) is called from the interactive prompt, __ is bound to load (filename) while the file is being processed.

See also _ and %.

Examples:

```(%i1) print("I was called as", __);
I was called as print(I was called as, __)
(%o1) print(I was called as, __)
(%i2) foo(__);
(%o2) foo(foo(__))
(%i3) g (x) := (print("Current input expression =", __), 0);
(%o3) g(x) := (print("Current input expression =", __), 0)
(%i4) [aa : 1, bb : 2, cc : 3];
(%o4) [1, 2, 3]
(%i5) (aa + bb + cc)/(dd + ee + g(x));
      cc + bb + aa
Current input expression = -----------
     g(x) + ee + dd
     6
(%o5) -------------
        ee + dd
```

_ is the most recent input expression (e.g., %i1, %i2, %i3, ...).

_ is assigned the input expression before the input is simplified or evaluated. However, the value of _ is simplified (but not evaluated) when it is displayed.

_ is recognized by batch and load. In a file processed by batch, _ has the same meaning as at the interactive prompt. In a file processed by load, _ is bound to the input expression most recently evaluated at the interactive prompt or in a batch file; _ is not bound to the input expressions in the file being processed.
See also __ and %.

Examples:

```
(%i1) 13 + 29;
   (%o1) 42

(%i2) :lisp $_
   ((MPLUS) 13 29)
   (%i2) ;
   (%o2) 42

(%i3) sin (\pi/2);
   (%o3) 1

(%i4) :lisp $_
   ((%SIN) ((MQUOTIENT) $\pi 2))
   (%i4) ;
   (%o4) 1

(%i5) a: 13$
(%i6) b: 29$
(%i7) a + b;
   (%o7) 42

(%i8) :lisp $_
   ((MPLUS) $A $B)
   (%i8) ;
   (%o8) b + a

(%i9) a + b;
   (%o9) 42

(%i10) ev (_);
   (%o10) 42
```

% [System variable]

% is the output expression (e.g., %o1, %o2, %o3, ...) most recently computed by Maxima, whether or not it was displayed.

% is recognized by batch and load. In a file processed by batch, % has the same meaning as at the interactive prompt. In a file processed by load, % is bound to the output expression most recently computed at the interactive prompt or in a batch file; % is not bound to output expressions in the file being processed.

See also __, __%, and %th.

%% [System variable]

In compound statements, namely block, lambda, or (s_1, ... , s_n), % is the value of the previous statement.

At the first statement in a compound statement, or outside of a compound statement, % is undefined.

%% is recognized by batch and load, and it has the same meaning as at the interactive prompt.

See also %.

Examples:
The following two examples yield the same result.

```
(%i1) block (integrate (x^5, x), ev (%%, x=2) - ev (%%, x=1));
    21
(%o1) --
    2
(%i2) block ([prev], prev: integrate (x^5, x),
    ev (prev, x=2) - ev (prev, x=1));
    21
(%o2) --
    2
```

A compound statement may comprise other compound statements. Whether a statement be simple or compound, `%` is the value of the previous statement.

```
(%i3) block (block (a^n, %%*42), %%/6);
     n
(%o3) 7 a
```

Within a compound statement, the value of `%` may be inspected at a break prompt, which is opened by executing the `break` function. For example, entering `%`; in the following example yields 42.

```
(%i4) block (a: 42, break ())$

Entering a Maxima break point. Type 'exit;' to resume.

_%%;
42
```

### `%th (i)`

The value of the \( i \)'th previous output expression. That is, if the next expression to be computed is the \( n \)'th output, \( %th \ (m) \) is the \( (n - m) \)'th output.

`%th` is recognized by `batch` and `load`. In a file processed by `batch`, `%th` has the same meaning as at the interactive prompt. In a file processed by `load`, `%th` refers to output expressions most recently computed at the interactive prompt or in a batch file; `%th` does not refer to output expressions in the file being processed.

See also `%` and `%%`.

Example:

`%th` is useful in `batch` files or for referring to a group of output expressions. This example sets \( s \) to the sum of the last five output expressions.

```
(%i1) 1;2;3;4;5;
(%o1) 1
(%o2) 2
(%o3) 3
(%o4) 4
(%o5) 5
(%i6) block (s: 0, for i:1 thru 5 do s: s + %th(i), s);
(%o6) 15
```
? [Special symbol]
As prefix to a function or variable name, ? signifies that the name is a Lisp name, not a Maxima name. For example, ?round signifies the Lisp function ROUND. See Section 37.1 [Lisp and Maxima], page 661, for more on this point.
The notation ? word (a question mark followed a word, separated by whitespace) is equivalent to describe("word"). The question mark must occur at the beginning of an input line; otherwise it is not recognized as a request for documentation. See also describe.

?? [Special symbol]
The notation ?? word (?? followed a word, separated by whitespace) is equivalent to describe("word", inexact). The question mark must occur at the beginning of an input line; otherwise it is not recognized as a request for documentation. See also describe.

$ [Input terminator]
The dollar sign $ terminates an input expression, and the most recent output % and an output label, e.g. %o1, are assigned the result, but the result is not displayed.
See also ;.
Example:

(%i1) 1 + 2 + 3 $
(%i2) %;
(%o2) 6
(%i3) %o1;
(%o3) 6

; [Input terminator]
The semicolon ; terminates an input expression, and the resulting output is displayed.
See also $.
Example:

(%i1) 1 + 2 + 3;
(%o1) 6

inchar [Option variable]
Default value: %i
inchar is the prefix of the labels of expressions entered by the user. Maxima automatically constructs a label for each input expression by concatenating inchar and linenum.
inchar may be assigned any string or symbol, not necessarily a single character. Because Maxima internally takes into account only the first char of the prefix, the prefixes inchar, outchar, and linechar should have a different first char. Otherwise some commands like kill(inlabels) do not work as expected.
See also labels.
Example:

(%i1) inchar: "input";
(%o1) input
\begin{verbatim}
(%i2) expand((a+b)^3);
\end{verbatim}

\begin{verbatim}
3 2 2 3
b + 3 a b + 3 a b + a
\end{verbatim}

**infolists**

Default value: []

`infolists` is a list of the names of all of the information lists in Maxima. These are:

- **labels** All bound `%i`, `%o`, and `%t` labels.
- **values** All bound atoms which are user variables, not Maxima options or switches, created by `:` or `::` or functional binding.
- **functions** All user-defined functions, created by `:=` or `define`.
- **arrays** All arrays, hashed arrays and memoizing functions.
- **macros** All user-defined macro functions, created by `::=`.
- **myoptions** All options ever reset by the user (whether or not they are later reset to their default values).
- **rules** All user-defined pattern matching and simplification rules, created by `tellsimp`, `tellsimpafter`, `defmatch`, or `defrule`.
- **aliases** All atoms which have a user-defined alias, created by the `alias`, `ordergreat`, `orderless` functions or by declaring the atom as a noun with `declare`.
- **dependencies** All atoms which have functional dependencies, created by the `depends`, `dependencies`, or `gradef` functions.
- **gradefs** All functions which have user-defined derivatives, created by the `gradef` function.
- **props** All atoms which have any property other than those mentioned above, such as properties established by `atvalue` or `matchdeclare`, etc., as well as properties established in the `declare` function.
- **structures** All structs defined using `defstruct`.
- **let_rule_packages** All user-defined let rule packages plus the special package `default_let_rule_package`. (`default_let_rule_package` is the name of the rule package used when one is not explicitly set by the user.)
**kill**

*kill (a_1, . . ., a_n)*  
*kill (labels)*  
*kill (inlabels, outlabels, linelabels)*  
*kill (n)*  
*kill ([m, n])*  
*kill (values, functions, arrays, . . .)*  
*kill (all)*  
*kill (allbut (a_1, . . ., a_n))*

Removes all bindings (value, function, array, or rule) from the arguments $a_1, . . ., a_n$. An argument $a_k$ may be a symbol or a single array element. When $a_k$ is a single array element, `kill` unbinds that element without affecting any other elements of the array.

Several special arguments are recognized. Different kinds of arguments may be combined, e.g., `kill (inlabels, functions, allbut (foo, bar))`.

- `kill (labels)` unbinds all input, output, and intermediate expression labels created so far. `kill (inlabels)` unbinds only input labels which begin with the current value of `inchar`. Likewise, `kill (outlabels)` unbinds only output labels which begin with the current value of `outchar`, and `kill (linelabels)` unbinds only intermediate expression labels which begin with the current value of `linechar`.
- `kill (n)`, where $n$ is an integer, unbinds the $n$ most recent input and output labels.
- `kill ([m, n])` unbinds input and output labels $m$ through $n$.
- `kill (infolist)`, where `infolist` is any item in `infolists` (such as `values`, `functions`, or `arrays`) unbinds all items in `infolist`. See also `infolists`.
- `kill (all)` unbinds all items on all infolists. `kill (all)` does not reset global variables to their default values; see `reset` on this point.
- `kill (allbut (a_1, . . ., a_n))` unbinds all items on all infolists except for $a_1, . . ., a_n$. `kill (allbut (infolist))` unbinds all items except for the ones on `infolist`, where `infolist` is `values`, `functions`, `arrays`, etc.

The memory taken up by a bound property is not released until all symbols are unbound from it. In particular, to release the memory taken up by the value of a symbol, one unbinds the output label which shows the bound value, as well as unbinding the symbol itself.

`kill` quotes its arguments. The quote-quote operator `''` defeats quotation.

- `kill (symbol)` unbinds all properties of `symbol`. In contrast, the functions `remvalue`, `remfunction`, `remarray`, and `remrule` unbind a specific property. Note that facts declared by `assume` don’t require a symbol they apply to, therefore aren’t stored as properties of symbols and therefore aren’t affected by `kill`.

`kill` always returns `done`, even if an argument has no binding.

**labels (symbol)**  
*Returns* the list of input, output, or intermediate expression labels which begin with `symbol`. Typically `symbol` is the value of `inchar`, `outchar`, or `linechar`. If no labels begin with `symbol`, `labels` returns an empty list.
By default, Maxima displays the result of each user input expression, giving the result an output label. The output display is suppressed by terminating the input with $ (dollar sign) instead of ; (semicolon). An output label is constructed and bound to the result, but not displayed, and the label may be referenced in the same way as displayed output labels. See also %, %%, and %th.

Intermediate expression labels can be generated by some functions. The option variable `programmode` controls whether `solve` and some other functions generate intermediate expression labels instead of returning a list of expressions. Some other functions, such as `ldisplay`, always generate intermediate expression labels.

See also `inchar`, `outchar`, `linechar`, and `infolists`.

**labels**

[System variable]

The variable `labels` is the list of input, output, and intermediate expression labels, including all previous labels if `inchar`, `outchar`, or `linechar` were redefined.

**linechar**

[Option variable]

Default value: %t

`linechar` is the prefix of the labels of intermediate expressions generated by Maxima. Maxima constructs a label for each intermediate expression (if displayed) by concatenating `linechar` and `linenum`.

`linechar` may be assigned any string or symbol, not necessarily a single character. Because Maxima internally takes into account only the first char of the prefix, the prefixes `inchar`, `outchar`, and `linechar` should have a different first char. Otherwise some commands like `kill(inlabels)` do not work as expected.

Intermediate expressions might or might not be displayed. See `programmode` and `labels`.

**linenum**

[System variable]

The line number of the current pair of input and output expressions.

**myoptions**

[System variable]

Default value: []

`myoptions` is the list of all options ever reset by the user, whether or not they get reset to their default value.

**nolabels**

[Option variable]

Default value: false

When `nolabels` is true, input and output result labels (%i and %o, respectively) are displayed, but the labels are not bound to results, and the labels are not appended to the `labels` list. Since labels are not bound to results, garbage collection can recover the memory taken up by the results.

Otherwise input and output result labels are bound to results, and the labels are appended to the `labels` list.

Intermediate expression labels (%t) are not affected by `nolabels`; whether `nolabels` is true or false, intermediate expression labels are bound and appended to the `labels` list.

See also `batch`, `load`, and `labels`. 
optionset

Default value: false

When optionset is true, Maxima prints out a message whenever a Maxima option is reset. This is useful if the user is doubtful of the spelling of some option and wants to make sure that the variable he assigned a value to was truly an option variable.

Example:

```lisp
(%i1) optionset:true;
assignment: assigning to option optionset
(%o1) true
(%i2) gamma_expand:true;
assignment: assigning to option gamma_expand
(%o2) true
```

outchar

Default value: %o

outchar is the prefix of the labels of expressions computed by Maxima. Maxima automatically constructs a label for each computed expression by concatenating outchar and linenum.

outchar may be assigned any string or symbol, not necessarily a single character. Because Maxima internally takes into account only the first char of the prefix, the prefixes inchar, outchar and linechar should have a different first char. Otherwise some commands like kill(inlabels) do not work as expected.

See also labels.

Example:

```lisp
(%i1) outchar: "output";
(output1) output
(%i2) expand((a+b)^3);
3 2 2 3
  b + 3 a b + 3 a b + a
```

playback

playback ()
playback (n)
playback ([m, n])
playback ([m])
playback (input)
playback (slow)
playback (time)
playback (grind)

Displays input, output, and intermediate expressions, without recomputing them. playback only displays the expressions bound to labels; any other output (such as text printed by print or describe, or error messages) is not displayed. See also labels.

playback quotes its arguments. The quote-quote operator '' defeats quotation. playback always returns done.
playback () (with no arguments) displays all input, output, and intermediate expressions generated so far. An output expression is displayed even if it was suppressed by the $ terminator when it was originally computed.

playback (n) displays the most recent n expressions. Each input, output, and intermediate expression counts as one.

playback ([m, n]) displays input, output, and intermediate expressions with numbers from m through n, inclusive.

playback ([m]) is equivalent to playback ([m, m]); this usually prints one pair of input and output expressions.

playback (input) displays all input expressions generated so far.

playback (slow) pauses between expressions and waits for the user to press enter. This behavior is similar to demo. playback (slow) is useful in conjunction with save or stringout when creating a secondary-storage file in order to pick out useful expressions.

playback (time) displays the computation time for each expression.

playback (grind) displays input expressions in the same format as the grind function. Output expressions are not affected by the grind option. See grind.

Arguments may be combined, e.g., playback ([5, 10], grind, time, slow).

prompt [Option variable]

Default value: _

prompt is the prompt symbol of the demo function, playback (slow) mode, and the Maxima break loop (as invoked by break).

quit ([exit-code]) [Function]
Terminates the Maxima session. Note that the function must be invoked as quit(); or quit()$, not quit by itself. quit supports returning an exit code to the shell for Lisps and OSes that support exit codes. The default exit code is 0 (usually indicating no errors encountered). Thus quit(1) indicates to the shell that maxima exited with some kind of failure. This is useful in scripts where maxima can indicate to the shell that maxima failed to compute something or some other bad thing happened.

To stop a lengthy computation, type control-C. The default action is to return to the Maxima prompt. If *debugger-hook* is nil, control-C opens the Lisp debugger. See also Chapter 38 [Debugging], page 677.

read (expr_1, ..., expr_n) [Function]
Prints expr_1, ..., expr_n, then reads one expression from the console and returns the evaluated expression. The expression is terminated with a semicolon ; or dollar sign $.

See also readonly

Example:

(\%i1) foo: 42$
(\%i2) foo: read ("foo is", foo, " -- enter new value.")$
foo is 42 -- enter new value.
(a+b)^3;
(%i3) foo;  
3
(%o3) (b + a)

readonly (expr_1, ..., expr_n)  [Function]
Prints expr_1, ..., expr_n, then reads one expression from the console and returns the
expression (without evaluation). The expression is terminated with a ; (semicolon)
or $ (dollar sign).
See also read.
Examples:
(%i1) aa: 7$
(%i2) foo: readonly ("Enter an expression: ");
Enter an expression:
2^aa;

   aa
(%o2) 2
(%i3) foo: read ("Enter an expression: ");
Enter an expression:
2^aa;

   (%o3) 128

reset ()  [Function]
Resets many global variables and options, and some other variables, to their default
values.
reset processes the variables on the Lisp list *variable-initial-values*. The
Lisp macro defmvar puts variables on this list (among other actions). Many, but not
all, global variables and options are defined by defmvar, and some variables defined
by defmvar are not global variables or options.

showtime  [Option variable]
Default value: false
When showtime is true, the computation time and elapsed time is printed with each
output expression.
The computation time is always recorded, so time and playback can display the
computation time even when showtime is false.
See also timer.

to_lisp ()  [Function]
Enteres the Lisp system under Maxima. (to-maxima) returns to Maxima.
Example:
Define a function and enter the Lisp system under Maxima. The definition is inspected
on the property list, then the function definition is extracted, factored and stored in
the variable $result$. The variable can be used in Maxima after returning to Maxima.
(%i1) f(x):=x^2+x;

   2
(%o1) f(x) := x + x
(%i2) to_lisp();
Type (to-maxima) to restart, ($quit) to quit Maxima.
MAXIMA> (symbol-plist '$f)
(MPROPS (NIL MEXPR ((LAMBDA) ((MLIST) $X)
                                ((MPLUS) ((MEXPT) $X 2) $X))))
MAXIMA> (setq $result ($factor (caddr (mget '$f 'mexpr))))
((MTIMES SIMP FACTORED) $X ((MPLUS SIMP IRREDUCIBLE) 1 $X))
MAXIMA> (to-maxima)
Returning to Maxima
(%o2) true
(%i3) result;
(%o3) x (x + 1)

eval_string_lisp (str) [Function]
Sequentially read lisp forms from the string str and evaluate them. Any values produced from the last form are returned as a Maxima list.

Examples:

(%i1) eval_string_lisp ("");
(%o1) []
(%i2) eval_string_lisp ("(values)");
(%o2) []
(%i3) eval_string_lisp ("69");
(%o3) [69]
(%i4) eval_string_lisp ("1 2 3");
(%o4) [3]
(%i5) eval_string_lisp ("(values 1 2 3)");
(%o5) [1,2,3]
(%i6) eval_string_lisp ("(defun $foo (x) (* 2 x))");
(%o6) [foo]
(%i7) foo (5);
(%o7) 10

See also [eval_string], page 1201.

values [System variable]
Initial value: []
values is a list of all bound user variables (not Maxima options or switches). The list comprises symbols bound by :, or ::.

If the value of a variable is removed with the commands kill, remove, or remvalue the variable is deleted from values.

See functions for a list of user defined functions.

Examples:
First, values shows the symbols a, b, and c, but not d, it is not bound to a value, and not the user function f. The values are removed from the variables. values is the empty list.
(%i1) [a:99, b:: a-90, c:a-b, d, f(x):=x^2];
(%o1) [99, 90, a-b, d, f(x) := x^2]
(%i2) values;
(%o2) [a, b, c]
(%i3) [kill(a), remove(b,value), remvalue(c)];
(%o3) [done, done, [c]]
(%i4) values;
(%o4) []

4.3 Functions and Variables for Display

%edispflag [Option variable]
Default value: false
When %edispflag is true, Maxima displays %e to a negative exponent as a quotient. For example, %e^-x is displayed as 1/%e^x. See also exptdispflag.
Example:
(%i1) %e^-10;
(%o1) %e^-10
(%i2) %edispflag:true$
(%i3) %e^-10;
1
----
10
%e

absboxchar [Option variable]
Default value: !
absboxchar is the character used to draw absolute value signs around expressions which are more than one line tall.
Example:
(%i1) abs((x^3+1));
(%o1) abs(x^3+1)

declare_index_properties (a, [p_1, p_2, p_3, ...]) [Function]
declare_index_properties ([a, b, c, ...], [p_1, p_2, p_3, ...]) [Function]
postsubscript [Symbol]
postsuperscript [Symbol]
presuperscript [Symbol]
presubscript [Symbol]
Declares the properties of indices applied to the symbol a or each of the of symbols a, b, c, .... If multiple symbols are given, the whole list of properties applies to each symbol.
Given a symbol with indices, \( a[i_1, i_2, i_3, \ldots] \), the \( k \)-th property \( p_k \) applies to the \( k \)-th index \( i_k \). There may be any number of index properties, in any order.

Each property \( p_k \) must one of these four recognized properties: \texttt{postsubscript}, \texttt{postsuperscript}, \texttt{presuperscript}, or \texttt{presubscript}, to denote indices which are displayed, respectively, to the right and below, to the right and above, to the left and above, or to the left and below.

Index properties apply only to the 2-dimensional display of indexed variables (i.e., when \texttt{display2d} is \texttt{true}) and TeX output via \texttt{tex}. Otherwise, index properties are ignored. Index properties do not change the input of indexed variables, do not change the algebraic properties of indexed variables, and do not change the 1-dimensional display of indexed variables.

\texttt{declare_index_properties} quotes (does not evaluate) its arguments.

\texttt{remove_index_properties} removes index properties. \texttt{kill} also removes index properties (and all other properties).

\texttt{get_index_properties} retrieves index properties.

Examples:

Given a symbol with indices, \( a[i_1, i_2, i_3, \ldots] \), the \( k \)-th property \( p_k \) applies to the \( k \)-th index \( i_k \). There may be any number of index properties, in any order.

\begin{verbatim}
(%i1) declare_index_properties (A, [presubscript, postsubscript]);
(%o1) done
(%i2) declare_index_properties (B, [postsuperscript, postsuperscript, presuperscript, presubscript]);
(%o2) done
(%i3) declare_index_properties (C, [postsuperscript, presubscript, presuperscript, presubscript, presuperscript]);
(%o3) done
(%i4) A[w, x];
    A
    w x
(%o4)
(%i5) B[w, x, y];
    y w, x
(%o5)
(%i6) C[w, x, y, z];
    z w
    x, y
(%o6)
\end{verbatim}

Index properties apply only to the 2-dimensional display of indexed variables and TeX output. Otherwise, index properties are ignored.

\begin{verbatim}
(%i1) declare_index_properties (A, [presubscript, postsubscript]);
(%o1) done
(%i2) A[w, x];
    A
    w x
(%o2)
\end{verbatim}
(\%i3) tex (A[w, x]);
\$\{_wA_{x}\}$
(\%o3) false
(\%i4) display2d: false $
(\%i5) A[w, x];
(\%o5) A[w, x]
(\%i6) display2d: true $%
(\%i7) grind (A[w, x]);
A[w,x]$%
(\%o7) done
(\%i8) stringdisp: true $%
(\%i9) string (A[w, x]);
(\%o9) "A[w,x]"

get_index_properties (a)                \[Function\]
Returns the properties for \(a\) established by declare_index_properties.
See also remove_index_properties.

remove_index_properties (a, b, c, ...)  \[Function\]
Removes the properties established by declare_index_properties. All index properties are removed from each symbol \(a\), \(b\), \(c\), ....
remove_index_properties quotes (does not evaluate) its arguments.

display_index_separator                \[Symbol property\]
When a symbol \(A\) has index display properties declared via declare_index_properties, the value of the property display_index_separator is the string or other expression which is displayed between indices.
The value of display_index_separator is assigned by put(\(A\), \(S\), display_index_separator), where \(S\) is a string or other expression. The assigned value is retrieved by get(\(A\), display_index_separator).
The display index separator \(S\) can be a string, including an empty string, or false, indicating the default separator, or any expression. If not a string and not false, the property value is coerced to a string via string.
If no display index separator is assigned, the default separator is used. The default separator is a comma. There is no way to change the default separator.
Each symbol has its own value of display_index_separator.
See also put, get, and declare_index_properties.

Examples:
When a symbol \(A\) has index display properties, the value of the property display_index_separator is the string or other expression which is displayed between indices.
The value is assigned by put(\(A\), \(S\), display_index_separator),
(\%i1) declare_index_properties (A, [postsuperscript, postsuperscript, presubscript, presubscript]);
(\%o1) done
(\%i2) put (A, ";\", display_index_separator);
(\%o2) ;
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(%i3) A[w, x, y, z];
    w; x
(%o3)      A
   y; z

The assigned value is retrieved by \texttt{get(A, display\_index\_separator)}.

(%i1) declare_index_properties (A, [postsuperscript, postsuperscript, presubscript, presubscript]);
(%o1) done
(%i2) put (A, ";", display\_index\_separator);
(%o2) ;
(%i3) get (A, display\_index\_separator);
(%o3) ;

The display index separator \texttt{S} can be a string, including an empty string, or \texttt{false}, indicating the default separator, or any expression.

(%i1) declare_index_properties (A, [postsuperscript, postsuperscript, presubscript, presubscript]);
(%o1) done
(%i2) A[w, x, y, z];
    w, x
(%o2)      A
   y, z
(%i3) put (A, ";", display\_index\_separator);
(%o3) ;
(%i4) A[w, x, y, z];
    w-x
(%o4)      A
   y-z
(%i5) put (A, " ", display\_index\_separator);
(%o5) 
(%i6) A[w, x, y, z];
    w x
(%o6)      A
   y z
(%i7) put (A, ",", display\_index\_separator);
(%o7) 
(%i8) A[w, x, y, z];
    wx
(%o8)      A
   y z
(%i9) put (A, false, display\_index\_separator);
(%o9) false
(%i10) A[w, x, y, z];
    w, x
(%o10)      A
   y, z
If no display index separator is assigned, the default separator is used. The default separator is a comma.

```
(%i1) declare_index_properties (A, [postsuperscript, postsuperscript, presubscript, presubscript]);
   (%o1) done
(%i2) A[w, x, y, z];
   w, x
   A
   y, z
```

Each symbol has its own value of `display_index_separator`.

```
(%i1) declare_index_properties (A, [postsuperscript, postsuperscript, presubscript, presubscript]);
   (%o1) done
(%i2) put (A, " ", display_index_separator);
   (%o2)
(%i3) declare_index_properties (B, [postsuperscript, postsuperscript, presubscript, presubscript]);
   (%o3) done
(%i4) put (B, ";", display_index_separator);
   (%o4)
(%i5) A[w, x, y, z] + B[w, x, y, z];
   w;x w x
   B + A
   y;z y z
```

`disp (expr_1, expr_2, ...)`  [Function]

is like `display` but only the value of the arguments are displayed rather than equations. This is useful for complicated arguments which don't have names or where only the value of the argument is of interest and not the name.

See also `ldisp` and `print`.

Example:

```
(%i1) b[1,2]:x-x^2$
(%i2) x:123$
```
display (expr_1, expr_2, ...) [Function]
Displays equations whose left side is expr_i unevaluated, and whose right side is the value of the expression centered on the line. This function is useful in blocks and for statements in order to have intermediate results displayed. The arguments to display are usually atoms, subscripted variables, or function calls.
See also ldisplay, disp, and ldisp.
Example:
  (%i1) b[1,2]:x-x^2$
  (%i2) x:123$
  (%i3) display(x, b[1,2], sin(1.0));
     123
   2
   x - x
0.8414709848078965
(%o3) done

display2d [Option variable]
Default value: true
When display2d is true, the console display is an attempt to present mathematical expressions as they might appear in books and articles, using only letters, numbers, and some punctuation characters. This display is sometimes called the "pretty printer" display.
When display2d is true, Maxima attempts to honor the global variable for line length, linel. When an atom (symbol, number, or string) would otherwise cause a line to exceed linel, the atom may be printed in pieces on successive lines, with a continuation character (backslash, \) at the end of the leading piece; however, in some cases, such atoms are printed without a line break, and the length of the line is greater than linel.
When display2d is false, the console display is a 1-dimensional or linear form which is the same as the output produced by grind.
When display2d is false, the value of stringdisp is ignored, and strings are always displayed with quote marks.
When `display2d` is `false`, Maxima attempts to honor `linel`, but atoms are not broken across lines, and the actual length of an output line may exceed `linel`.

See also `leftjust` to switch between a left justified and a centered display of equations.

Example:

```maxima
(%i1) x/(x^2+1);
     x
(%o1) ----- 
        2
   x + 1
(%i2) display2d:false$
(%i3) x/(x^2+1);
(%o3) x/(x^2+1)
```

### `display_format_internal` [Option variable]

Default value: `false`

When `display_format_internal` is `true`, expressions are displayed without being transformed in ways that hide the internal mathematical representation. The display then corresponds to what `inpart` returns rather than `part`.

Examples:

```maxima
User       part     inpart
a-b;   a - b   a + (- 1) b
       a       - 1
a/b;       - a    b
       b       1/2
sqrt(x); sqrt(x)  x
       x
X*4/3;     ---    - X
       4      3
```

### `with_default_2d_display(expr)` [Function]

While maxima by default realizes 2d Output using ASCII-Art some frontend change that to TeX, MathML or a specific XML dialect that better suits the needs for this specific frontend. `with_default_2d_display` temporarily switches maxima to the default 2D ASCII Art formatter for outputting the result of `expr`.

See also `set_alt_display` and `display2d`.

### `dispterms(expr)` [Function]

Displays `expr` in parts one below the other. That is, first the operator of `expr` is displayed, then each term in a sum, or factor in a product, or part of a more general expression is displayed separately. This is useful if `expr` is too large to be otherwise displayed. For example if `P1, P2, ...` are very large expressions then the display program may run out of storage space in trying to display `P1 + P2 + ...` all at once.
However, `dispterms (P1 + P2 + ...)` displays P1, then below it P2, etc. When not using `dispterms`, if an exponential expression is too wide to be displayed as $A^B$ it appears as `expt (A, B)` (or as `ncexpt (A, B)` in the case of $A^^B$).

Example:

```latex
(\%i1) dispterms(2*a*sin(x)+%e^x);
```

```
+ 2 a sin(x) x %e
(\%o1) done
```

`expt (a, b)`  
[Special symbol]  
`ncexpt (a, b)`  
[Special symbol]

If an exponential expression is too wide to be displayed as $a^b$ it appears as `expt (a, b)` (or as `ncexpt (a, b)` in the case of $a^^b$).

`expt` and `ncexpt` are not recognized in input.

`exptdispflag`  
[Option variable]

Default value: `true`

When `exptdispflag` is `true`, Maxima displays expressions with negative exponents using quotients. See also `%edispflag`.

Example:

```latex
(\%i1) exptdispflag:true;
```

```
(\%o1) true
(\%i2) 10^-x;
```

```
1 ---
x 10
(\%o2) 10^-x;
```

```
(\%i3) exptdispflag:false;
```

```
(\%o3) false
(\%i4) 10^-x;
```

```
(- x 10
(\%o4) 10
```

`grind (expr)`  
[Function]

The function `grind` prints `expr` to the console in a form suitable for input to Maxima. `grind` always returns `done`.

When `expr` is the name of a function or macro, `grind` prints the function or macro definition instead of just the name.

See also `string`, which returns a string instead of printing its output. `grind` attempts to print the expression in a manner which makes it slightly easier to read than the output of `string`. 
\texttt{grind} evaluates its argument.

Examples:

\begin{verbatim}
(\%i1)  aa + 1729;
(\%o1)  aa + 1729
(\%i2)  grind (%);
         aa+1729$
(\%o2)  done
(\%i3)  [aa, 1729, aa + 1729];
(\%o3)  [aa, 1729, aa + 1729]
(\%i4)  grind (%);
         [aa,1729,aa+1729]$  
(\%o4)  done
(\%i5)  matrix ([aa, 17], [29, bb]);
         [ aa 17 ]
(\%o5)  [ ]
         [ 29 bb ]
(\%i6)  grind (%);
         matrix([aa,17],[29,bb])$
(\%o6)  done
(\%i7)  set (aa, 17, 29, bb);
         {17, 29, aa, bb}
(\%o7)  done
(\%i8)  grind (%);
         {17,29,aa,bb}$
(\%o8)  done
(\%i9)  exp (aa / (bb + 17)^29);
         aa  
         ----------
         29
         (bb + 17)
(\%o9)  %e
(\%i10) grind (%);
          %e^(aa/(bb+17)^29)$
(\%o10) done
(\%i11) expr: expand ((aa + bb)^10);
          10 9 2 8 3 7 4 6
          10 bb + 10 aa bb + 45 aa bb + 120 aa bb + 210 aa bb
          5 5 6 4 7 3 8 2
          + 252 aa bb + 210 aa bb + 120 aa bb + 45 aa bb
          9 10
          + 10 aa bb + aa
(\%i12) grind (expr);
          bb^10+10*aa*bb^9+45*aa^2*bb^8+120*aa^3*bb^7+210*aa^4*bb^6
          +252*aa^5*bb^5+210*aa^6*bb^4+120*aa^7*bb^3+45*aa^8*bb^2
          +10*aa^9*bb+aa^10$
(\%o12) done
\end{verbatim}
(%i13) string (expr);

(%o13) bb^{10} + 10*aa*bb^{9} + 45*aa^{2}*bb^{8} + 120*aa^{3}*bb^{7} + 210*aa^{4}*bb^{6} + 
     252*aa^{5}*bb^{5} + 210*aa^{6}*bb^{4} + 120*aa^{7}*bb^{3} + 45*aa^{8}*bb^{2} + 10*aa^{9}*bb + aa^{10}

(%i14) cholesky (A):= block ([n : length (A), L : copysmatrix (A),
     p : makelist (0, i, 1, length (A))],
     for i thru n do for j : i thru n do
     (x : L[i, j], x : x - sum (L[j, k] * L[i, k], k, 1, i - 1),
      if i = j then p[i] : 1 / sqrt(x) else L[j, i] : x * p[i]),
     for i thru n do L[i, i] : 1 / p[i],
     for i thru n do for j : i + 1 thru n do L[i, j] : 0), L)

define: warning: redefining the built-in function cholesky

(%i15) grind (cholesky);

cholesky(A):=block(
  [n:length(A),L:copymatrix(A),
   p:makelist(0,i,1,length(A))],
  for i thru n do (for j from i thru n do 
    (x:L[i,j],x:x-sum(L[j,k]*L[i,k],k,1,i-1),
     if i = j then p[i]:1/sqrt(x) 
     else L[j,i]:x*p[i])),
  for i thru n do L[i,i]:1/p[i],
  for i thru n do (for j from i+1 thru n do L[i,j]:0),L)

(%o15) done

(%i16) string (fundef (cholesky));

(%o16) cholesky(A):=block([n:length(A),L:copymatrix(A),p:makelist(0,i,1,length(A))],
    for i thru n do (for j from i thru n do (x:L[i,j],x:x-sum(L[j,k]*L[i,k],k,1,i-1),
        if i = j then p[i]:1/sqrt(x) 
        else L[j,i]:x*p[i])),
    for i thru n do L[i,i]:1/p[i],for i thru n do (for j from i+1 thru n do L[i,j]:0),L)

[Option variable] grind

When the variable grind is true, the output of string and stringout has the same format as that of grind; otherwise no attempt is made to specially format the output of those functions. The default value of the variable grind is false.

grind can also be specified as an argument of playback. When grind is present, playback prints input expressions in the same format as the grind function. Otherwise, no attempt is made to specially format input expressions.

ibase

Default value: 10

ibase is the base for integers read by Maxima.

ibase may be assigned any integer between 2 and 36 (decimal), inclusive. When ibase is greater than 10, the numerals comprise the decimal numerals 0 through 9 plus letters of the alphabet A, B, C, ..., as needed to make ibase digits in all. Letters are interpreted as digits only if the first digit is 0 through 9.
Uppercase and lowercase letters are not distinguished. The numerals for base 36, the largest acceptable base, comprise 0 through 9 and A through Z.

Whatever the value of \texttt{ibase}, when an integer is terminated by a decimal point, it is interpreted in base 10.

See also \texttt{obase}.

Examples:
\texttt{ibase} less than 10 (for example binary numbers).

\begin{verbatim}
(%i1) ibase : 2 $
(%i2) obase;
(%o2) 10
(%i3) 1111111111111111;
(%o3) 65535
\end{verbatim}

\texttt{ibase} greater than 10. Letters are interpreted as digits only if the first digit is 0 through 9 which means that hexadecimal numbers might need to be prepended by a 0.

\begin{verbatim}
(%i1) ibase : 16 $
(%i2) obase;
(%o2) 10
(%i3) 1000;
(%o3) 4096
(%i4) abcd;
(%o4) abcd
(%i5) symbolp (abcd);
(%o5) true
(%i6) 0abcd;
(%o6) 43981
(%i7) symbolp (0abcd);
(%o7) false
\end{verbatim}

When an integer is terminated by a decimal point, it is interpreted in base 10.

\begin{verbatim}
(%i1) ibase : 36 $
(%i2) obase;
(%o2) 10
(%i3) 1234;
(%o3) 49360
(%i4) 1234.;
(%o4) 1234
\end{verbatim}

\texttt{ldisp} \((\texttt{expr}_1, \ldots, \texttt{expr}_n)\)

Displays expressions \texttt{expr}_1, \ldots, \texttt{expr}_n to the console as printed output. \texttt{ldisp} assigns an intermediate expression label to each argument and returns the list of labels.

See also \texttt{disp}, \texttt{display}, and \texttt{ldisplay}.

Examples:

\begin{verbatim}
(%i1) e: (a+b)^3;
\end{verbatim}
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\begin{verbatim}
3
(\%o1) (b + a)
(\%i2) f: expand (e);
3 2 2 3
(\%o2) b + 3 a b + 3 a b + a
(\%i3) ldisp (e, f);
3
(\%t3) (b + a)
3 2 2 3
(\%t4) b + 3 a b + 3 a b + a
(\%o4) [%t3, %t4]
(\%i4) %t3;
3
(\%o4) (b + a)
(\%i5) %t4;
3 2 2 3
(\%o5) b + 3 a b + 3 a b + a

ldisplay (expr_1, ..., expr_n) [Function]
Displays expressions expr_1, ..., expr_n to the console as printed output. Each expression is printed as an equation of the form lhs = rhs in which lhs is one of the arguments of ldisplay and rhs is its value. Typically each argument is a variable. ldisp assigns an intermediate expression label to each equation and returns the list of labels.

See also display, disp, and ldisp.

Examples:
(\%i1) e: (a+b)^3;
3
(\%o1) (b + a)
(\%i2) f: expand (e);
3 2 2 3
(\%o2) b + 3 a b + 3 a b + a
(\%i3) ldisplay (e, f);
3
(\%t3) e = (b + a)
3 2 2 3
(\%t4) f = b + 3 a b + 3 a b + a
(\%o4) [%t3, %t4]
(\%i4) %t3;
3
(\%o4) (b + a)
(\%i5) %t4;
3 2 2 3
\end{verbatim}
\[(\%o5) \quad f = b + 3 \, a \, b + 3 \, a \, b + a\]

**leftjust**

Default value: `false`

When *leftjust* is `true`, equations in 2D-display are drawn left justified rather than centered.

See also `display2d` to switch between 1D- and 2D-display.

Example:

\[
(\%i1) \quad \text{expand}((x+1)^3);
\]
\[
3 \quad 2
\]
\[
(\%o1) \quad x + 3 \, x + 3 \, x + 1
\]
\[
(\%i2) \quad \text{leftjust}: \text{true}$\$
\]
\[
(\%i3) \quad \text{expand}((x+1)^3);
\]
\[
3 \quad 2
\]
\[
(\%o3) \quad x + 3 \, x + 3 \, x + 1
\]

**linel**

Default value: 79

`linel` is the assumed width (in characters) of the console display for the purpose of displaying expressions. `linel` may be assigned any value by the user, although very small or very large values may be impractical. Text printed by built-in Maxima functions, such as error messages and the output of `describe`, is not affected by `linel`.

**lispdisp**

Default value: `false`

When *lispdisp* is `true`, Lisp symbols are displayed with a leading question mark `?`. Otherwise, Lisp symbols are displayed with no leading mark. This has the same effect for 1-d and 2-d display.

Examples:

\[
(\%i1) \quad \text{lispdisp}: \text{false}$\$
\]
\[
(\%i2) \quad ?\text{foo} + ?\text{bar};
\]
\[
(\%o2) \quad \text{foo} + \text{bar}
\]
\[
(\%i3) \quad \text{lispdisp}: \text{true}$\$
\]
\[
(\%i4) \quad ?\text{foo} + ?\text{bar};
\]
\[
(\%o4) \quad ?\text{foo} + ?\text{bar}
\]

**negsumdispflag**

Default value: `true`

When *negsumdispflag* is `true`, \(x - y\) displays as \(x - y\) instead of as \(-y + x\). Setting it to `false` causes the special check in display for the difference of two expressions to not be done. One application is that thus \(a + \%i*b\) and \(a - \%i*b\) may both be displayed the same way.

**obase**

Default value: 10
**obase** is the base for integers displayed by Maxima.

**obase** may be assigned any integer between 2 and 36 (decimal), inclusive. When **obase** is greater than 10, the numerals comprise the decimal numerals 0 through 9 plus capital letters of the alphabet A, B, C, ..., as needed. A leading 0 digit is displayed if the leading digit is otherwise a letter. The numerals for base 36, the largest acceptable base, comprise 0 through 9, and A through Z.

See also **ibase**.

Examples:

```
(%i1) obase : 2;
(%o1) 10
(%i10) 2^8 - 1;
(%o10) 11111111
(%i11) obase : 8;
(%o3) 10
(%i4) 8^8 - 1;
(%o4) 77777777
(%i5) obase : 16;
(%o5) 10
(%i6) 16^8 - 1;
(%o6) 0FFFFFFFF
(%i7) obase : 36;
(%o7) 10
(%i8) 36^8 - 1;
(%o8) 0ZZZZZZZZ
```

**pfeformat**

[Option variable]

Default value: false

When **pfeformat** is true, a ratio of integers is displayed with the solidus (forward slash) character, and an integer denominator n is displayed as a leading multiplicative term 1/n.

Examples:

```
(%i1) pfeformat: false$
(%i2) 2^16/7^3;
   65536
(%o2) -----        
   343
(%i3) (a+b)/8;
   b + a
(%o3) -----        
   8
(%i4) pfeformat: true$
(%i5) 2^16/7^3;
   65536/343
(%o5)                
(%i6) (a+b)/8;
   1/8 (b + a)
```
powerdisp

Default value: false

When powerdisp is true, a sum is displayed with its terms in order of increasing power. Thus a polynomial is displayed as a truncated power series, with the constant term first and the highest power last.

By default, terms of a sum are displayed in order of decreasing power.

Example:

```
(%i1) powerdisp: true;
(%o1) true
(%i2) x^2 + x^3 + x^4;
   2    3    4
(%o2) x + x + x
(%i3) powerdisp: false;
(%o3) false
(%i4) x^2 + x^3 + x^4;
   4    3    2
(%o4) x + x + x
```

print (expr_1, ..., expr_n)

Evaluates and displays expr_1, ..., expr_n one after another, from left to right, starting at the left edge of the console display.

The value returned by print is the value of its last argument. print does not generate intermediate expression labels.

See also display, disp, ldisplay, and ldisp. Those functions display one expression per line, while print attempts to display two or more expressions per line.

To display the contents of a file, see printfile.

Examples:

```
(%i1) r: print ("(a+b)^3 is", expand ((a+b)^3), "log (a^10/b) is",
    radcan (log (a^10/b)))$
   3  2  2  3
(a+b)^3 is b + 3 a b + 3 a b + a log (a^10/b) is
    10 log(a) - log(b)
(%i2) r;
(%o2) 10 log(a) - log(b)
(%i3) disp ("(a+b)^3 is", expand ((a+b)^3), "log (a^10/b) is",
    radcan (log (a^10/b)))$
(a+b)^3 is
   3    2    2    3
b + 3 a b + 3 a b + a
    log (a^10/b) is
   10 log(a) - log(b)
```
**sqrtdispflag**  
[Option variable]  
Default value: true  
When sqrtdispflag is false, causes sqrt to display with exponent 1/2.

**stardisp**  
[Option variable]  
Default value: false  
When stardisp is true, multiplication is displayed with an asterisk * between operands.

**ttyoff**  
[Option variable]  
Default value: false  
When ttyoff is true, output expressions are not displayed. Output expressions are still computed and assigned labels. See labels.  
Text printed by built-in Maxima functions, such as error messages and the output of describe, is not affected by ttyoff.
5 Data Types and Structures

5.1 Numbers

5.1.1 Introduction to Numbers

Complex numbers

A complex expression is specified in Maxima by adding the real part of the expression to %i times the imaginary part. Thus the roots of the equation \( x^2 - 4x + 13 = 0 \) are \( 2 + 3\%i \) and \( 2 - 3\%i \). Note that simplification of products of complex expressions can be effected by expanding the product. Simplification of quotients, roots, and other functions of complex expressions can usually be accomplished by using the \texttt{realpart}, \texttt{imagpart}, \texttt{rectform}, \texttt{polarform}, \texttt{abs}, and \texttt{carg} functions.

5.1.2 Functions and Variables for Numbers

\texttt{bfloat (expr)}

\texttt{bfloat} replaces integers, rationals, floating point numbers, and some symbolic constants in \texttt{expr} with bigfloat (variable-precision floating point) numbers.

The constants \%e, \%gamma, \%phi, and \%pi are replaced by a numerical approximation. However, \%e in \%e\^x is not replaced by a numeric value unless \texttt{bfloat(x)} is a number. \texttt{bfloat} also causes numerical evaluation of some built-in functions, namely trigonometric functions, exponential functions, \texttt{abs}, and \texttt{log}.

The number of significant digits in the resulting bigfloats is specified by the global variable \texttt{fpprec}. Bigfloats already present in \texttt{expr} are replaced with values which have precision specified by the current value of \texttt{fpprec}.

When \texttt{float2bf} is \texttt{false}, a warning message is printed when a floating point number is replaced by a bigfloat number with less precision.

Examples:

\texttt{bfloat} replaces integers, rationals, floating point numbers, and some symbolic constants in \texttt{expr} with bigfloat numbers.

\begin{verbatim}
(\%i1) bfloat([123, 17/29, 1.75]);
(\%o1) [1.23b2, 5.862068965517241b-1, 1.75b0]
(\%i2) bfloat([\%e, \%gamma, \%phi, \%pi]);
(\%o2) [2.718281828459045b0, 5.772156649015329b-1,
  1.618033988749895b0, 3.141592653589793b0]
(\%i3) bfloat((f(123) + g(h(17/29)))/(x + \%gamma));
    1.0b0 (g(h(5.862068965517241b-1)) + f(1.23b2))
(\%o3) ----------------------------------------------
        x + 5.772156649015329b-1
\end{verbatim}

\texttt{bfloat} also causes numerical evaluation of some built-in functions.

\begin{verbatim}
(\%i1) bfloat(sin(17/29));
(\%o1) 5.532051841609784b-1
(\%i2) bfloat(exp(\%pi));
\end{verbatim}
\[(%o2)\quad 2.314069263277927b1\]
\[(%i3)\ bfloat(\text{abs}(-\gamma));\]
\[(%o3)\quad 5.772156649015329b-1\]
\[(%i4)\ bfloat(\text{log}(\phi));\]
\[(%o4)\quad 4.812118250596035b-1\]

\textbf{bfloatp (expr)} \quad \text{[Function]}

Returns \texttt{true} if \texttt{expr} is a bigfloat number, otherwise \texttt{false}.

\textbf{bftorat} \quad \text{[Option variable]}

Default value: \texttt{false}

\texttt{bftorat} controls the conversion of \texttt{bfloat}s to rational numbers. When \texttt{bftorat} is \texttt{false}, \texttt{ratepsilon} will be used to control the conversion (this results in relatively small rational numbers). When \texttt{bftorat} is \texttt{true}, the rational number generated will accurately represent the \texttt{bfloat}.

Note: \texttt{bftorat} has no effect on the transformation to rational numbers with the function \texttt{rationalize}.

Example:

\[(%i1)\ \texttt{ratepsilon:1e-4;}\]
\[(%o1)\quad 1.0e-4\]
\[(%i2)\ \texttt{rat(bfloat(11111/11111)), bftorat:false;}\]
\[\text{‘rat’ replaced 9.99990999991B-2 by } 1/10 = 1.0B-1\]
\[1\]
\[(%o2)/R/\]
\[--\]
\[10\]
\[(%i3)\ \texttt{rat(bfloat(11111/11111)), bftorat:true;}\]
\[\text{‘rat’ replaced 9.99990999991B-2 by } 11111/11111 = 9.99990999991B-2\]
\[11111\]
\[(%o3)/R/\]
\[------\]
\[111111\]

\textbf{bftrunc} \quad \text{[Option variable]}

Default value: \texttt{true}

\texttt{bftrunc} causes trailing zeroes in non-zero bigfloat numbers not to be displayed. Thus, if \texttt{bftrunc} is \texttt{false}, \texttt{bfloat (1)} displays as \texttt{1.000000000000000B0}. Otherwise, this is displayed as \texttt{1.0B0}.

\textbf{evenp (expr)} \quad \text{[Function]}

Returns \texttt{true} if \texttt{expr} is a literal even integer, otherwise \texttt{false}.

\texttt{evenp} returns \texttt{false} if \texttt{expr} is a symbol, even if \texttt{expr} is declared \texttt{even}.

\textbf{float (expr)} \quad \text{[Function]}

Converts integers, rational numbers and bigfloats in \texttt{expr} to floating point numbers. It is also an \texttt{evflag}. \texttt{float} causes non-integral rational numbers and bigfloat numbers to be converted to floating point.
**float2bf**  
*Option variable*  
Default value: true  
When float2bf is false, a warning message is printed when a floating point number is replaced by a bigfloat number with less precision.

**floatnump (expr)**  
*Function*  
Returns true if expr is a floating point number, otherwise false.

**fpprec**  
*Option variable*  
Default value: 16  
fpprec is the number of significant digits for arithmetic on bigfloat numbers. fpprec does not affect computations on ordinary floating point numbers.  See also bfloat and fpprintprec.

**fpprintprec**  
*Option variable*  
Default value: 0  
fpprintprec is the number of digits to print when printing an ordinary float or bigfloat number.  
For ordinary floating point numbers, when fpprintprec has a value between 2 and 16 (inclusive), the number of digits printed is equal to fpprintprec. Otherwise, fpprintprec is 0, or greater than 16, and the number of digits printed is 16.  
For bigfloat numbers, when fpprintprec has a value between 2 and fpprec (inclusive), the number of digits printed is equal to fpprintprec. Otherwise, fpprintprec is 0, or greater than fpprec, and the number of digits printed is equal to fpprec.  
For both ordinary floats and bigfloats, trailing zero digits are suppressed. The actual number of digits printed is less than fpprintprec if there are trailing zero digits.  
fpprintprec cannot be 1.

**integerp (expr)**  
*Function*  
Returns true if expr is a literal numeric integer, otherwise false.  
integerp returns false if expr is a symbol, even if expr is declared integer.  
Examples:

```
(%i1) integerp (0);                           (%o1) true
(%i2) integerp (1);                           (%o2) true
(%i3) integerp (-17);                         (%o3) true
(%i4) integerp (0.0);                         (%o4) false
(%i5) integerp (1.0);                         (%o5) false
(%i6) integerp (%pi);                         (%o6) false
(%i7) integerp (n);                           (%o7) false
```

(\%i8) declare (n, integer);
(\%o8) done
(\%i9) integerp (n);
(\%o9) false

\textbf{m1pbranch} \quad [\text{Option variable}]

Default value: false

\texttt{m1pbranch} is the principal branch for \(-1\) to a power. Quantities such as \((-1)^{\frac{1}{3}}\) (that is, an "odd" rational exponent) and \((-1)^{\frac{1}{4}}\) (that is, an "even" rational exponent) are handled as follows:

\textbf{domain:real}

\((-1)^{\frac{1}{3}}\): -1
\((-1)^{\frac{1}{4}}\): \((-1)^{\frac{1}{4}}\)

\textbf{domain:complex}

\texttt{m1pbranch:}false \quad \texttt{m1pbranch:}true

\((-1)^{\frac{1}{3}}\): \(\frac{1}{2} + \frac{i}{2}\text{sqrt}(3)\)
\((-1)^{\frac{1}{4}}\): \(\frac{\text{sqrt}(2)}{2} + \frac{i}{2}\text{sqrt}(2)\)

\textbf{nonnegintegerp (n)} \quad [\text{Function}]

Return true if and only if \(n \geq 0\) and \(n\) is an integer.

\textbf{numberp (expr)} \quad [\text{Function}]

Returns true if \(expr\) is a literal integer, rational number, floating point number, or bigfloat, otherwise false.

\texttt{numberp} returns false if \(expr\) is a symbol, even if \(expr\) is a symbolic number such as \%pi or \%i, or declared to be even, odd, integer, rational, irrational, real, imaginary, or complex.

Examples:

(\%i11) numberp (42);
(\%o11) true
(\%i12) numberp (-13/19);
(\%o12) true
(\%i13) numberp (3.14159);
(\%o13) true
(\%i14) numberp (-1729b-4);
(\%o14) true
(\%i15) map (numberp, \[%e, \%pi, \%i, \%phi, inf, minf\]);
(\%o15) [false, false, false, false, false, false]
(\%i16) declare (a, even, b, odd, c, integer, d, rational, 
\text{e, irrational, f, real, g, imaginary, h, complex});
(\%o16) done
(\%i17) map (numberp, [a, b, c, d, e, f, g, h]);
(\%o17) [false, false, false, false, false, false, false, false]
**numer**  

The option variable `numer` causes some mathematical functions (including exponentiation) with numerical arguments to be evaluated in floating point. It causes variables in `expr` which have been given numerals to be replaced by their values. It also sets the `float` switch on.

See also `%enumer`.

Examples:

```lisp
(%i1) [sqrt(2), sin(1), 1/(1+sqrt(3))];
(%o1) [sqrt(2), sin(1), -----------]
      sqrt(3) + 1
(%i2) [sqrt(2), sin(1), 1/(1+sqrt(3))],numer;
(%o2) [1.414213562373095, 0.8414709848078965, 0.3660254037844387]
```

**numer_pbranch**  

The option variable `numer_pbranch` controls the numerical evaluation of the power of a negative integer, rational, or floating point number. When `numer_pbranch` is `true` and the exponent is a floating point number or the option variable `numer` is `true` too, Maxima evaluates the numerical result using the principal branch. Otherwise a simplified, but not an evaluated result is returned.

Examples:

```lisp
(%i1) (-2)^0.75;
(%o1) (- 2)
(%i2) (-2)^0.75,numer_pbranch:true;
(%o2) 1.189207115002721 %i - 1.189207115002721
(%i3) (-2)^(3/4);
(%o3) (- 1)^(3/4)
(%i4) (-2)^(3/4),numer;
(%o4) 1.681792830507429 (- 1)
(%i5) (-2)^(3/4),numer,numer_pbranch:true;
(%o5) 1.189207115002721 %i - 1.189207115002721
```

**numerval**

Declares the variables `x_1`, ..., `x_n` to have numeric values equal to `expr_1`, ..., `expr_n`. The numeric value is evaluated and substituted for the variable in any expressions in which the variable occurs if the `numer` flag is `true`. See also `ev`.

The expressions `expr_1`, ..., `expr_n` can be any expressions, not necessarily numeric.

**oddp**

Returns `true` if `expr` is a literal odd integer, otherwise `false`.

`oddp` returns `false` if `expr` is a symbol, even if `expr` is declared `odd`. 
ratepsilon

[Option variable]

Default value: 2.0e-15

ratepsilon is the tolerance used in the conversion of floating point numbers to rational numbers, when the option variable bftorat has the value false. See bftorat for an example.

rationalize (expr)

[Function]

Convert all double floats and big floats in the Maxima expression expr to their exact rational equivalents. If you are not familiar with the binary representation of floating point numbers, you might be surprised that rationalize (0.1) does not equal 1/10. This behavior isn't special to Maxima – the number 1/10 has a repeating, not a terminating, binary representation.

(%i1) rationalize (0.5);
   1
(%o1) -
   2
(%i2) rationalize (0.1);
   3602879701896397
(%o2) ------------------
   36028797018963968
(%i3) fpprec : 5$
(%i4) rationalize (0.1b0);
   209715
(%o4) -------
   2097152
(%i5) fpprec : 20$
(%i6) rationalize (0.1b0);
   236118324143482260685
(%o6) ----------------------------
   2361183241434822606848
(%i7) rationalize (sin (0.1*x + 5.6));
   3602879701896397 x
(%o7) sin(------------------ + ----------------)
   36028797018963968

ratnump (expr)

[Function]

Returns true if expr is a literal integer or ratio of literal integers, otherwise false.
5.2 Strings

5.2.1 Introduction to Strings

Strings (quoted character sequences) are enclosed in double quote marks " for input, and displayed with or without the quote marks, depending on the global variable stringdisp.

Strings may contain any characters, including embedded tab, newline, and carriage return characters. The sequence \" is recognized as a literal double quote, and \ as a literal backslash. When backslash appears at the end of a line, the backslash and the line termination (either newline or carriage return and newline) are ignored, so that the string continues with the next line. No other special combinations of backslash with another character are recognized; when backslash appears before any character other than ", \, or a line termination, the backslash is ignored. There is no way to represent a special character (such as tab, newline, or carriage return) except by embedding the literal character in the string.

There is no character type in Maxima; a single character is represented as a one-character string.

The stringproc add-on package contains many functions for working with strings.

Examples:

```
(%i1) s_1 : "This is a string.";
  (%o1) This is a string.
(%i2) s_2 : "Embedded \"double quotes\" and backslash \ characters.";
  (%o2) Embedded "double quotes" and backslash \ characters.
(%i3) s_3 : "Embedded line termination in this string.";
  (%o3) Embedded line termination in this string.
(%i4) s_4 : "Ignore the \ line termination \ characters in \ this string.";
  (%o4) Ignore the line termination characters in this string.
(%i5) stringdisp : false;
  (%o5) false
(%i6) s_1;
  (%o6) This is a string.
(%i7) stringdisp : true;
  (%o7) true
(%i8) s_1;
  (%o8) "This is a string."
```

5.2.2 Functions and Variables for Strings

concat (arg_1, arg_2, . . .)  [Function]
Concatenates its arguments. The arguments must evaluate to atoms. The return value is a symbol if the first argument is a symbol and a string otherwise.
concat evaluates its arguments. The single quote ' prevents evaluation.
See also `sconcat`, that works on non-atoms, too, `simplode`, `string` and `eval_string`. For complex string conversions see also `printf`.

```
(%i1) y: 7$
(%i2) z: 88$
(%i3) concat (y, z/2);
    744
(%o3)
(%i4) concat ('y, z/2);
    y44
(%o4)
```

A symbol constructed by `concat` may be assigned a value and appear in expressions. The `::` (double colon) assignment operator evaluates its left-hand side.

```
(%i5) a: concat ('y, z/2);
    y44
(%o5)
(%i6) a:: 123;
    123
(%o6)
(%i7) y44;
    123
(%o7)
(%i8) b^a;
    y44
(%o8)
(%i9) %, numer;
    123
(%o9)
```

Note that although `concat (1, 2)` looks like a number, it is a string.

```
(%i10) concat (1, 2) + 3;
    12 + 3
(%o10)
```

`sconcat (arg_1, arg_2, . . .)`  
[Function]  
Concatenates its arguments into a string. Unlike `concat`, the arguments do not need to be atoms.

See also `concat, simplode, string` and `eval_string`. For complex string conversions see also `printf`.

```
(%i11) sconcat ("xx[", 3, "]::", expand ((x+y)^3));
    xx[3]:y^3+3*x*y^2+3*x^2*y+x^3
(%o11)
```

Another purpose for `sconcat` is to convert arbitrary objects to strings.

```
(%i12) sconcat (x);
    x
(%o12)
(%i13) stringp(%);
    true
(%o13)
```

`string (expr)`  
[Function]  
Converts `expr` to Maxima’s linear notation just as if it had been typed in.

The return value of `string` is a string, and thus it cannot be used in a computation. See also `concat, sconcat, simplode` and `eval_string`.  

stringdisp

[Option variable]

Default value: false

When stringdisp is true, strings are displayed enclosed in double quote marks. Otherwise, quote marks are not displayed.

stringdisp is always true when displaying a function definition.

Examples:

(\%i1) stringdisp: false$
(\%i2) "This is an example string."
(\%o2) This is an example string.
(\%i3) foo () :=
  print ("This is a string in a function definition.");
(\%o3) foo() :=
  print("This is a string in a function definition.")
(\%i4) stringdisp: true$
(\%i5) "This is an example string."
(\%o5) "This is an example string."
5.3 Constants

5.3.1 Functions and Variables for Constants

%e
%e represents the base of the natural logarithm, also known as Euler’s number. The numeric value of %e is the double-precision floating-point value 2.718281828459045d0. (See A&S eqn 4.1.16, A&S 4.1.17.)

%i
%i represents the imaginary unit, \(\sqrt{-1}\).

false
false represents the Boolean constant of the same name. Maxima implements false by the value NIL in Lisp.

%gamma
The Euler-Mascheroni constant, 0.5772156649015329.... It is defined by (A&S eqn 6.1.3 and DLMF 5.2.ii)

\[ \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) \]

ind
ind represents a bounded, indefinite result.
See also limit.
Example:

(%i1) limit (sin(1/x), x, 0);
(%o1) ind

inf
inf represents real positive infinity.

infinity
infinity represents complex infinity.

minf
minf represents real minus (i.e., negative) infinity.

%phi
%phi represents the so-called golden mean, \((1 + \sqrt{5})/2\). The numeric value of %phi is the double-precision floating-point value 1.618033988749895d0.

fibtophi expresses Fibonacci numbers \(\text{fib}(n)\) in terms of %phi.

By default, Maxima does not know the algebraic properties of %phi. After evaluating \texttt{tellrat(\%phi^2 - \%phi - 1)} and \texttt{algebraic: true, ratsimp} can simplify some expressions containing %phi.

Examples:
fibtophi expresses Fibonacci numbers \(\text{fib}(n)\) in terms of \(\%phi\).

\[
\begin{align*}
\text{(i1)} \quad & \text{fibtophi (fib (n));} \\
& \frac{\%phi^n}{\%phi + 1} - \frac{(1 - \%phi)^n}{\%phi - 1} \\
\text{(o1)} \quad & \text{fib (n-1) + fib (n) - fib (n+1);} \\
& - \text{fib}(n + 1) + \text{fib}(n) + \text{fib}(n - 1) \\
\text{(i2)} \quad & \text{fibtophi (\%;)} \\
& \frac{\%phi^{n+1} - (1 - \%phi)^{n+1}}{\%phi - 1} + \frac{\%phi^n - (1 - \%phi)^n}{\%phi - 1} \\
& \frac{\%phi^{n-1} - (1 - \%phi)^{n-1}}{\%phi - 1} \\
\text{(o3)} \quad & \text{ratsimp (\%;)} \\
& 0 \\
\text{(i4)} \quad & \text{By default, Maxima does not know the algebraic properties of \%phi. After evaluating tellrat (\%phi^2 - \%phi - 1) and algebraic: true, ratsimp can simplify some expressions containing \%phi.}
\end{align*}
\]

\[
\begin{align*}
\text{(i1)} \quad & \text{e : expand ((\%phi^2 - \%phi - 1) * (A + 1));} \\
& 2 A - \%phi A - A + \%phi - \%phi - 1 \\
\text{(o1)} \quad & \text{ratsimp (e);} \\
& 2 A + \%phi - \%phi - 1 \\
\text{(i2)} \quad & \text{tellrat (\%phi^2 - \%phi - 1);} \\
& \frac{\%phi^2 - \%phi - 1}{\%phi - 1} \\
\text{(i3)} \quad & \text{algebraic : true;} \\
& \text{true} \\
\text{(i4)} \quad & \text{ratsimp (e);} \\
& 0 \\
\end{align*}
\]

\(\%pi\) [Constant]

\(\%pi\) represents the ratio of the perimeter of a circle to its diameter. The numeric value of \(\%pi\) is the double-precision floating-point value 3.141592653589793d0.

\text{true} [Constant]

\text{true} represents the Boolean constant of the same name. Maxima implements \text{true} by the value \text{T} in Lisp.

\text{und} [Constant]

\text{und} represents an undefined result.

See also \text{limit}.
Example:

(%i1) limit (x*sin(x), x, inf);
(%o1) und

zeroa

zeroa represents an infinitesimal above zero. zeroa can be used in expressions. limit
simplifies expressions which contain infinitesimals.
See also zeroz and limit.
Example:

limit simplifies expressions which contain infinitesimals:

(%i1) limit(zeroa);
(%o1) 0
(%i2) limit(x+zeroa);
(%o2) x

zerob

zerob represents an infinitesimal below zero. zerob can be used in expressions. limit
simplifies expressions which contain infinitesimals.
See also zeroa and limit.
5.4 Lists

5.4.1 Introduction to Lists
Lists are the basic building block for Maxima and Lisp. All data types other than arrays, hashed arrays and numbers are represented as Lisp lists. These Lisp lists have the form

\((\text{MLIST}) \text{a} 2\)

to indicate an expression \(\text{a}+2\). At Maxima level one would see the infix notation \(\text{a}+2\). Maxima also has lists which are printed as

\([1, 2, 7, \text{x+y}]\)

for a list with 4 elements. Internally this corresponds to a Lisp list of the form

\((\text{MLIST}) 1 2 7 ((\text{MPLUS}) \text{x} \text{y}))\)

The flag which denotes the type field of the Maxima expression is a list itself, since after it has been through the simplifier the list would become

\((\text{MLIST SIMP}) 1 2 7 ((\text{MPLUS SIMP}) \text{x} \text{y}))\)

5.4.2 Functions and Variables for Lists

[ and ] mark the beginning and end, respectively, of a list. [ and ] also enclose the subscripts of a list, array, hashed array, or memoizing function. Note that other than for arrays accessing the \(n\)th element of a list may need an amount of time that is roughly proportional to \(n\). See Section 5.4.3 [Performance considerations for Lists], page 74.

Note that if an element of a subscripted variable is written to before a list or an array of this name is declared a hashed array (see Section 5.5 [Arrays], page 76) is created, not a list.

Examples:

\((%i1) \text{x: [a, b, c]};\)
\((%o1) \text{[a, b, c]}\)
\((%i2) \text{x[3];}\)
\((%o2) \text{c}\)
\((%i3) \text{array (y, fixnum, 3);}\)
\((%o3) \text{y}\)
\((%i4) \text{y[2]: %pi;}\)
\((%o4) \text{%pi}\)
\((%i5) \text{y[2];}\)
\((%o5) \text{%pi}\)
\((%i6) \text{z['foo]: 'bar;}\)
\((%o6) \text{'bar}\)
\((%i7) \text{z['foo];}\)
\((%o7) \text{bar}\)
\[
\%i8 \text{ g[k] := 1/(k^2+1);}
\]
\[
\%o8 \quad \frac{1}{k^2+1}
\]
\[
\%i9 \text{ g[10];}
\]
\[
\%o9 \quad \frac{1}{101}
\]

append (list_1, ..., list_n) [Function]

Returns a single list of the elements of list_1 followed by the elements of list_2, ... append also works on general expressions, e.g. append (f(a,b), f(c,d,e)); yields f(a,b,c,d,e).

See also addrow, addcol and join.
Do example(append); for an example.

assoc [Function]

assoc (key, e, default)
assoc (key, e)

assoc searches for key as the first part of an argument of e and returns the second part of the first match, if any.

key may be any expression. e must be a nonatomic expression, and every argument of e must have exactly two parts. assoc returns the second part of the first matching argument of e. Matches are determined by is(key = first(a)) where a is an argument of e.

If there are two or more matches, only the first is returned. If there are no matches, default is returned, if specified. Otherwise, false is returned.

See also sublist and member.

Examples:

key may be any expression. e must be a nonatomic expression, and every argument of e must have exactly two parts. assoc returns the second part of the first matching argument of e.

\[
\%i11 \text{ assoc (f(x), foo(g(x) = y, f(x) = z + 1, h(x) = 2*u));}
\]
\[
\%o11 \quad z + 1
\]

If there are two or more matches, only the first is returned.

\[
\%i11 \text{ assoc (yy, [xx = 111, yy = 222, yy = 333, yy = 444]);}
\]
\[
\%o11 \quad 222
\]

If there are no matches, default is returned, if specified. Otherwise, false is returned.

\[
\%i11 \text{ assoc (abc, [[x, 111], [y, 222], [z, 333]], none);}
\]
\[
\%o11 \quad \text{none}
\]
\[
\%i12 \text{ assoc (abc, [[x, 111], [y, 222], [z, 333]]);}
\]
\[
\%o12 \quad \text{false}
\]
cons

cons (expr, list)
cons (expr_1, expr_2)
cons (expr, list) returns a new list constructed of the element expr as its first element, followed by the elements of list. This is analogous to the Lisp language construction operation "cons".

The Maxima function cons can also be used where the second argument is other than a list and this might be useful. In this case, cons (expr_1, expr_2) returns an expression with same operator as expr_2 but with argument cons(expr_1, args(expr_2)).

Examples:

(%i1) cons(a,[b,c,d]);
  
(%o1) [a, b, c, d]

(%i2) cons(a,f(b,c,d));
  
(%o2) f(a, b, c, d)

In general, cons applied to a nonlist doesn’t make sense. For instance, cons(a,b^c) results in an illegal expression, since '^' cannot take three arguments.

When inflag is true, cons operates on the internal structure of an expression, otherwise cons operates on the displayed form. Especially when inflag is true, cons applied to a nonlist sometimes gives a surprising result; for example

(%i1) cons(a,-a), inflag: true;
   
   2

(%o1) - a

(%i2) cons(a,-a), inflag: false;
   
   0

copylist (list)

Returns a copy of the list list.

create_list (form, x_1, list_1, ..., x_n, list_n)

Create a list by evaluating form with x_1 bound to each element of list_1, and for each such binding bind x_2 to each element of list_2, ... The number of elements in the result will be the product of the number of elements in each list. Each variable x_i must actually be a symbol – it will not be evaluated. The list arguments will be evaluated once at the beginning of the iteration.

(%i1) create_list (x^i, i, [1, 3, 7]);
   
   3 7

(%o1) [x, x , x ]

With a double iteration:

(%i1) create_list ([i, j], i, [a, b], j, [e, f, h]);
   
   [a, e], [a, f], [a, h], [b, e], [b, f], [b, h]

Instead of list_i two args may be supplied each of which should evaluate to a number. These will be the inclusive lower and upper bounds for the iteration.

(%i1) create_list ([i, j], i, [1, 2, 3], j, 1, i);
   
   [[i, 1], [2, 1], [2, 2], [3, 1], [3, 2], [3, 3]]

Note that the limits or list for the j variable can depend on the current value of i.
delete

delete(expr_1, expr_2)
delete(expr_1, expr_2, n)

delete(expr_1, expr_2) removes from expr_2 any arguments of its top-level operator which are the same (as determined by "=") as expr_1. Note that "=" tests for formal equality, not equivalence. Note also that arguments of subexpressions are not affected.

expr_1 may be an atom or a non-atomic expression. expr_2 may be any non-atomic expression. delete returns a new expression; it does not modify expr_2.

delete(expr_1, expr_2, n) removes from expr_2 the first n arguments of the top-level operator which are the same as expr_1. If there are fewer than n such arguments, then all such arguments are removed.

Examples:
Removing elements from a list.
(%i1) delete(y, [w, x, y, z, y, x, w]);
(%o1) [w, x, z, z, x, w]

Removing terms from a sum.
(%i1) delete(sin(x), x + sin(x) + y);
(%o1) y + x

Removing factors from a product.
(%i1) delete(u - x, (u - w)*(u - x)*(u - y)*(u - z));
(%o1) (u - w) (u - y) (u - z)

Removing arguments from an arbitrary expression.
(%i1) delete(a, foo(a, b, c, d, a));
(%o1) foo(b, c, d)

Limit the number of removed arguments.
(%i1) delete(a, foo(a, b, a, c, d, a), 2);
(%o1) foo(b, c, d)

Whether arguments are the same as expr_1 is determined by ".=". Arguments which are equal but not "=" are not removed.

(%i1) [is (equal (0, 0)), is (equal (0, 0.0)), is (equal (0, 0b0))];
(%o1) [true, true, true]
(%i2) [is (0 = 0), is (0 = 0.0), is (0 = 0b0)];
(%o2) [true, false, false]
(%i3) delete(0, [0, 0.0, 0b0]);
(%o3) [0.0, 0.0b0]
(%i4) is (equal ((x + y)*(x - y), x^2 - y^2));
(%o4) true
(%i5) is ((x + y)*(x - y) = x^2 - y^2);
(%o5) false
(%i6) delete((x + y)*(x - y), [(x + y)*(x - y), x^2 - y^2]);
(%o6) [x - y]
eighth (expr)

Returns the 8th item of expression or list expr. See first for more details.

endcons

endcons (expr, list)
endcons (expr_1, expr_2)

endcons (expr, list) returns a new list constructed of the elements of list followed by expr. The Maxima function endcons can also be used where the second argument is other than a list and this might be useful. In this case, endcons (expr_1, expr_2) returns an expression with same operator as expr_2 but with argument endcons(expr_1, args(expr_2)). Examples:

(%i1) endcons(a,[b,c,d]);
(b, c, d, a)
(%i2) endcons(a,f(b,c,d));
f(b, c, d, a)

In general, endcons applied to a nonlist doesn’t make sense. For instance, endcons(a,b^c) results in an illegal expression, since ‘^’ cannot take three arguments.

When inflag is true, endcons operates on the internal structure of an expression, otherwise endcons operates on the displayed form. Especially when inflag is true, endcons applied to a nonlist sometimes gives a surprising result; for example

(%i1) endcons(a,-a), inflag : true;
2
(%o1) - a
(%i2) endcons(a,-a), inflag : false;
0

fifth (expr)

Returns the 5th item of expression or list expr. See first for more details.

first (expr)

Returns the first part of expr which may result in the first element of a list, the first row of a matrix, the first term of a sum, etc.:
Note that \texttt{first} and its related functions, \texttt{rest} and \texttt{last}, work on the form of \texttt{expr} which is displayed not the form which is typed on input. If the variable \texttt{inflag} is set to \texttt{true} however, these functions will look at the internal form of \texttt{expr}. One reason why this may make a difference is that the simplifier re-orders expressions:

\begin{verbatim}
(%i1) x+y;
(%o1) y+1
(%i2) first(x+y),inflag : true;
(%o2) x
(%i3) first(x+y),inflag : false;
(%o3) y
\end{verbatim}

The functions \texttt{second} ... \texttt{tenth} yield the second through the tenth part of their input argument.

See also \texttt{firstn} and \texttt{part}.

\texttt{firstn (expr, count)} \hspace{1cm} \textbf{[Function]}

Returns the first \texttt{count} arguments of \texttt{expr}, if \texttt{expr} has at least \texttt{count} arguments. Returns \texttt{expr} if \texttt{expr} has less than \texttt{count} arguments.

\texttt{expr} may be any nonatomic expression. When \texttt{expr} is something other than a list, \texttt{firstn} returns an expression which has the same operator as \texttt{expr}. \texttt{count} must be a nonnegative integer.

\texttt{firstn} honors the global flag \texttt{inflag}, which governs whether the internal form of an expression is processed (when \texttt{inflag} is true) or the displayed form (when \texttt{inflag} is false).

Note that \texttt{firstn(expr, 1)}, which returns a nonatomic expression containing the first argument, is not the same as \texttt{first(expr)}, which returns the first argument by itself.

See also \texttt{lastn} and \texttt{rest}.

Examples:

\texttt{firstn} returns the first \texttt{count} elements of \texttt{expr}, if \texttt{expr} has at least \texttt{count} elements.

\begin{verbatim}
(%i1) mylist : [1, a, 2, b, 3, x, 4 - y, 2*z + sin(u)];
(%o1) [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
(%i2) firstn (mylist, 0);
(%o2) []
(%i3) firstn (mylist, 1);
(%o3) [1]
(%i4) firstn (mylist, 2);
(%o4) [1, a]
(%i5) firstn (mylist, 7);
(%o5) [1, a, 2, b, 3, x, 4 - y]
\end{verbatim}

\texttt{firstn} returns \texttt{expr} if \texttt{expr} has less than \texttt{count} elements.

\begin{verbatim}
(%i1) mylist : [1, a, 2, b, 3, x, 4 - y, 2*z + sin(u)];
(%o1) [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
(%i2) firstn (mylist, 100);
(%o2) [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
\end{verbatim}
expr may be any nonatomic expression.

(%i1) myfoo : foo(1, a, 2, b, 3, x, 4 - y, 2*z + sin(u));
(%o1) foo(1, a, 2, b, 3, x, 4 - y, 2 z + sin(u))
(%i2) firstn (myfoo, 4);
(%o2) foo(1, a, 2, b)

(%i3) mybar : bar[m, n](1, a, 2, b, 3, x, 4 - y, 2*z + sin(u));
(%o3) bar (1, a, 2, b, 3, x, 4 - y, 2 z + sin(u))

(%i4) firstn (mybar, 4);
(%o4) bar (1, a, 2, b)

(%i5) mymatrix : genmatrix (lambda ([i, j], 10*i + j), 10, 4) $
(%i6) firstn (mymatrix, 3);

(%o6) [11 12 13 14]
[21 22 23 24]
[31 32 33 34]

firstn honors the global flag inflag.

(%i1) myexpr : a + b + c + d + e;
(%o1) e + d + c + b + a
(%i2) firstn (myexpr, 3), inflag=true;
(%o2) c + b + a
(%i3) firstn (myexpr, 3), inflag=false;
(%o3) e + d + c

Note that firstn(expr, 1) is not the same as first(expr).

(%i1) firstn ([w, x, y, z], 1);
(%o1) [w]
(%i2) first ([w, x, y, z]);
(%o2) w

fourth (expr)  [Function]
Returns the 4th item of expression or list expr. See first for more details.

join (l, m)  [Function]
Creates a new list containing the elements of lists l and m, interspersed. The result has elements [l[1], m[1], l[2], m[2], ...]. The lists l and m may contain any type of elements.

If the lists are different lengths, join ignores elements of the longer list.
Maxima complains if l or m is not a list.
See also append.

Examples:

(%i1) L1: [a, sin(b), c!, d - 1];
(%o1) [a, sin(b), c!, d - 1]
(%i2) join (L1, [1, 2, 3, 4]);
(%o2) [a, 1, sin(b), 2, c!, 3, d - 1, 4]
(\%i3) join ([L1], [aa, bb, cc, dd, ee, ff]);
(\%o3)    [a, aa, sin(b), bb, c!, cc, d - 1, dd]

last (expr) [Function]
Returns the last part (term, row, element, etc.) of the expr.
See also lastn.

lastn (expr, count) [Function]
Returns the last count arguments of expr, if expr has at least count arguments. Returns expr if expr has less than count arguments.
expr may be any nonatomic expression. When expr is something other than a list, lastn returns an expression which has the same operator as expr. count must be a nonnegative integer.
lastn honors the global flag inflag, which governs whether the internal form of an expression is processed (when inflag is true) or the displayed form (when inflag is false).
Note that lastn(expr, 1), which returns a nonatomic expression containing the last argument, is not the same as last(expr), which returns the last argument by itself.
See also firstn and rest.
Examples:
lastn returns the last count elements of expr, if expr has at least count elements.
(\%i1) mylist : [1, a, 2, b, 3, x, 4 - y, 2*z + sin(u)];
(\%o1)   [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
(\%i2) lastn (mylist, 0);
(\%o2)    []
(\%i3) lastn (mylist, 1);
(\%o3)    [2 z + sin(u)]
(\%i4) lastn (mylist, 2);
(\%o4)    [4 - y, 2 z + sin(u)]
(\%i5) lastn (mylist, 7);
(\%o5)    [a, 2, b, 3, x, 4 - y, 2 z + sin(u)]

lastn returns expr if expr has less than count elements.
(\%i1) mylist : [1, a, 2, b, 3, x, 4 - y, 2*z + sin(u)];
(\%o1)   [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
(\%i2) lastn (mylist, 100);
(\%o2)    [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]

expr may be any nonatomic expression.
(\%i1) myfoo : foo(1, a, 2, b, 3, x, 4 - y, 2*z + sin(u));
(\%o1)    foo(1, a, 2, b, 3, x, 4 - y, 2 z + sin(u))
(\%i2) lastn (myfoo, 4);
(\%o2)    foo(3, x, 4 - y, 2 z + sin(u))
(\%i3) mybar : bar[m, n](1, a, 2, b, 3, x, 4 - y, 2*z + sin(u));
(\%o3)    bar (1, a, 2, b, 3, x, 4 - y, 2 z + sin(u))
(%i4) lastn (mybar, 4);
(%o4)    bar (3, x, 4 - y, 2 z + sin(u))

m, n
(%i5) mymatrix : genmatrix (lambda ([i, j], 10*i + j), 10, 4) $
(%i6) lastn (mymatrix, 3);

\[
\begin{bmatrix}
81 & 82 & 83 & 84 \\
91 & 92 & 93 & 94 \\
101 & 102 & 103 & 104
\end{bmatrix}
\]

lastn honors the global flag inflag.

(%i1) myexpr : a + b + c + d + e;
(%o1) e + d + c + b + a
(%i2) lastn (myexpr, 3), inflag=true;
(%o2) e + d + c
(%i3) lastn (myexpr, 3), inflag=false;
(%o3) c + b + a

Note that lastn(expr, 1) is not the same as last(expr).

(%i1) lastn ([w, x, y, z], 1);
(%o1) [z]
(%i2) last ([w, x, y, z]);
(%o2) z

length (expr)
[Function]
Returns (by default) the number of parts in the external (displayed) form of expr. For lists this is the number of elements, for matrices it is the number of rows, and for sums it is the number of terms (see dispform).

The length command is affected by the inflag switch. So, e.g. length(a/(b*c)); gives 2 if inflag is false (Assuming exptdispflag is true), but 3 if inflag is true (the internal representation is essentially a*b^-1*c^-1).

Determining a list’s length typically needs an amount of time proportional to the number of elements in the list. If the length of a list is used inside a loop it therefore might drastically increase the performance if the length is calculated outside the loop instead.

lstarith
[Option variable]
Default value: true
If false causes any arithmetic operations with lists to be suppressed; when true, list-matrix operations are contagious causing lists to be converted to matrices yielding a result which is always a matrix. However, list-list operations should return lists.

listp (expr)
[Function]
Returns true if expr is a list else false.

lreduce
[Function]
\[
lreduce (F, s) \\
lreduce (F, s, s_0)
\]
Extends the binary function F to an n-ary function by composition, where s is a list.
lreduce($F$, $s$) returns $F(\ldots F(F(s_1, s_2), s_3), \ldots s_n)$. When the optional argument $s_0$ is present, the result is equivalent to $\text{lreduce}(F, \text{cons}(s_0, s))$.

The function $F$ is first applied to the leftmost list elements, thus the name "lreduce".

See also $\text{rreduce}$, $\text{xreduce}$, and $\text{tree_reduce}$.

Examples:

lreduce without the optional argument.

(%%i1) lreduce (f, [1, 2, 3]);
(%%o1) f(f(1, 2), 3)
(%%i2) lreduce (f, [1, 2, 3, 4]);
(%%o2) f(f(f(1, 2), 3), 4)

lreduce with the optional argument.

(%%i1) lreduce (f, [1, 2, 3], 4);
(%%o1) f(f(f(4, 1), 2), 3)

lreduce applied to built-in binary operators. / is the division operator.

(%%i1) lreduce ("^", args ({a, b, c, d}));
(%%o1) ((a ) )
(%%i2) lreduce ("/", args ({a, b, c, d}));
(%%o2) ----
           b c d

makelist

makelist ()
makelist (expr, n)
makelist (expr, i, i_max)
makelist (expr, i, i_0, i_max)
makelist (expr, i, i_0, i_max, step)
makelist (expr, x, list)

The first form, $\text{makelist}()$, creates an empty list. The second form, $\text{makelist}$ ($expr$), creates a list with $expr$ as its single element. $\text{makelist}$ ($expr$, $n$) creates a list of $n$ elements generated from $expr$.

The most general form, $\text{makelist}$ ($expr$, $i$, $i_0$, $i$ _max_, $step$), returns the list of elements obtained when $\text{ev}$ ($expr$, $i=j$) is applied to the elements $j$ of the sequence: $i_0$, $i_0 + step$, $i_0 + 2*step$, ..., with $|j|$ less than or equal to $|i$ _max_ |.

The increment $step$ can be a number (positive or negative) or an expression. If it is omitted, the default value 1 will be used. If both $i_0$ and $step$ are omitted, they will both have a default value of 1.

$\text{makelist}$ ($expr$, $x$, $list$) returns a list, the $j$th element of which is equal to $\text{ev}$ ($expr$, $x=list[j]$) for $j$ equal to 1 through $\text{length}$ ($list$).

Examples:

(%%i1) makelist (concat (x,i), i, 6);
(%%o1) [x1, x2, x3, x4, x5, x6]
(%%i2) makelist (x=y, y, [a, b, c]);
(%%o2) [x = a, x = b, x = c]
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(%i3) makelist (x^2, x, 3, 2*%pi, 2); (%o3) [9, 25]
(%i4) makelist (random(6), 4); (%o4) [2, 0, 2, 5]
(%i5) flatten (makelist (makelist (i^2, 3), i, 4)); (%o5) [1, 1, 1, 4, 4, 4, 4, 9, 9, 9, 16, 16, 16]
(%i6) flatten (makelist (makelist (i^2, i, 3), 4)); (%o6) [1, 4, 9, 1, 4, 9, 1, 4, 9]

(member (expr_1, expr_2)) [Function]
Returns true if is(expr_1 = a) for some element a in args(expr_2), otherwise returns false.

expr_2 is typically a list, in which case args(expr_2) = expr_2 and is(expr_1 = a)
for some element a in expr_2 is the test.

member does not inspect parts of the arguments of expr_2, so it may return false
even if expr_1 is a part of some argument of expr_2.

See also elementp.

Examples:

(%i1) member (8, [8, 8.0, 8b0]); (%o1) true
(%i2) member (8, [8.0, 8b0]); (%o2) false
(%i3) member (b, [a, b, c]); (%o3) true
(%i4) member (b, [[a, b], [b, c]]); (%o4) false
(%i5) member ([b, c], [[a, b], [b, c]]); (%o5) true
(%i6) F (1, 1/2, 1/4, 1/8); (%o6) F(1, 1/2, 1/4, 1/8)

1 1 1
F(1, 1, 1)

(%i7) member (1/8, %); (%o7) true
(%i8) member ("ab", ["aa", "ab", sin(1), a + b]); (%o8) true

ninth (expr) [Function]
Returns the 9th item of expression or list expr. See first for more details.

pop (list) [Function]
pop removes and returns the first element from the list list. The argument list must
be a mapatom that is bound to a nonempty list. If the argument list is not bound to
a nonempty list, Maxima signals an error. For examples, see push.

push (item, list) [Function]
push prepends the item item to the list list and returns a copy of the new list. The
second argument list must be a mapatom that is bound to a list. The first argument
item can be any Maxima symbol or expression. If the argument list is not bound to a list, Maxima signals an error.

To remove the first item from a list, see pop.

Examples:

(%i1) ll: [];
(%o1) []
(%i2) push (x, ll);  
(%o2) [x]
(%i3) push (x^2+y, ll);  
2
(%o3) [y + x , x]
(%i4) a: push ("string", ll);  
2
(%o4) [string, y + x , x]
(%i5) pop (ll);  
(%o5) string
(%i6) pop (ll);  
2
(%o6) y + x
(%i7) pop (ll);  
(%o7) x
(%i8) ll;  
(%o8) []
(%i9) a;  
(%o9) [string, y + x , x]

rest

rest (expr, n)
rest (expr)

Returns expr with its first n elements removed if n is positive and its last - n elements removed if n is negative. If n is 1 it may be omitted. The first argument expr may be a list, matrix, or other expression. When expr is an atom, rest signals an error; when expr is an empty list and partswitch is false, rest signals an error. When expr is an empty list and partswitch is true, rest returns end.

Applying rest to expression such as f(a,b,c) returns f(b,c). In general, applying rest to a nonlist doesn't make sense. For example, because '^' requires two arguments, rest(a^b) results in an error message. The functions args and op may be useful as well, since args(a^b) returns [a,b] and op(a^b) returns ^.

See also firstn and lastn.

(%i1) rest(a+b+c);  
(%o1) b+a
(%i2) rest(a+b+c,2);  
(%o2) a
(%i3) rest(a+b+c,-2);  
(%o3) c
**reverse** ([list])

Reverses the order of the members of the list (not the members themselves). `reverse` also works on general expressions, e.g. `reverse(a=b)`; gives `b=a`.

See also `sreverse`.

**rreduce**

```
 rreduce (F, s)
rreduce (F, s, s_{n+1})
```

Extends the binary function \( F \) to an \( n \)-ary function by composition, where \( s \) is a list. \( rreduce(F, s) \) returns \( F(s_1, \ldots, F(s_{n-2}, F(s_{n-1}, s_n))) \). When the optional argument \( s_{n+1} \) is present, the result is equivalent to \( rreduce(F, \text{endcons}(s_{n+1}, s)) \).

The function \( F \) is first applied to the rightmost list elements, thus the name "rreduce".

See also `lreduce`, `tree_reduce`, and `xreduce`.

Examples:

- **rreduce** without the optional argument.
  ```
  (%i1) rreduce (f, [1, 2, 3]);
  (%o1) f(1, f(2, 3))
  (%i2) rreduce (f, [1, 2, 3, 4]);
  (%o2) f(1, f(2, f(3, 4)))
  ```

- **rreduce** with the optional argument.
  ```
  (%i1) rreduce (f, [1, 2, 3], 4);
  (%o1) f(1, f(2, f(3, 4)))
  ```

- **rreduce** applied to built-in binary operators. `/` is the division operator.
  ```
  (%i1) rreduce ("-", args ({a, b, c, d}));
  d
c
  (%o1)     b
  a
  (%i2) rreduce ("/", args ({a, b, c, d}));
  a c
  ---
  b d
  ```

**second** ([expr])

Returns the 2nd item of expression or list \( expr \). See `first` for more details.

**seventh** ([expr])

Returns the 7th item of expression or list \( expr \). See `first` for more details.

**sixth** ([expr])

Returns the 6th item of expression or list \( expr \). See `first` for more details.

**sort**

```
 sort (L, P)
sort (L)
```

\( \text{sort}(L, P) \) sorts a list \( L \) according to a predicate \( P \) of two arguments which defines a strict weak order on the elements of \( L \). If \( P(a, b) \) is `true`, then \( a \) appears before \( b \).
in the result. If neither $P(a, b)$ nor $P(b, a)$ are true, then $a$ and $b$ are equivalent, and appear in the result in the same order as in the input. That is, sort is a stable sort.

If $P(a, b)$ and $P(b, a)$ are both true for some elements of $L$, then $P$ is not a valid sort predicate, and the result is undefined. If $P(a, b)$ is something other than true or false, sort signals an error.

The predicate may be specified as the name of a function or binary infix operator, or as a lambda expression. If specified as the name of an operator, the name must be enclosed in double quotes.

The sorted list is returned as a new object; the argument $L$ is not modified. sort($L$) is equivalent to sort($L$, orderlessp).

The default sorting order is ascending, as determined by orderlessp. The predicate ordergreatp sorts a list in descending order.

All Maxima atoms and expressions are comparable under orderlessp and ordergreatp.

Operators < and > order numbers, constants, and constant expressions by magnitude. Note that orderlessp and ordergreatp do not order numbers, constants, and constant expressions by magnitude.

ordermagnitudep orders numbers, constants, and constant expressions the same as <, and all other elements the same as orderlessp.

Examples:

sort sorts a list according to a predicate of two arguments which defines a strict weak order on the elements of the list.

(%i1) sort ([1, a, b, 2, 3, c], 'orderlessp); (%o1) [1, 2, 3, a, b, c]

(%i2) sort ([1, a, b, 2, 3, c], 'ordergreatp); (%o2) [c, b, a, 3, 2, 1]

The predicate may be specified as the name of a function or binary infix operator, or as a lambda expression. If specified as the name of an operator, the name must be enclosed in double quotes.

(%i1) L : [[1, x], [3, y], [4, w], [2, z]]

(%i2) foo (a, b) := a[1] > b[1]; (%o2) foo(a, b) := a > b

(%i3) sort (L, 'foo); (%o3) [[4, w], [3, y], [2, z], [1, x]]

(%i4) infix (">>"); (%o4) >>

(%i5) a >> b := a[1] > b[1]; (%o5) (a >> b) := a > b

(%i6) sort (L, ">>"); (%o6) [[4, w], [3, y], [2, z], [1, x]]
(\%i7) sort (L, lambda ([a, b], a[1] > b[1]));
(\%o7) [[4, w], [3, y], [2, z], [1, x]]

sort(L) is equivalent to sort(L, orderlessp).

(%i11) L : [a, 2*b, -5, 7, 1 + %e, %pi];
(%o11) [a, 2 b, - 5, 7, %e + 1, %pi]
(%i12) sort (L);
(%o12) [- 5, 7, %e + 1, %pi, a, 2 b]
(%i13) sort (L, 'orderlessp);
(%o13) [- 5, 7, %e + 1, %pi, a, 2 b]

The default sorting order is ascending, as determined by orderlessp. The predicate ordergreatp sorts a list in descending order.

(%i11) L : [a, 2*b, -5, 7, 1 + %e, %pi];
(%o11) [a, 2 b, - 5, 7, %e + 1, %pi]
(%i12) sort (L, 'ordergreatp);
(%o12) [2 b, a, %pi, %e + 1, 7, - 5]

All Maxima atoms and expressions are comparable under orderlessp and ordergreatp.

(%i1) L : [%pi, 3, 4, %e, %gamma];
(%o1) [%pi, 3, 4, %e, %gamma]
(%i2) sort (L, orderlessp);
(%o2) [3, 4, %e, %gamma, %pi, 2]
(%i3) sort (L, ordergreatp);
(%o3) [%pi, %gamma, %e, 4, 3]

ordermagnitudep orders numbers, constants, and constant expressions the same as <, and all other elements the same as orderlessp.

(%i11) L: [%i, 1+%i, 2*x, minf, inf, %e, sin(1), 0.1, 2, 3, 1.0, 1.0b0];
(%o11) [%i, %i + 1, 2 x, minf, inf, %e, sin(1), 0, 1, 2, 3, 1.0, 1.0b0]
(%i2) sort (L, ordermagnitudep);
(%o2) [minf, 0, sin(1), 1, 1.0, 1.0b0, 2, %e, 3, inf, %i, %i + 1, 2 x]
(%i3) sort (L, orderlessp);
(%o3) [0, 1, 1.0, 2, 3, sin(1), 1.0b0, %e, %i, %i + 1, inf, minf, 2 x]

sublist (list, p)

Returns the list of elements of list for which the predicate p returns true.

Example:
(%i1) L: [1, 2, 3, 4, 5, 6];
(%o1) [1, 2, 3, 4, 5, 6]
(%i2) sublist (L, evenp);
(%o2) [2, 4, 6]

sublist_indices (L, P)

Returns the indices of the elements x of the list L for which the predicate maybe(P(x)) returns true; this excludes unknown as well as false. P may be the name of a function or a lambda expression. L must be a literal list.

Examples:
(%i1) sublist_indices ('[a, b, b, c, 1, 2, b, 3, b], lambda ([x], x='b));
(%o1) [2, 3, 7, 9]
(%i2) sublist_indices ('[a, b, b, c, 1, 2, b, 3, b], symbolp);
(%o2) [1, 2, 3, 4, 7, 9]
(%i3) sublist_indices ([1 > 0, 1 < 0, 2 < 1, 2 > 1, 2 > 0], identity);
(%o3) [1, 4, 5]
(%i4) assume (x < -1);
(%o4) [x < - 1]
(%i5) map (maybe, [x > 0, x < 0, x < -2]);
(%o5) [false, true, unknown]
(%i6) sublist_indices ([x > 0, x < 0, x < -2], identity);
(%o6) [2]

tenth (expr)

Returns the 10th item of expression or list expr. See first for more details.

third (expr)

Returns the 3rd item of expression or list expr. See first for more details.

tree_reduce

tree_reduce (F, s)

Extends the binary function F to an n-ary function by composition, where s is a set or list.

tree_reduce is equivalent to the following: Apply F to successive pairs of elements to form a new list [F(s_1, s_2), F(s_3, s_4), ...], carrying the final element
unchanged if there are an odd number of elements. Then repeat until the list is reduced to a single element, which is the return value.

When the optional argument \( s_0 \) is present, the result is equivalent to \( \text{tree_reduce}(F, \text{cons}(s_0, s)) \).

For addition of floating point numbers, \( \text{tree_reduce} \) may return a sum that has a smaller rounding error than either \( \text{rreduce} \) or \( \text{lreduce} \).

The elements of \( s \) and the partial results may be arranged in a minimum-depth binary tree, thus the name "tree_reduce".

Examples:

\( \text{tree_reduce} \) applied to a list with an even number of elements.

\[
(\%i1) \quad \text{tree_reduce}\left(f, [a, b, c, d]\right);
(\%o1) \quad f(f(a, b), f(c, d))
\]

\( \text{tree_reduce} \) applied to a list with an odd number of elements.

\[
(\%i1) \quad \text{tree_reduce}\left(f, [a, b, c, d, e]\right);
(\%o1) \quad f(f(f(a, b), f(c, d)), e)
\]

\text{unique}(L) \quad [\text{Function}]

Returns the unique elements of the list \( L \).

When all the elements of \( L \) are unique, \text{unique} returns a shallow copy of \( L \), not \( L \) itself.

If \( L \) is not a list, \text{unique} returns \( L \).

Example:

\[
(\%i1) \quad \text{unique}\left([1, \%pi, a + b, 2, 1, \%e, \%pi, a + b, [1]]\right);
(\%o1) \quad [1, 2, \%e, \%pi, [1], b + a]
\]

\text{xreduce} \quad [\text{Function}]

\[
\text{xreduce}(F, s)
\]

\[
\text{xreduce}(F, s, s_0)
\]

Extends the function \( F \) to an n-ary function by composition, or, if \( F \) is already n-ary, applies \( F \) to \( s \). When \( F \) is not n-ary, \text{xreduce} is the same as \text{lreduce}. The argument \( s \) is a list.

Functions known to be n-ary include addition +, multiplication *, and or, max, min, and append. Functions may also be declared n-ary by \text{declare}(F, nary). For these functions, \text{xreduce} is expected to be faster than either \text{rreduce} or \text{lreduce}.

When the optional argument \( s_0 \) is present, the result is equivalent to \( \text{xreduce}(s, \text{cons}(s_0, s)) \).

Floating point addition is not exactly associative; be that as it may, \text{xreduce} applies Maxima’s n-ary addition when \( s \) contains floating point numbers.

Examples:

\( \text{xreduce} \) applied to a function known to be n-ary. \( F \) is called once, with all arguments.

\[
(\%i1) \quad \text{declare}\left(F, \text{nary}\right);
(\%o1) \quad \text{done}
(\%i2) \quad F([L]) := L;
(\%o2) \quad F([L]) := L
\]
(%i3) xreduce (F, [a, b, c, d, e]);

(%o3)                      [a, b, c, d, e]

xreduce applied to a function not known to be n-ary. G is called several times, with
two arguments each time.

(%i1) G ([L]) := L;

(%o1)                         G([L]) := L

(%i2) xreduce (G, [a, b, c, d, e]);

(%o2)                      [[[a, b], c], d], e]

(%i3) lreduce (G, [a, b, c, d, e]);

(%o3)                      [[[a, b], c], d], e]

5.4.3 Performance considerations for Lists

Lists provide efficient ways of appending and removing elements. They can be created
without knowing their final dimensions. Lisp provides efficient means of copying and han-
dling lists. Also nested lists do not need to be strictly rectangular. These advantages over
declared arrays come with the drawback that the amount of time needed for accessing a
random element within a list may be roughly proportional to the element’s distance from
its beginning. Efficient traversal of lists is still possible, though, by using the list as a stack
or a fifo:

(%i1) l: [Test, 1, 2, 3, 4];

(%o1)                     [Test, 1, 2, 3, 4]

(%i2) while l # [] do
    disp(pop(l));

    Test
    1
    2
    3
    4

(%o2)  done

Another even faster example would be:

(%i1) l: [Test, 1, 2, 3, 4];

(%o1)                     [Test, 1, 2, 3, 4]

(%i2) for i in l do
    disp(pop(l));

    Test
    1
    2
    3
Beginning traversal with the last element of a list is possible after reversing the list using `reverse ()`. If the elements of a long list need to be processed in a different order performance might be increased by converting the list into a declared array first.

Note also that the ending condition of `for` loops is tested for every iteration which means that the result of a `length` should be cached if it is used in the ending condition:

```
(%i1) l: makelist (i, i, 1, 100000)$
(%i2) length : length (l);
(%o2) 100000
(%i3) x : 1;
(%o3) 1
(%i4) for i : 1 thru length do 
   x : x + 1$
(%i5) x;
(%o5) 100001
```
5.5 Arrays

Maxima supports 3 array-like constructs:

- If one tries to write to an indexed variable without creating a list first an undeclared array (also named hashed array) is created that grows dynamically and allows numbers, symbols and strings as indices:

```lisp
(%i1) a["feww"] : 1;
(%o1) 1
(%i2) a[qqwdqwd] : 3;
(%o2) 3
(%i3) a[5] : 99;
(%o3) 99
(%i4) a[qqwdqwd];
(%o4) 3
(%i5) a[5];
(%o5) 99
(%i6) a["feww"];
(%o6) 1
```

Since lisp handles hashed arrays and memoizing functions similar to arrays many of the functions that can be applied to arrays can be applied to them, as well.

- Lists (see `makelist`) allow for fast addition and removal of elements, can be created without knowing their final size.

- Declared arrays that allow fast access to random elements at the cost that their size needs to be known at construction time. (See Section 5.4.3 [Performance considerations for Lists], page 74.)

5.5.1 Functions and Variables for Arrays

`array` [Function]

- `array (name, dim_1, ..., dim_n)`
- `array (name, type, dim_1, ..., dim_n)`
- `array ([name_1, ..., name_m], dim_1, ..., dim_n)`

Creates an n-dimensional array. n may be less than or equal to 5. The subscripts for the i'th dimension are the integers running from 0 to dim_i.

`array (name, dim_1, ..., dim_n)` creates a general array.

`array (name, type, dim_1, ..., dim_n)` creates an array, with elements of a specified type. type can be `fixnum` for integers of limited size or `flonum` for floating-point numbers.

`array ([name_1, ..., name_m], dim_1, ..., dim_n)` creates m arrays, all of the same dimensions.

See also `arraymake`, `arrayinfo` and `make_array`.

`arrayapply (A, [i_1, ..., i_n])` [Function]

Evaluates `A [i_1, ..., i_n]`, where `A` is an array and `i_1, ..., i_n` are integers.

This is reminiscent of `apply`, except the first argument is an array instead of a function.
arrayinfo \((A)\) [Function]

Returns information about the array \(A\). The argument \(A\) may be a declared array, a hashed array, a memoizing function, or a subscripted function.

For declared arrays, \texttt{arrayinfo} returns a list comprising the atom \texttt{declared}, the number of dimensions, and the size of each dimension. The elements of the array, both bound and unbound, are returned by \texttt{listarray}.

For undeclared arrays (hashed arrays), \texttt{arrayinfo} returns a list comprising the atom \texttt{hashed}, the number of subscripts, and the subscripts of every element which has a value. The values are returned by \texttt{listarray}.

For memoizing functions, \texttt{arrayinfo} returns a list comprising the atom \texttt{hashed}, the number of subscripts, and any subscript values for which there are stored function values. The stored function values are returned by \texttt{listarray}.

For subscripted functions, \texttt{arrayinfo} returns a list comprising the atom \texttt{hashed}, the number of subscripts, and any subscript values for which there are lambda expressions. The lambda expressions are returned by \texttt{listarray}.

See also \texttt{listarray}.

Examples:

\texttt{arrayinfo} and \texttt{listarray} applied to a declared array.

\begin{verbatim}
(\%i1) array (aa, 2, 3);
(\%o1) aa
(\%i2) aa [2, 3] : %pi;
(\%o2) %pi
(\%i3) aa [1, 2] : %e;
(\%o3) %e
(\%i4) arrayinfo (aa);
(\%o4) [declared, 2, [2, 3]]
(\%i5) listarray (aa);
(\%o5) [#####, #####, #####, #####, #####, #####, %e, #####,

#####, #####, #####, %pi]
\end{verbatim}

\texttt{arrayinfo} and \texttt{listarray} applied to an undeclared array (hashed array.).

\begin{verbatim}
(\%i1) bb [FOO] : (a + b)^2;
(\%o1) (b + a)
(\%i2) bb [BAR] : (c - d)^3;
(\%o2) (c - d)
(\%i3) arrayinfo (bb);
(\%o3) [hashed, 1, [BAR], [FOO]]
(\%i4) listarray (bb);
(\%o4) [(c - d), (b + a)]
\end{verbatim}

\texttt{arrayinfo} and \texttt{listarray} applied to a memoizing function.
Using \texttt{arrayinfo} in order to convert an undeclared array to a declared array:

\begin{verbatim}
(%i1) for i:0 thru 10 do a[i]:i^2$
(%i2) indices:map(first,rest(rest(arrayinfo(a))))$
(%i3) array(A,fixnum,length(indices)-1)$
(%i4) fillarray(A,map(lambda([x],a[x]),indices))$
(%i5) listarray(A);
(%o5) [0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100]
\end{verbatim}

\texttt{arrayinfo} and \texttt{listarray} applied to a subscripted function.

\begin{verbatim}
(%i1) dd [x] (y) := y ^ x;
(%o1) dd (y) := y
(%i2) dd [a + b];
(%o2) lambda([y], y )
(%i3) dd [v - u];
(%o3) lambda([y], y )
(%i4) arrayinfo (dd);
(%o4) [hashed, 1, [b + a], [v - u]]
(%i5) listarray (dd);
(%o5) [lambda([y], y ), lambda([y], y )]
\end{verbatim}

\texttt{arraymake} \hspace{1em} \texttt{(A, \{i_1, \ldots, i_n\})} \hspace{1em} \text{[Function]}

Returns the expression \texttt{A[i_1, \ldots, i_n]}. The result is an unevaluated array reference.
arraymake is reminiscent of funmake, except the return value is an unevaluated array reference instead of an unevaluated function call.

Examples:

```
(%i1) arraymake (A, [1]);
(%o1) A

(%i2) arraymake (A, [k]);
(%o2) A

(%i3) arraymake (A, [i, j, 3]);
(%o3) A

(%i4) array (A, fixnum, 10);
(%o4) A

(%i5) fillarray (A, makelist (i^2, i, 1, 11));
(%o5) A

(%i6) arraymake (A, [5]);
(%o6) A

(%i7) ''%;
(%o7) 36

(%i8) L : [a, b, c, d, e];
(%o8) [a, b, c, d, e]

(%i9) arraymake ('L, [n]);
(%o9) L

(%i10) ''%, n = 3;
(%o10) c

(%i11) A2 : make_array (fixnum, 10);
(%o11) {Lisp Array: #(0 0 0 0 0 0 0 0 0 0)}

(%i12) fillarray (A2, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]);
(%o12) {Lisp Array: #(1 2 3 4 5 6 7 8 9 10)}

(%i13) arraymake ('A2, [8]);
(%o13) A2

(%i14) ''%;
(%o14) 9
```

arrays

Default value: []

arrays is a list of arrays that have been allocated. These comprise arrays declared by array, hashed arrays that can be constructed by implicit definition (assigning something to an element that isn’t yet declared as a list or an array), and memoizing functions defined by := and define. Arrays defined by make_array are not included.

See also array, arrayapply, arrayinfo, arraymake, fillarray, listarray, and rearray.
Examples:

```
(%i1) array (aa, 5, 7);
(%o1) aa

(%i2) bb [FOO] : (a + b)^2;
(%o2) (b + a)

(%i3) cc [x] := x/100;
(%o3) cc := ---
     x 100

(%i4) dd : make_array ('any, 7);
(%o4) {Lisp Array: #(NIL NIL NIL NIL NIL NIL NIL)}

(%i5) arrays;
(%o5) [aa, bb, cc]
```

`arraysetapply (A, [i_1, ..., i_n], x)`  
Assigns `x` to `A[i_1, ..., i_n]`, where `A` is an array and `i_1, ..., i_n` are integers. `arraysetapply` evaluates its arguments.

`fillarray (A, B)`  
Fills array `A` from `B`, which is a list or an array.

If a specific type was declared for `A` when it was created, it can only be filled with elements of that same type; it is an error if an attempt is made to copy an element of a different type.

If the dimensions of the arrays `A` and `B` are different, `A` is filled in row-major order. If there are not enough elements in `B` the last element is used to fill out the rest of `A`. If there are too many, the remaining ones are ignored.

`fillarray` returns its first argument.

Examples:

Create an array of 9 elements and fill it from a list.

```
(%i1) array (a1, fixnum, 8);
(%o1) a1

(%i2) listarray (a1);
(%o2) [0, 0, 0, 0, 0, 0, 0, 0, 0]

(%i3) fillarray (a1, [1, 2, 3, 4, 5, 6, 7, 8, 9]);
(%o3) a1

(%i4) listarray (a1);
(%o4) [1, 2, 3, 4, 5, 6, 7, 8, 9]
```

When there are too few elements to fill the array, the last element is repeated. When there are too many elements, the extra elements are ignored.

```
(%i1) a2 : make_array (fixnum, 8);
(%o1) {Lisp Array: #(0 0 0 0 0 0 0 0)}

(%i2) fillarray (a2, [1, 2, 3, 4, 5]);
(%o2) {Lisp Array: #(1 2 3 4 5 5 5 5)}

(%i3) fillarray (a2, [4]);
(%o3) {Lisp Array: #(4 4 4 4 4 4 4 4)}
```
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\[
\text{(%i4) fillarray (a2, makelist (i, i, 1, 100));}
\]
\[
\text{(Lisp Array: #(1 2 3 4 5 6 7 8))}
\]

Multiple-dimension arrays are filled in row-major order.

\[
\text{(%i11) a3 : make_array (fixnum, 2, 5);} \\
\text{(Lisp Array: #2A((0 0 0 0 0) (0 0 0 0 0)))}
\]
\[
\text{(%i12) fillarray (a3, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]);}
\]
\[
\text{(Lisp Array: #2A((1 2 3 4 5) (6 7 8 9 10)))}
\]
\[
\text{(%i13) a4 : make_array (fixnum, 5, 2);} \\
\text{(Lisp Array: #2A((0 0) (0 0) (0 0) (0 0) (0 0)))}
\]
\[
\text{(%i14) fillarray (a4, a3);} \\
\text{(Lisp Array: #2A((1 2) (3 4) (5 6) (7 8) (9 10)))}
\]

\textbf{listarray (A)}

Returns a list of the elements of the array A. The argument A may be an array, an undeclared array (hashed array), a memoizing function, or a subscripted function.

Elements are listed in row-major order. That is, elements are sorted according to the first index, then according to the second index, and so on. The sorting order of index values is the same as the order established by \texttt{orderless}.

For undeclared arrays (hashed arrays), memoizing functions, and subscripted functions, the elements correspond to the index values returned by \texttt{arrayinfo}.

Unbound elements of general arrays (that is, not \texttt{fixnum} and not \texttt{flonum}) are returned as #####. Unbound elements of \texttt{fixnum} or \texttt{flonum} arrays are returned as 0 or 0.0, respectively. Unbound elements of hashed arrays, memoizing functions, and subscripted functions are not returned.

Examples:

\texttt{listarray} and \texttt{arrayinfo} applied to a declared array.

\[
\text{(%i11) array (aa, 2, 3);} \\
\text{(%i1) aa} \\
\text{(%i2) aa [2, 3] : %pi;}
\]
\[
\text{(%i3) aa [1, 2] : %e;}
\]
\[
\text{(%i4) listarray (aa);} \\
\text{(%o4) [#####, ##### , ###### , ###### , ###### , ###### , %e, ###### ,}
\text{##### , %pi]}
\]
\[
\text{(%i5) arrayinfo (aa);} \\
\text{(declared, 2, [2, 3])}
\]

\texttt{listarray} and \texttt{arrayinfo} applied to an undeclared array (hashed array).

\[
\text{(%i11) bb [FOO] : (a + b)^2;}
\]
\[
\text{(%o1) (b + a)}
\]
\[
\text{(%i12) bb [BAR] : (c - d)^3;}
\]
\[
\text{(%o2) (c - d)}
\]
listarray and arrayinfo applied to a memoizing function.

```maxima
(%i1) cc [x, y] := y / x;
    y
   cc := -
       x
(%i2) cc [u, v];
    v
   -
   u
(%i3) cc [4, z];
    z
   -
   4
(%i4) listarray (cc);
    z
   [-]
   4
(%i5) arrayinfo (cc);
    [hashed, 2, [4, z], [u, v]]
```

listarray and arrayinfo applied to a subscripted function.

```maxima
(%i1) dd [x] (y) := y ^ x;
   x
   dd (y) := y
(%i2) dd [a + b];
   b + a
   lambda([y], y   )
(%i3) dd [v - u];
   v - u
   lambda([y], y   )
(%i4) listarray (dd);
   b + a
   v - u
   lambda([y], y   ), lambda([y], y   )
(%i5) arrayinfo (dd);
    [hashed, 1, [b + a], [v - u]]
```

**make_array**

`make_array (type, dim_1, ..., dim_n)`

[Function]

Creates and returns a Lisp array. *type* may be any, flonum, fixnum, hashed or functional. There are *n* indices, and the *i*’th index runs from 0 to `dim_i - 1`.

The advantage of `make_array` over `array` is that the return value doesn’t have a name, and once a pointer to it goes away, it will also go away. For example, if `y`:
make_array (...) then y points to an object which takes up space, but after y: false, y no longer points to that object, so the object can be garbage collected.

Examples:

```lisp
(%i1) A1 : make_array (fixnum, 10);
(%o1) {Lisp Array: #(0 0 0 0 0 0 0 0 0 0)}
1729
(%i3) A1; {Lisp Array: #(0 0 0 0 0 0 1729 0 0 0)}
(%i4) A2 : make_array (flonum, 10);
(%o4) {Lisp Array: #(0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0)}
2.718281828
(%i6) A2; {Lisp Array: #(0.0 0.0 2.718281828 0.0 0.0 0.0 0.0 0.0 0.0 0.0)}
(%i7) A3 : make_array (any, 10);
(%o7) {Lisp Array: #(NIL NIL NIL NIL NIL NIL NIL NIL NIL)}
(%i8) A3 [4] : x - y - z;
(%i9) A3; {Lisp Array: #(NIL NIL NIL NIL
((MPLUS SIMP) $X ((MTIMES SIMP) -1 $Y) ((MTIMES SIMP) -1 $Z))
NIL NIL NIL NIL)}
(%i10) A4 : make_array (fixnum, 2, 3, 5);
(%o10) {Lisp Array: #3A(((0 0 0 0 0) (0 0 0 0 0) (0 0 0 0 0))
((0 0 0 0 0) (0 0 0 0 0) (0 0 0 0 0)))}
(%i11) fillarray (A4, makelist (i, i, 1, 2*3*5));
(%o11) {Lisp Array: #3A(((1 2 3 4 5) (6 7 8 9 10) (11 12 13 14 15))
((16 17 18 19 20) (21 22 23 24 25) (26 27 28 29 30)))}
(%i12) A4 [0, 2, 1];
12
```

rearray \(A, \text{dim}_1, \ldots, \text{dim}_n\)  
[Function]  
Changes the dimensions of an array. The new array will be filled with the elements of the old one in row-major order. If the old array was too small, the remaining elements are filled with false, 0.0 or 0, depending on the type of the array. The type of the array cannot be changed.
remarray

remarray (A_1, . . ., A_n)
remarray (all)

Removes arrays and array associated functions and frees the storage occupied. The arguments may be declared arrays, hashed arrays, array functions, and subscripted functions.

remarray (all) removes all items in the global list arrays.

It may be necessary to use this function if it is desired to clear the cache of a memoizing function.

remarray returns the list of arrays removed.

remarray quotes its arguments.

subvar (x, i)

Evaluates the subscripted expression x[i].

subvar evaluates its arguments.

arraymake (x, [i]) constructs the expression x[i], but does not evaluate it.

Examples:

(%i1) x : foo $
(%i2) i : 3 $
(%i3) subvar (x, i);
(%o3) foo
3
(%i4) foo : [aa, bb, cc, dd, ee]$  
(%i5) subvar (x, i);
(%o5) cc
(%i6) arraymake (x, [i]);  
(%o6) foo
3
(%i7) ''%;  
(%o7) cc

subvarp (expr)

Returns true if expr is a subscripted variable, for example a[i].

use_fast_arrays

Default value: false

When use_fast_arrays is true, arrays declared by array are values instead of properties, and undeclared arrays (hashed arrays) are implemented as Lisp hashed arrays.

When use_fast_arrays is false, arrays declared by array are properties, and undeclared arrays are implemented with Maxima’s own hashed array implementation.

Note that the code use_fast_arrays switches to is not necessarily faster than the default one; Arrays created by make_array are not affected by use_fast_arrays.

See also translate_fast_arrays.
translate_fast_arrays

[Option variable]

Default value: false

When translate_fast_arrays is true, the Maxima-to-Lisp translator generates code that assumes arrays are values instead of properties, as if use_fast_arrays were true.

When translate_fast_arrays is false, the Maxima-to-Lisp translator generates code that assumes arrays are properties, as if use_fast_arrays were false.
5.6 Structures

5.6.1 Introduction to Structures

Maxima provides a simple data aggregate called a structure. A structure is an expression in which arguments are identified by name (the field name) and the expression as a whole is identified by its operator (the structure name). A field value can be any expression.

A structure is defined by the `defstruct` function; the global variable `structures` is the list of user-defined structures. The function `new` creates instances of structures. The `@` operator refers to fields. `kill(S)` removes the structure definition `S`, and `kill(x@ a)` unbinds the field `a` of the structure instance `x`.

In the pretty-printing console display (with `display2d` equal to `true`), structure instances are displayed with the value of each field represented as an equation, with the field name on the left-hand side and the value on the right-hand side. (The equation is only a display construct; only the value is actually stored.) In 1-dimensional display (via `grind` or with `display2d` equal to `false`), structure instances are displayed without the field names.

There is no way to use a field name as a function name, although a field value can be a lambda expression. Nor can the values of fields be restricted to certain types; any field can be assigned any kind of expression. There is no way to make some fields accessible or inaccessible in different contexts; all fields are always visible.

5.6.2 Functions and Variables for Structures

`structures` [Global variable]
`structures` is the list of user-defined structures defined by `defstruct`.

`defstruct` [Function]

```
defstruct (S(a_1, ..., a_n))
defstruct (S(a_1 = v_1, ..., a_n = v_n))
```

Define a structure, which is a list of named fields `a_1`, ..., `a_n` associated with a symbol `S`. An instance of a structure is just an expression which has operator `S` and exactly `n` arguments. `new(S)` creates a new instance of structure `S`.

An argument which is just a symbol `a` specifies the name of a field. An argument which is an equation `a = v` specifies the field name `a` and its default value `v`. The default value can be any expression.

`defstruct` puts `S` on the list of user-defined structures, `structures`.

`kill(S)` removes `S` from the list of user-defined structures, and removes the structure definition.

Examples:
```
(%i1) defstruct (foo (a, b, c));
(%o1) [foo(a, b, c)]
(%i2) structures;
(%o2) [foo(a, b, c)]
(%i3) new (foo);
(%o3) foo(a, b, c)
(%i4) defstruct (bar (v, w, x = 123, y = %pi));
```
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(%o4) bar(v, w, x = 123, y = %pi)
(%i5) structures;
(%o5) [foo(a, b, c), bar(v, w, x = 123, y = %pi)]
(%i6) new (bar);
(%o6) bar(v, w, x = 123, y = %pi)
(%i7) kill (foo);
(%o7) done
(%i8) structures;
(%o8) [bar(v, w, x = 123, y = %pi)]

new

new (S)
new (S(v_1, ..., v_n))
new creates new instances of structures.
new(S) creates a new instance of structure S in which each field is assigned its default value, if any, or no value at all if no default was specified in the structure definition.
new(S(v_1, ..., v_n)) creates a new instance of S in which fields are assigned the values v_1, ..., v_n.

Examples:
(%i1) defstruct (foo (w, x = %e, y = 42, z));
(%o1) [foo(w, x = %e, y = 42, z)]
(%i2) new (foo);
(%o2) foo(w, x = %e, y = 42, z)
(%i3) new (foo (1, 2, 4, 8));
(%o3) foo(w = 1, x = 2, y = 4, z = 8)

@

@ is the structure field access operator. The expression x@ a refers to the value of field a of the structure instance x. The field name is not evaluated.
If the field a in x has not been assigned a value, x@ a evaluates to itself.
kill(x@ a) removes the value of field a in x.

Examples:
(%i1) defstruct (foo (x, y, z));
(%o1) [foo(x, y, z)]
(%i2) u : new (foo (123, a - b, %pi));
(%o2) foo(x = 123, y = a - b, z = %pi)
(%i3) u@z;
(%o3) %pi
(%i4) u@z : %e;
(%o4) %e
(%i5) u;
(%o5) foo(x = 123, y = a - b, z = %e)
(%i6) kill (u@z);
(%o6) done
(%i7) u;
(%o7) foo(x = 123, y = a - b, z)
(%i8) u@z;
(%o8) u@z

The field name is not evaluated.

(%i1) defstruct (bar (g, h));
(%o1) [bar(g, h)]
(%i2) x : new (bar);
(%o2) bar(g, h)
(%i3) x@h : 42;
(%o3) 42
(%i4) h : 123;
(%o4) 123
(%i5) x@h;
(%o5) 42
(%i6) x@h : 19;
(%o6) 19
(%i7) x;
(%o7) bar(g, h = 19)
(%i8) h;
(%o8) 123
6 Expressions

6.1 Introduction to Expressions

There are a number of reserved words which should not be used as variable names. Their
use would cause a possibly cryptic syntax error.

<table>
<thead>
<tr>
<th>integrate</th>
<th>next</th>
<th>from</th>
<th>diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>in</td>
<td>at</td>
<td>limit</td>
<td>sum</td>
</tr>
<tr>
<td>for</td>
<td>and</td>
<td>elseif</td>
<td>then</td>
</tr>
<tr>
<td>else</td>
<td>do</td>
<td>or</td>
<td>if</td>
</tr>
<tr>
<td>unless</td>
<td>product</td>
<td>while</td>
<td>thru</td>
</tr>
</tbody>
</table>

Most things in Maxima are expressions. A sequence of expressions can be made into an
expression by separating them by commas and putting parentheses around them. This is
similar to the C comma expression.

(%i1) x: 3$
(%i2) (x: x+1, x: x^2);
(%o2) 16
(%i3) (if (x > 17) then 2 else 4);
(%o3) 4
(%i4) (if (x > 17) then x: 2 else y: 4, y+x);
(%o4) 20

Even loops in Maxima are expressions, although the value they return is the not too
useful done.

(%i1) y: (x: 1, for i from 1 thru 10 do (x: x*i))$
(%i2) y;
(%o2) done

Whereas what you really want is probably to include a third term in the comma expres-
sion which actually gives back the value.

(%i3) y: (x: 1, for i from 1 thru 10 do (x: x*i), x)$
(%i4) y;
(%o4) 3628800

6.2 Nouns and Verbs

Maxima distinguishes between operators which are "nouns" and operators which are
"verbs". A verb is an operator which can be executed. A noun is an operator which
appears as a symbol in an expression, without being executed. By default, function names
are verbs. A verb can be changed into a noun by quoting the function name or applying
the nounify function. A noun can be changed into a verb by applying the verbify
function. The evaluation flag nouns causes ev to evaluate nouns in an expression.

The verb form is distinguished by a leading dollar sign $ on the corresponding Lisp
symbol. In contrast, the noun form is distinguished by a leading percent sign % on the
corresponding Lisp symbol. Some nouns have special display properties, such as 'integrate
and 'derivative (returned by diff), but most do not. By default, the noun and verb
forms of a function are identical when displayed. The global flag `noundisp` causes Maxima to display nouns with a leading quote mark `'
``.

See also `noun`, `nouns`, `nounify`, and `verbify`.

Examples:

```maxima
(%i1) foo (x) := x^2;
2
(%o1) foo(x) := x
(%i2) foo (42);
(%o2) 1764
(%i3) 'foo (42);
(%o3) foo(42)
(%i4) 'foo (42), nouns;
(%o4) 1764
(%i5) declare (bar, noun);
(%o5) done
(%i6) bar (x) := x/17;
(%o6) bar(x) := --
     17
(%i7) bar (52);
(%o7) bar(52)
(%i8) bar (52), nouns;
(%o8) bar(52)
(%i9) integrate (1/x, x, 1, 42);
(%o9) log(42)
(%i10) 'integrate (1/x, x, 1, 42);
 (%o10)
 42
 / [ 1
 \ ] x
 1
(%i11) ev (%);  
(%o11) log(42)
```

### 6.3 Identifiers

Maxima identifiers may comprise alphabetic characters, plus the numerals 0 through 9, plus any other character preceded by the backslash `\` character.

A numeral may be the first character of an identifier if it is preceded by a backslash. Numerals which are the second or later characters need not be preceded by a backslash.

The alphabetic characters are initially `%`, `_`, and all characters for which the Lisp function `ALPHA-CHAR-P` returns `true`. Characters may be declared alphabetic by the `declare` function. If so declared, they need not be preceded by a backslash in an identifier.
Maxima is case-sensitive. The identifiers foo, F00, and Foo are distinct. See Section 37.1 [Lisp and Maxima], page 661, for more on this point.

A Maxima identifier is a Lisp symbol which begins with a dollar sign $. Any other Lisp symbol is preceded by a question mark ? when it appears in Maxima. See Section 37.1 [Lisp and Maxima], page 661, for more on this point.

Examples:

```
(%i1) %an_ordinary_identifier42;
(%o1) %an_ordinary_identifier42
(%i2) embedded\ spaces\ in\ an\ identifier;
(%o2) embedded spaces in an identifier
(%i3) symbolp (%);
(%o3) true
(%i4) [foo+bar, foo\+bar];
(%o4) [foo + bar, foo+bar]
(%i5) [1729, \1729];
(%o5) [1729, 1729]
(%i6) [symbolp (foo\+bar), symbolp (\1729)];
(%o6) [true, true]
(%i7) [is (foo\+bar = foo+bar), is (\1729 = 1729)];
(%o7) [false, false]
(%i8) baz\~quux;
(%o8) baz~quux
(%i9) declare ("~", alphabetic);
(%o9) done
(%i10) baz\~quux;
(%o10) baz~quux
(%i11) [is (foo = FOO), is (F00 = Foo), is (Foo = foo)];
(%o11) [false, false, false]
(%i12) :lisp (defvar *my-lisp-variable* '$foo)
*MY-LISP-VARIABLE*
(%i12) ?*my\-lisp\-variable*
(%o12) foo
```

### 6.4 Inequality

Maxima has the inequality operators <, <=, >=, >, #, and notequal. See [if], page 669, for a description of conditional expressions.

### 6.5 Functions and Variables for Expressions

**alias** (new_name_1, old_name_1, ..., new_name_n, old_name_n) [Function]  
provides an alternate name for a (user or system) function, variable, array, etc. Any even number of arguments may be used.

**aliases** [System variable]  
Default value: []
aliases is the list of atoms which have a user defined alias (set up by the alias, ordergreat, orderless functions or by declaring the atom a noun with declare.)

allbut [Keyword]
works with the part commands (i.e. part, inpart, substpart, substinpart, dpart, and lpart). For example,

```maxima
(%i1) expr : e + d + c + b + a;
(%o1) e + d + c + b + a
(%i2) part (expr, [2, 5]);
(%o2) d + a
```
while

```maxima
(%i1) expr : e + d + c + b + a;
(%o1) e + d + c + b + a
(%i2) part (expr, allbut (2, 5));
(%o2) e + c + b
```

allbut is also recognized by kill.

```maxima
(%i1) [aa : 11, bb : 22, cc : 33, dd : 44, ee : 55];
(%o1) [11, 22, 33, 44, 55]
(%i2) kill (allbut (cc, dd));
(%o0) done
(%i1) [aa, bb, cc, dd];
(%o1) [aa, bb, 33, 44]
```

kill(allbut(a_1, a_2, ...)) has the effect of kill(all) except that it does not kill the symbols a_1, a_2, ...

args (expr) [Function]
Returns the list of arguments of expr, which may be any kind of expression other than an atom. Only the arguments of the top-level operator are extracted; subexpressions of expr appear as elements or subexpressions of elements of the list of arguments.
The order of the items in the list may depend on the global flag inflag.

args (expr) is equivalent to substpart ("[", expr, 0). See also substpart, apply, funmake, and op.

How to convert a matrix to a nested list:

```maxima
(%i1) M:matrix([1,2],[3,4]);
   [ 1  2 ]
(%o1) [     ]
   [ 3  4 ]
(%i2) args(M);
(%o2) [[1, 2], [3, 4]]
```

Since maxima internally treats a sum of n terms as a summation command with n arguments args() can extract the list of terms in a sum:

```maxima
(%i1) a+b+c;
(%o1) c + b + a
(%i2) args(%);
(%o2) [c, b, a]
```
atom (expr)  [Function]
Returns true if expr is atomic (i.e. a number, name or string) else false. Thus atom(5) is true while atom(a[1]) and atom(sin(x)) are false (assuming a[1] and x are unbound).

box  [Function]
box (expr)
box (expr, a)
Returns expr enclosed in a box. The return value is an expression with box as the operator and expr as the argument. A box is drawn on the display when display2d is true.
box (expr, a) encloses expr in a box labelled by the symbol a. The label is truncated if it is longer than the width of the box.
box evaluates its argument. However, a boxed expression does not evaluate to its content, so boxed expressions are effectively excluded from computations. rembox removes the box again.
boxchar is the character used to draw the box in box and in the dpart and lpart functions.
See also rembox, dpart and lpart.

Examples:
(%i1) box (a^2 + b^2);

(%o1) (b + a^2)

(%i2) a : 1234;
(%o2) 1234

(%i3) b : c - d;
(%o3) c - d

(%i4) box (a^2 + b^2);

(%o4) (c - d) + 1522756

(%i5) box (a^2 + b^2, term_1);

(%o5) (c - d) + 1522756

(%i6) 1729 - box (1729);  

(%o6) 1729 - "1729"

(%i7) boxchar: ";

(%o7) "

(%i8) boxchar: ";

(%o8) ";"
(%i8) box (sin(x) + cos(y));

- cos(y) + sin(x) -

boxchar

[Option variable]

Default value: "

boxchar is the character used to draw the box in the box and in the dpart and lpart functions.

All boxes in an expression are drawn with the current value of boxchar; the drawing character is not stored with the box expression.

collapse (expr)

[Function]
Collapses expr by causing all of its common (i.e., equal) subexpressions to share (i.e., use the same cells), thereby saving space. (collapse is a subroutine used by the optimize command.) Thus, calling collapse may be useful after loading in a save file. You can collapse several expressions together by using collapse ([expr_1, ..., expr_n]). Similarly, you can collapse the elements of the array A by doing collapse (listarray ('A)).

copy (e)

[Function]
Return a copy of the Maxima expression e. Although e can be any Maxima expression, the copy function is the most useful when e is either a list or a matrix; consider:

(%i1) m : [1,[2,3]]$
(%i2) mm : m$
(%i3) mm[2][1] : x$
(%i4) m;
(%o4) [1, [x, 3]]
(%i5) mm;
(%o5) [1, [x, 3]]

Let’s try the same experiment, but this time let mm be a copy of m

(%i1) m : [1,[2,3]]$
(%i2) mm : copy(m)$
(%i3) mm[2][1] : x$
(%i4) m;
(%o4) [1, [2, 3]]
(%i5) mm;
(%o5) [1, [x, 3]]

This time, the assignment to mm does not change the value of m.

disolate (expr, x_1, ..., x_n)

[Function]
is similar to isolate (expr, x) except that it enables the user to isolate more than one variable simultaneously. This might be useful, for example, if one were attempting to change variables in a multiple integration, and that variable change involved two or more of the integration variables. This function is autoloaded from simplification/disol.mac. A demo is available by demo("disol")$.
dispform \[\text{dispform} (\text{expr})\]
\[\text{dispform} (\text{expr}, \text{all})\]
Returns the external representation of \text{expr}.

\text{dispform} (\text{expr})\] returns the external representation with respect to the main (top-level) operator. \text{dispform} (\text{expr}, \text{all}) returns the external representation with respect to all operators in \text{expr}.

See also \text{part}, \text{inpart}, and \text{inflag}.

Examples:

The internal representation of \(- x\) is "negative one times \(x\)" while the external representation is "minus \(x\)".

\[
\begin{align*}
(\%i1) & \quad - x; \\
(\%o1) & \quad - x \\
(\%i2) & \quad \text{?format (true, "\text{-}S\%", \%)}; \\
& \quad ((\text{MTIMES SIMP}) -1 \$X) \\
(\%o2) & \quad \text{false} \\
(\%i3) & \quad \text{dispform (- x)}; \\
(\%o3) & \quad - x \\
(\%i4) & \quad \text{?format (true, "\text{-}S\%", \%)}; \\
& \quad ((\text{MMINUS SIMP}) \$X) \\
(\%o4) & \quad \text{false}
\end{align*}
\]

The internal representation of \(\sqrt{x}\) is "\(x\) to the power 1/2" while the external representation is "square root of \(x\)".

\[
\begin{align*}
(\%i1) & \quad \text{sqrt} (x); \\
(\%o1) & \quad \text{sqrt}(x) \\
(\%i2) & \quad \text{?format (true, "\text{-}S\%", \%)}; \\
& \quad ((\text{MEXPT SIMP}) \$X (\text{((\text{RAT SIMP}) 1 2)))} \\
(\%o2) & \quad \text{false} \\
(\%i3) & \quad \text{dispform (sqrt (x))}; \\
(\%o3) & \quad \text{sqrt}(x) \\
(\%i4) & \quad \text{?format (true, "\text{-}S\%", \%)}; \\
& \quad ((\text{%SQRT SIMP}) \$X) \\
(\%o4) & \quad \text{false}
\end{align*}
\]

Use of the optional argument \text{all}.

\[
\begin{align*}
(\%i1) & \quad \text{expr : sin} (\text{sqrt} (x)); \\
(\%o1) & \quad \text{sin} (\text{sqrt}(x)) \\
(\%i2) & \quad \text{freeof} (\text{sqrt}, \text{expr}); \\
(\%o2) & \quad \text{true} \\
(\%i3) & \quad \text{freeof} (\text{sqrt}, \text{dispform} (\text{expr})); \\
(\%o3) & \quad \text{true} \\
(\%i4) & \quad \text{freeof} (\text{sqrt}, \text{dispform} (\text{expr, all})); \\
(\%o4) & \quad \text{false}
\end{align*}
\]
**dpart (expr, n_1, ..., n_k)**

Selects the same subexpression as `part`, but instead of just returning that subexpression as its value, it returns the whole expression with the selected subexpression displayed inside a box. The box is actually part of the expression.

```
(%i1) dpart (x+y/z^2, 1, 2, 1);
y
---- + x
2
"z"
""
```

**exptisolate**

Default value: `false`

`exptisolate`, when `true`, causes `isolate (expr, var)` to examine exponents of atoms (such as `%e`) which contain `var`.

**exptsubst**

Default value: `false`

`exptsubst`, when `true`, permits substitutions such as `y` for `%e^-x` in `%e^- (a x)`.

```
(%i1) %e^- (a*x);
a x
(%o1) %e
(%i2) exptsubst;
(%o2) false
(%i3) subst(y, %e^-x, %e^- (a*x));
a x
(%o3) %e
(%i4) exptsubst: not exptsubst;
(%o4) true
(%i5) subst(y, %e^-x, %e^- (a*x));
a
(%o5) y
```

**freeof (x_1, ..., x_n, expr)**

`freeof (x_1, expr)` returns `true` if no subexpression of `expr` is equal to `x_1` or if `x_1` occurs only as a dummy variable in `expr`, or if `x_1` is neither the noun nor verb form of any operator in `expr`, and returns `false` otherwise.

`freeof (x_1, ..., x_n, expr)` is equivalent to `freeof (x_1, expr) and ... and freeof (x_n, expr)`.

The arguments `x_1, ..., x_n` may be names of functions and variables, subscripted names, operators (enclosed in double quotes), or general expressions. `freeof` evaluates its arguments.

`freeof` operates only on `expr` as it stands (after simplification and evaluation) and does not attempt to determine if some equivalent expression would give a different result. In particular, simplification may yield an equivalent but different expression which comprises some different elements than the original form of `expr`.
A variable is a dummy variable in an expression if it has no binding outside of the expression. Dummy variables recognized by `freeof` are the index of a sum or product, the limit variable in `limit`, the integration variable in the definite integral form of `integrate`, the original variable in `laplace`, formal variables in `at` expressions, and arguments in `lambda` expressions.

The indefinite form of `integrate` is not free of its variable of integration.

Examples:

Arguments are names of functions, variables, subscripted names, operators, and expressions. `freeof (a, b, expr)` is equivalent to `freeof (a, expr)` and `freeof (b, expr)`.

```lisp
(%i1) expr: z^3 * cos (a[1]) * b^(c+d);
   3
d + c    3
(%o1) cos(a ) b z
     1

(%i2) freeof (z, expr);
(%o2) false

(%i3) freeof (cos, expr);
(%o3) false

(%i4) freeof (a[1], expr);
(%o4) false

(%i5) freeof (cos (a[1]), expr);
(%o5) false

(%i6) freeof (b^(c+d), expr);
(%o6) false

(%i7) freeof ("^", expr);
(%o7) false

(%i8) freeof (w, sin, a[2], sin (a[2]), b*(c+d), expr);
(%o8) true
```

`freeof` evaluates its arguments.

```lisp
(%i11) expr: (a+b)^5$

(%i2) c: a$

(%i3) freeof (c, expr);
(%o3) false
```

`freeof` does not consider equivalent expressions. Simplification may yield an equivalent but different expression.

```lisp
(%i11) expr: (a+b)^5$
(%i2) expand (expr);
   5  4  2  3  3  2  4  5
(%o2) b + 5 a b + 10 a b + 10 a b + 5 a b + a

(%i3) freeof (a+b, %);
(%o3) true

(%i4) freeof (a+b, expr);
(%o4) false

(%i5) exp (x);
     x
```
A summation or definite integral is free of its dummy variable. An indefinite integral is not free of its variable of integration.

(%i1) freeof (i, 'sum (f(i), i, 0, n));
(%o1) true
(%i2) freeof (x, 'integrate (x^2, x, 0, 1));
(%o2) true
(%i3) freeof (x, 'integrate (x^2, x));
(%o3) false

inflag

Default value: false

When inflag is true, functions for part extraction inspect the internal form of expr.

Note that the simplifier re-orders expressions. Thus first (x + y) returns x if inflag is true and y if inflag is false. (first (y + x) gives the same results.)

Also, setting inflag to true and calling part or substpart is the same as calling inpart or substinpart.

Functions affected by the setting of inflag are: part, substpart, first, rest, last, length, the for ... in construct, map, fullmap, maplist, reveal and pickapart.

inpart (expr, n_1, ..., n_k)

is similar to part but works on the internal representation of the expression rather than the displayed form and thus may be faster since no formatting is done. Care should be taken with respect to the order of subexpressions in sums and products (since the order of variables in the internal form is often different from that in the displayed form) and in dealing with unary minus, subtraction, and division (since these operators are removed from the expression). part (x+y, 0) or inpart (x+y, 0) yield +, though in order to refer to the operator it must be enclosed in "s. For example ... if inpart (%o9,0) = "+" then ....

Examples:

(%i1) x + y + w*z;
(%o1) w z + y + x
(%i2) inpart (%o1, 3, 2);
(%o2) z
(%i3) part (%o2, 4, 1, 2);
(%o3) z
(%i4) 'limit (f(x)^g(x+1), x, 0, minus);
(%o4) limit f(x)
     x -> 0-
(%i5) inpart (%o4, 1, 2);
(%o5) g(x + 1)
isolate (expr, x)  [Function]

Returns expr with subexpressions which are sums and which do not contain var replaced by intermediate expression labels (these being atomic symbols like %t1, %t2, ...). This is often useful to avoid unnecessary expansion of subexpressions which don’t contain the variable of interest. Since the intermediate labels are bound to the subexpressions they can all be substituted back by evaluating the expression in which they occur.

exptisolate (default value: false) if true will cause isolate to examine exponents of atoms (like %e) which contain var.

isolate_wrt_times if true, then isolate will also isolate with respect to products. See isolate_wrt_times. See also disolate.

Do example (isolate) for examples.

isolate_wrt_times  [Option variable]

Default value: false

When isolate_wrt_times is true, isolate will also isolate with respect to products. E.g. compare both settings of the switch on

(%i1) isolate_wrt_times: true$
(%i2) isolate (expand ((a+b+c)^2), c);

(%%t2) 2 a

(%t3) 2 b

(%t4) b + 2 a b + a

(%%o4) c + %t3 c + %t2 c + %t4
(%i4) isolate_wrt_times: false$
(%i5) isolate (expand ((a+b+c)^2), c);

(%%o5) c + 2 b c + 2 a c + %t4

listconstvars  [Option variable]

Default value: false

When listconstvars is true the list returned by listofvars contains constant variables, such as %e, %pi, %i or any variables declared as constant that occur in expr. A variable is declared as constant type via declare, and constantp returns true for all variables declared as constant. The default is to omit constant variables from listofvars return value.

listdummyvars  [Option variable]

Default value: true

When listdummyvars is false, "dummy variables" in the expression will not be included in the list returned by listofvars. (The meaning of "dummy variables" is
as given in \texttt{freeof}. "Dummy variables" are mathematical things like the index of a sum or product, the limit variable, and the definite integration variable.)

Example:

\begin{verbatim}
(%i1) listdummyvars: true$
(%i2) listofvars ('sum(f(i), i, 0, n));
[ i, n]
(%i3) listdummyvars: false$
(%i4) listofvars ('sum(f(i), i, 0, n));
[n]
\end{verbatim}

\textbf{listofvars (expr)}

Returns a list of the variables in \textit{expr}.

\textbf{listconstvars} if \textit{true} causes \textit{listofvars} to include \%e, \%pi, \%i, and any variables declared constant in the list it returns if they appear in \textit{expr}. The default is to omit these.

See also the option variable \textit{listdummyvars} to exclude or include "dummy variables" in the list of variables.

\begin{verbatim}
(%i1) listofvars (f (x[1]+y) / g^(2+a));
[g, a, x , y]
\end{verbatim}

\textbf{lfreeof (list, expr)}

For each member \textit{m} of \textit{list}, calls \textit{freeof} \( (m, expr) \). It returns \textit{false} if any call to \textit{freeof} does and \textit{true} otherwise.

Example:

\begin{verbatim}
(%i1) lfreeof ([ a, x], x^2+b);
false
(%i2) lfreeof ([ b, x], x^2+b);
false
(%i3) lfreeof ([ a, y], x^2+b);
true
\end{verbatim}

\textbf{lpart (label, expr, n_1, \ldots, n_k)}

is similar to \textbf{dpart} but uses a labelled box. A labelled box is similar to the one produced by \textbf{dpart} but it has a name in the top line.

\textbf{mainvar}

You may declare variables to be \textit{mainvar}. The ordering scale for atoms is essentially: numbers \textless{} constants (e.g., \%e, \%pi) \textless{} scalars \textless{} other variables \textless{} mainvars. E.g., compare \texttt{expand ((x+y)^4)} with \texttt{(declare (x, mainvar), expand ((x+y)^4))}. (Note: Care should be taken if you elect to use the above feature. E.g., if you subtract an expression in which \textit{x} is a \textit{mainvar} from one in which \textit{x} isn’t a \textit{mainvar}, resimplification e.g. with \texttt{ev (expr, simp)} may be necessary if cancellation is to occur. Also, if you save an expression in which \textit{x} is a \textit{mainvar}, you probably should also save \textit{x}.)

\textbf{noun}

\textit{noun} is one of the options of the \textbf{declare} command. It makes a function so declared a "noun", meaning that it won’t be evaluated automatically.
Example:

(%i1) factor (12345678);  
   2
(%o1) 2 3 47 14593
(%i2) declare (factor, noun);  
   done
(%i3) factor (12345678);  
   factor(12345678)
(%i4) ''%, nouns;  
   2
(%o4) 2 3 47 14593

noundisp

Default value: false

When noundisp is true, nouns display with a single quote. This switch is always true when displaying function definitions.

nounify (f)

Returns the noun form of the function name f. This is needed if one wishes to refer to the name of a verb function as if it were a noun. Note that some verb functions will return their noun forms if they can’t be evaluated for certain arguments. This is also the form returned if a function call is preceded by a quote.

See also verbify.

nterms (expr)

Returns the number of terms that expr would have if it were fully expanded out and no cancellations or combination of terms occurred. Note that expressions like \texttt{sin (expr)}, \texttt{sqrt (expr)}, \texttt{exp (expr)}, etc. count as just one term regardless of how many terms expr has (if it is a sum).

op (expr)

Returns the main operator of the expression expr. \texttt{op (expr)} is equivalent to \texttt{part (expr, 0)}.

\texttt{op} returns a string if the main operator is a built-in or user-defined prefix, binary or n-ary infix, postfix, matchfix, or nofix operator. Otherwise, if expr is a subscripted function expression, \texttt{op} returns the subscripted function; in this case the return value is not an atom. Otherwise, expr is a \texttt{memoizing function} or ordinary function expression, and \texttt{op} returns a symbol.

\texttt{op} observes the value of the global flag \texttt{inflag}.

\texttt{op} evaluates it argument.

See also \texttt{args}.

Examples:

(%i1) stringdisp: true$
(%i2) op (a * b * c);  
   "*"
(%i3) op (a * b + c);  
   "+"
(%i4) op ('sin (a + b));
   sin
(%o4) sin
(%i5) op (a!);
   "!
(%o5) !
(%i6) op (-a);
   "-
(%o6) -
(%i7) op ([a, b, c]);
   "[
(%o7) [a, b, c]
(%i8) op ('(if a > b then c else d));
   "if
(%o8) if
(%i9) op ('foo (a));
   "foo
(%o9) foo
(%i10) prefix (foo);
   "foo
(%o10) foo
(%i11) op (foo a);
   "foo
(%o11) foo
(%i12) op (F [x, y] (a, b, c));
   F
   x, y
(%o12) F
   x, y
(%i13) op (G [u, v, w]);
   "G
(%o13) G

operatorp

operatorp (expr, op)
operatorp (expr, [op_1, ..., op_n])
operatorp (expr, op) returns true if op is equal to the operator of expr.
operatorp (expr, [op_1, ..., op_n]) returns true if some element op_1, ..., op_n is equal to the operator of expr.

opsubst

Default value: true

When opsubst is false, subst does not attempt to substitute into the operator of an expression. E.g., (opsubst: false, subst (x^2, r, r+r[0])) will work.

(%i11) r+r[0];
   r + r
(%o11) r + r
(%i12) opsubst;
   true
(%i13) subst (x^2, r, r+r[0]);
   2   2
   x + (x )
   0
(%o13) x + (x )
(%i14) opsubst: not opsubst;
   false
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(%i5) subst (x^2, r, r+r[0]);
    2
(%o5) x + r

optimize (expr) [Function]
Returns an expression that produces the same value and side effects as expr but
does so more efficiently by avoiding the recomputation of common subexpressions.
optimize also has the side effect of "collapsing" its argument so that all common
subexpressions are shared. Do example (optimize) for examples.

optimprefix [Option variable]
Default value: %

optimprefix is the prefix used for generated symbols by the optimize command.

ordergreatp (expr_1, expr_2) [Function]
orderlessp (expr_1, expr_2) [Function]
ordergreatp returns true if expr_1 succeeds expr_2 in the canonical ordering of
Maxima expressions, and false otherwise.

orderlessp returns true if expr_1 precedes expr_2 in the canonical ordering of Max-
ima expressions, and false otherwise.

All Maxima atoms and expressions are comparable under ordergreatp and
orderlessp, although there are isolated examples of expressions for which these
predicates are not transitive; that is a bug.

The canonical ordering of atoms (symbols, literal numbers, and strings) is the follow-
ing.

(integers and floats) precede (bigfloats) precede (declared constants) precede (strings)
precede (declared scalars) precede (first argument to orderless) precedes ... precedes
(last argument to orderless) precedes (other symbols) precede (last argument to order-
great) precedes ... precedes (first argument to ordergreat) precedes (decl-
ared main variables)

For non-atomic expressions, the canonical ordering is derived from the ordering for
atoms. For the built-in + * and ^ operators, the ordering is not easily summarized.
For other built-in operators and all other functions and operators, expressions are ordered by their arguments (beginning with the first argument), then by the name of the operator or function. In the case of subscripted expressions, the subscripted symbol is considered the operator and the subscript is considered an argument.

The canonical ordering of expressions is modified by the functions `ordergreat` and `orderless`, and the `mainvar`, `constant`, and `scalar` declarations.

See also `sort`.

Examples:

Ordering ordinary symbols and constants. Note that `%pi` is not ordered according to its numerical value.

```maxima
(%i1) stringdisp : true;
(%o1) true
(%i2) sort ([%pi, 3b0, 3.0, x, X, "foo", 3, a, 4, "bar", 4.0, 4b0]);
(%o2) [3, 3.0, 4, 4.0, 3.0b0, 4.0b0, %pi, "bar", "foo", X, a, x]
```

Effect of `ordergreat` and `orderless` functions.

```maxima
(%i1) sort ([M, H, K, T, E, W, G, A, P, J, S]);
(%i2) ordergreat (S, J);
(%o2) done
(%i3) orderless (M, H);
(%o3) done
(%i4) sort ([M, H, K, T, E, W, G, A, P, J, S]);
```

Effect of `mainvar`, `constant`, and `scalar` declarations.

```maxima
(%i1) sort ([aa, foo, bar, bb, baz, quux, cc, dd, A1, B1, C1]);
(%o1) [A1, B1, C1, aa, bar, baz, bb, cc, dd, foo, quux]
(%i2) declare (aa, mainvar);
(%o2) done
(%i3) declare ([baz, quux], constant);
(%o3) done
(%i4) declare ([A1, B1], scalar);
(%o4) done
(%i5) sort ([aa, foo, bar, bb, baz, quux, cc, dd, A1, B1, C1]);
(%o5) [baz, quux, A1, B1, C1, bar, bb, cc, dd, foo, aa]
```

Ordering non-atomic expressions.

```maxima
(%i1) sort ([1, 2, n, f(1), f(2), f(2, 1), g(1), g(1, 2), g(n), f(n, 1)]);
(%o1) [1, 2, f(1), g(1), g(1, 2), f(2), f(2, 1), n, g(n), f(n, 1)]
(%i2) sort ([foo(1), X[1], X[k], foo(k), 1, k]);
(%o2) [1, X, foo(1), k, X, foo(k)]
```
part \( (\text{expr}, n_1, \ldots, n_k) \)  

Returns parts of the displayed form of \( \text{expr} \). It obtains the part of \( \text{expr} \) as specified by the indices \( n_1, \ldots, n_k \). First part \( n_1 \) of \( \text{expr} \) is obtained, then part \( n_2 \) of that, etc. The result is part \( n_k \) of . . . part \( n_2 \) of part \( n_1 \) of \( \text{expr} \). If no indices are specified \( \text{expr} \) is returned.

\text{part} can be used to obtain an element of a list, a row of a matrix, etc.

If the last argument to a \text{part} function is a list of indices then several subexpressions are picked out, each one corresponding to an index of the list. Thus \text{part} \( (x + y + z, [1, 3]) \) is \( z+x \).

\text{piece} holds the last expression selected when using the \text{part} functions. It is set during the execution of the function and thus may be referred to in the function itself as shown below.

If \text{partswitch} is set to \text{true} then \text{end} is returned when a selected part of an expression doesn’t exist, otherwise an error message is given.

See also \text{inpart}, \text{substpart}, \text{substinpart}, \text{dpart}, and \text{lpart}.

Examples:

```plaintext
(%i1) part(z+2*y+a,2);
(%o1) 2 y
(%i2) part(z+2*y+a,[1,3]);
(%o2) z + a
(%i3) part(z+2*y+a,2,1);  
(%o3) 2
```

\text{example (part)} displays additional examples.

partition \( \text{(expr, x)} \)  

Returns a list of two expressions. They are (1) the factors of \( \text{expr} \) (if it is a product), the terms of \( \text{expr} \) (if it is a sum), or the list (if it is a list) which don’t contain \( \text{x} \) and, (2) the factors, terms, or list which do.

Examples:

```plaintext
(%i1) partition (2*a*x*f(x), x);
(%o1) [2 a, x f(x)]
(%i2) partition (a+b, x);
(%o2) [b + a, 0]
(%i3) partition ([a, b, f(a), c], a);
(%o3) [[b, c], [a, f(a)]]
```

partswitch  

Default value: \text{false}

When \text{partswitch} is \text{true}, \text{end} is returned when a selected part of an expression doesn’t exist, otherwise an error message is given.

pickapart \( \text{(expr, n)} \)  

Assigns intermediate expression labels to subexpressions of \( \text{expr} \) at depth \( n \), an integer. Subexpressions at greater or lesser depths are not assigned labels. \text{pickapart} returns an expression in terms of intermediate expressions equivalent to the original expression \( \text{expr} \).
See also \texttt{part}, \texttt{dpart}, \texttt{lpart}, \texttt{inpart}, and \texttt{reveal}.

Examples:

\begin{verbatim}
(\%i1) expr: (a+b)/2 + sin (x^2)/3 - log (1 + sqrt(x+1));
       2
  sin(x)  b + a
(\%o1) - log(sqrt(x + 1) + 1) + ------- + ----- 
          3  2
(\%i2) pickapart (expr, 0);
       2
  sin(x)  b + a
(\%t2) - log(sqrt(x + 1) + 1) + ------- + ----- 
          3  2
(\%o2) \%t2
(\%i3) pickapart (expr, 1);

(\%t3) - log(sqrt(x + 1) + 1)

       2
  sin(x)
(\%t4) ------- 
          3

(\%t5) b + a 
 ----- 
    2
(\%o5) \%t5 + \%t4 + \%t3
(\%i5) pickapart (expr, 2);

(\%t6) log(sqrt(x + 1) + 1)

       2
  sin(x)
(\%t7) ------- 
          3

(\%t8) b + a

       2  3
\%t8 \%t7
(\%o8) ---- + ---- - \%t6
(\%i8) pickapart (expr, 3);

(\%t9) sqrt(x + 1) + 1
\end{verbatim}
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\[
\begin{align*}
\text{(\%t10)} & \quad \frac{2}{x} \\
\text{(\%o10)} & \quad \frac{b + a}{2} \sin(\%t10) - \log(\%t9) + \frac{\text{---------}}{3} \\
\text{(\%i10)} & \quad \text{pickapart (expr, 4);} \\
\text{(\%t11)} & \quad \sqrt{x + 1} \\
\text{(\%o11)} & \quad \frac{2}{\sin(x)} \frac{b + a}{3} + \frac{\text{---------}}{2} - \log(\%t11 + 1) \\
\text{(\%i11)} & \quad \text{pickapart (expr, 5);} \\
\text{(\%t12)} & \quad x + 1 \\
\text{(\%o12)} & \quad \frac{2}{\sin(x)} \frac{b + a}{3} + \frac{\text{---------}}{2} - \log(\sqrt{\%t12} + 1) \\
\text{(\%i12)} & \quad \text{pickapart (expr, 6);} \\
\end{align*}
\]

\text{piece} \qquad \text{[System variable]}

Holds the last expression selected when using the part functions. It is set during the execution of the function and thus may be referred to in the function itself.

\text{psubst} \qquad \text{[Function]}

\[
\begin{align*}
\text{psubst (list, expr)} \\
\text{psubst (a, b, expr)} \\
\text{psubst(a, b, expr) is similar to subst. See subst.} \\
\end{align*}
\]

In distinction from subst the function psubst makes parallel substitutions, if the first argument list is a list of equations.

See also \text{sublis} for making parallel substitutions and \text{let} and \text{letsimp} for others ways to do substitutions.

Example:

The first example shows parallel substitution with psubst. The second example shows the result for the function subst, which does a serial substitution.

\[
\begin{align*}
\text{(\%i11)} & \quad \text{psubst ([a^2=b, b=a], \sin(a^2) + \sin(b));} \\
\text{(\%o11)} & \quad \sin(b) + \sin(a)
\end{align*}
\]
(%i2) subst ([a^2=b, b=a], sin(a^2) + sin(b));
(%o2) 2 sin(a)

rembox
rembox (expr, unlabelled)
rembox (expr, label)
rembox (expr)
Removes boxes from expr.

rembox (expr, unlabelled) removes all unlabelled boxes from expr.
rembox (expr, label) removes only boxes bearing label.
rembox (expr) removes all boxes, labelled and unlabelled.

Boxes are drawn by the box, dpart, and lpart functions.

Examples:

(%i11) expr: (a*d - b*c)/h^2 + sin(%pi*x);
   a d - b c
(%o11) sin(%pi x) + ---------
     2
     h
(%i12) dpart (dpart (expr, 1, 1), 2, 2);
dpart: fell off the end.
-- an error. To debug this try: debugmode(true);
(%i13) expr2: lpart (BAR, lpart (FOO, %, 1), 2);
   BAR
   "a d - b c"
   "sin(%pi x)" + "---------"
   " 2 "
   " h "
   " "
(%o13) BAR
   "a d - b c"
   "sin(%pi x)" + "---------"
   " 2 "
   " h "
   " "
(%i14) rembox (expr2, unlabelled);
   BAR
   "a d - b c"
   "sin(%pi x)" + "---------"
   " 2 "
   " h "
   " "
(%o14) BAR
   "a d - b c"
   "sin(%pi x)" + "---------"
   " 2 "
   " h "
   " "
(%i15) rembox (expr2, FOO);
   BAR
   "a d - b c"
   "sin(%pi x)" + "---------"
   " 2 "
   " h "
   " "
(%o15) BAR
   "a d - b c"
   "sin(%pi x)" + "---------"
   " 2 "
   " h "
   " "
(%i6) rembox (expr2, BAR);
   FOO
   "sin(%pi x)" + ---------
   2
   h
(%o6) %pi

(%i7) rembox (expr2);
   a d - b c
(%o7) sin(%pi x) + ---------
   2
   h

reveal (expr, depth) [Function]
Replaces parts of expr at the specified integer depth with descriptive summaries.

- Sums and differences are replaced by Sum(n) where n is the number of operands of the sum.
- Products are replaced by Product(n) where n is the number of operands of the product.
- Exponentials are replaced by Expt.
- Quotients are replaced by Quotient.
- Unary negation is replaced by Negterm.
- Lists are replaced by List(n) where n is the number of elements of the list.

When depth is greater than or equal to the maximum depth of expr, reveal (expr, depth) returns expr unmodified.

reveal evaluates its arguments. reveal returns the summarized expression.

Example:

(%i1) e: expand ((a - b)^2)/expand ((exp(a) + exp(b))^2);
2 2
   b - 2 a b + a
(%o1) -----------------------------------
   b + a 2 b 2 a
   2 %e + %e + %e
(%i2) reveal (e, 1);
(%o2) Quotient
(%i3) reveal (e, 2);
(%o3) Sum(3)
(%i4) reveal (e, 3);
(%o4) Product(2) + Expt + Expt
(%i5) reveal (e, 4);
2 2
   b - Product(3) + a
(%o5) -----------------------------------
The function `sqrtdenest` denests square roots of simple, numerical, binomial surds, where possible. For example:

\[
\frac{\sqrt{\sqrt{3}/2 + 1}}{\sqrt{11\sqrt{2} - 12}}
\]

\[
\frac{\sqrt{3}}{\sqrt{\frac{3}{2} + 1}}
\]

\[
\frac{1}{\sqrt{11\sqrt{2} - 12}}
\]

\[
\frac{\sqrt{3}}{2} + \frac{1}{2}
\]

\[
\frac{3^{1/4}}{2} - \frac{2^{3/4}}{2}
\]

Substitution can sometimes help, especially when applying `sqrtdenest` more than once. For example:

\[
(19601 - 13860\sqrt{2})^{7/4}
\]

\[
\frac{1}{4} - \frac{3}{4}
\]

\[
3^{2/4} - 2^{3/4}
\]

The `sublis` function makes multiple parallel substitutions into an expression. For example:

\[
\text{sublis}([a=b, b=a], \sin(a) + \cos(b));
\]

\[
\sin(b) + \cos(a)
\]

The variable `sublis_apply_lambda` controls simplification after `sublis`. Its default value is `true`, meaning lambda's substituted are applied in simplification after `sublis` is used or whether you have to do an `ev` to get things to apply. `true` means do the application.
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subnumsimp

Default value: false

If true then the functions subst and psubst can substitute a subscripted variable \( f[x] \) with a number, when only the symbol \( f \) is given. See also subst.

(%i1) subst(100,g,g[x]+2);
subst: cannot substitute 100 for operator g in expression g
-- an error. To debug this try: debugmode(true);

(%i2) subst(100,g,g[x]+2),subnumsimp:true;
(%o2) 102

subst \((a, b, c)\)

Substitutes \( a \) for \( b \) in \( c \). \( b \) must be an atom or a complete subexpression of \( c \). For example, \( x+y+z \) is a complete subexpression of \( 2*(x+y+z)/w \) while \( x+y \) is not. When \( b \) does not have these characteristics, one may sometimes use substpart or ratsubst (see below). Alternatively, if \( b \) is of the form \( e/f \) then one could use subst \((a*f, e, c)\) while if \( b \) is of the form \( e^{1/f} \) then one could use subst \((a^f, e, c)\). The subst command also discerns the \( x^y \) in \( x^{-y} \) so that subst \((a, sqrt(x), 1/sqrt(x))\) yields \( 1/a \). \( a \) and \( b \) may also be operators of an expression enclosed in double-quotes " or they may be function names. If one wishes to substitute for the independent variable in derivative forms then the at function (see below) should be used.

subst is an alias for substitute.

The commands subst \((eq_1, expr)\) or subst \(([eq_1, \ldots, eq_k], expr)\) are other permissible forms. The \( eq_i \) are equations indicating substitutions to be made. For each equation, the right side will be substituted for the left in the expression expr. The equations are substituted in serial from left to right in expr. See the functions sublis and psubst for making parallel substitutions.

exptsubst if true permits substitutions like \( y \) for \( %e^x \) in \( %e^{a*x} \) to take place.

When opsubst is false, subst will not attempt to substitute into the operator of an expression. E.g. \((opsubst: false, subst (x^2, r, r+r[0]))\) will work.

See also at, ev and psubst, as well as let and lsetimp.

Examples:

(%i1) subst (a, x+y, x + (x+y)^2 + y);
2
(%o1) y + x + a

(%i2) subst (-%i, %i, a + b+%i);
(%o2) a - %i b

The substitution is done in serial for a list of equations. Compare this with a parallel substitution:

(%i1) subst([a=b, b=c], a+b);
(%o1) 2 c
(%i2) sublis([a=b, b=c], a+b);
(%o2) c + b

Single-character Operators like + and - have to be quoted in order to be replaced by subst. It is to note, though, that \(a+b-c\) might be expressed as \(a+b+(-1\times c)\) internally.

(%i3) subst(["+"="-"],a+b-c);
(%o3) c-b+a

The difference between subst and at can be seen in the following example:

(%i1) g1: y(t)=a*x(t)+b*diff(x(t),t);
   \[
   y(t) = b \left(\frac{d}{dt} (x(t))\right) + a \times (t) 
   \]
(%o1) \[
\begin{align*}
\frac{d}{dt} (x(t)) &= 1 \\
\end{align*}
\]

(%i2) subst('diff(x(t),t)=1,g1);
(%o2) y(t) = a \times (t) + b

(%i3) at(g1,'diff(x(t),t)=1);
\[
\begin{align*}
\frac{d}{dt} (x(t)) &= 1 \\
\end{align*}
\]

For further examples, do example (subst).

\textbf{substinp}art \((x, \text{expr}, n_1, \ldots, n_k)\)  

[Function]

Similar to substpart, but substinpart works on the internal representation of expr.

Examples:

(%i1) x . 'diff (f(x), x, 2);
\[
\begin{align*}
\frac{d}{dx} x . (-\frac{\partial}{\partial x} (f(x))) \\
\end{align*}
\]

(%o1) \[
\begin{align*}
\frac{d}{dx} x . (-\frac{\partial}{\partial x} (f(x))) \\
\end{align*}
\]

(%i2) substinpart (d^2, %, 2);
\[
\begin{align*}
\frac{d}{dx} x . \left(\frac{\partial}{\partial x} (f(x))\right) \\
\end{align*}
\]

(%o2) \[
\begin{align*}
\frac{d}{dx} x . \left(\frac{\partial}{\partial x} (f(x))\right) \\
\end{align*}
\]

(%i3) substinpart (f1, f[1](x + 1), 0);
(%o3) f1(x + 1)

If the last argument to a part function is a list of indices then several subexpressions are picked out, each one corresponding to an index of the list. Thus

(%i1) part (x + y + z, [1, 3]);
(%o1) z + x

\textbf{piece} holds the value of the last expression selected when using the part functions. It is set during the execution of the function and thus may be referred to in the function itself as shown below. If partswitch is set to true then end is returned when a selected part of an expression doesn’t exist, otherwise an error message is given.
Chapter 6: Expressions

(%i1) expr: 27*y^3 + 54*x*y^2 + 36*x^2*y + y + 8*x^3 + x + 1;
            3  2  2  3
(%o1)    27 y + 54 x y + 36 x y + y + 8 x + x + 1
(%i2) part (expr, 2, [1, 3]);
     2
(%o2)       54 y
(%i3) sqrt (piece/54);
(%o3) abs(y)
(%i4) substpart (factor (piece), expr, [1, 2, 3, 5]);
     3
(%o4) (3 y + 2 x) + y + x + 1
(%i5) expr: 1/x + y/x - 1/z;
     1 y 1
(%o5) (- -) + - + -
     z x x
(%i6) substpart (xthru (piece), expr, [2, 3]);
     y + 1 1
(%o6) ----- - -
     x z

Also, setting the option inflag to true and calling part or substpart is the same as calling inpart or substinpant.

substpart (x, expr, n_1, . . ., n_k)

Substitutes x for the subexpression picked out by the rest of the arguments as in part. It returns the new value of expr. x may be some operator to be substituted for an operator of expr. In some cases x needs to be enclosed in double-quotes " (e.g. substpart ("+", a*b, 0) yields b + a).

Example:

(%i1) 1/(x^2 + 2);
(%o1)
     1
-----
     2
     x + 2
(%i2) substpart (3/2, %, 2, 1, 2);
(%o2)
     1
-----
     3/2
     x + 2
(%i3) a*x + f(b, y);
(%o3) a x + f(b, y)
(%i4) substpart ("+", %, 1, 0);
(%o4) x + f(b, y) + a

Also, setting the option inflag to true and calling part or substpart is the same as calling inpart or substinpant.

symbolp (expr)

Returns true if expr is a symbol, else false.
See also Section 6.3 [Identifiers], page 90.

unorder () [Function]
Disables the aliasing created by the last use of the ordering commands ordergreat and orderless. ordergreat and orderless may not be used more than one time each without calling unorder. unorder does not substitute back in expressions the original symbols for the aliases introduced by ordergreat and orderless. Therefore, after execution of unorder the aliases appear in previous expressions.
See also ordergreat and orderless.
Examples:
ordergreat(a) introduces an alias for the symbol a. Therefore, the difference of %o2 and %o4 does not vanish. unorder does not substitute back the symbol a and the alias appears in the output %o7.

(%i1) unorder();
          []
(%i2) b*x + a^2;
      2
(%o2) b x + a
(%i3) ordergreat (a);
(%o3) done
(%i4) b*x + a^2;
2
(%o4) a + b x
(%i5) unorder();
          2 2
(%o5) a - a
(%i6) %th(2);
          [a]

verbify (f) [Function]
Returns the verb form of the function name f. See also verb, noun, and nounify.
Examples:

(%i1) verbify ('foo);
      foo
(%i2) :lisp $%
$FOO
(%i2) nounify (foo);
      foo
(%i3) :lisp $%
$FOO
7 Operators

7.1 Introduction to operators

It is possible to define new operators with specified precedence, to undefine existing operators, or to redefine the precedence of existing operators. An operator may be unary prefix or unary postfix, binary infix, n-ary infix, matchfix, or nofix. "Matchfix" means a pair of symbols which enclose their argument or arguments, and "nofix" means an operator which takes no arguments. As examples of the different types of operators, there are the following.

- unary prefix
  - negation - a
- unary postfix
  - factorial a!
- binary infix
  - exponentiation a^b
- n-ary infix
  - addition a + b
- matchfix
  - list construction [a, b]

(There are no built-in nofix operators; for an example of such an operator, see nofix.)

The mechanism to define a new operator is straightforward. It is only necessary to declare a function as an operator; the operator function might or might not be defined.

An example of user-defined operators is the following. Note that the explicit function call "dd" (a) is equivalent to dd a, likewise "<-" (a, b) is equivalent to a <- b. Note also that the functions 'dd" and "<-" are undefined in this example.

```
(%i1) prefix ("dd");
  prefix: $dd$
(%o1)  prefix

(%i2) dd a;
  d a
(%o2)  d a

(%i3) "dd" (a);
  "dd" a
(%o3)  "dd" a

(%i4) infix ("<-");
  infix: $<-$
(%o4)  infix

(%i5) a <- dd b;
  a <- d b
(%o5)  a <- d b

(%i6) "<-" (a, "dd" (b));
  "<-" a, "dd" b
(%o6)  "<-" a, "dd" b
```

The Maxima functions which define new operators are summarized in this table, stating the default left and right binding powers (lbp and rbp, respectively). (Binding power determines operator precedence. However, since left and right binding powers can differ, binding power is somewhat more complicated than precedence.) Some of the operation definition functions take additional arguments; see the function descriptions for details.

<table>
<thead>
<tr>
<th>operator</th>
<th>rbp</th>
<th>lbp</th>
</tr>
</thead>
<tbody>
<tr>
<td>prefix</td>
<td>180</td>
<td></td>
</tr>
<tr>
<td>postfix</td>
<td></td>
<td>180</td>
</tr>
</tbody>
</table>
infix \( \text{lbp}=180, \text{rbp}=180 \)

nary \( \text{lbp}=180, \text{rbp}=180 \)

matchfix (binding power not applicable)

nofix (binding power not applicable)

For comparison, here are some built-in operators and their left and right binding powers.

<table>
<thead>
<tr>
<th>Operator</th>
<th>lbp</th>
<th>rbp</th>
</tr>
</thead>
<tbody>
<tr>
<td>:</td>
<td>180</td>
<td>20</td>
</tr>
<tr>
<td>::</td>
<td>180</td>
<td>20</td>
</tr>
<tr>
<td>:=</td>
<td>180</td>
<td>20</td>
</tr>
<tr>
<td>::=</td>
<td>180</td>
<td>20</td>
</tr>
<tr>
<td>!</td>
<td>160</td>
<td></td>
</tr>
<tr>
<td>!!</td>
<td>160</td>
<td></td>
</tr>
<tr>
<td>~</td>
<td>140</td>
<td>139</td>
</tr>
<tr>
<td>.</td>
<td>130</td>
<td>129</td>
</tr>
<tr>
<td>*</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>/</td>
<td>120</td>
<td>120</td>
</tr>
<tr>
<td>+</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>-</td>
<td>100</td>
<td>134</td>
</tr>
<tr>
<td>=</td>
<td>80</td>
<td>80</td>
</tr>
<tr>
<td>#</td>
<td>80</td>
<td>80</td>
</tr>
<tr>
<td>&gt;</td>
<td>80</td>
<td>80</td>
</tr>
<tr>
<td>&gt;=</td>
<td>80</td>
<td>80</td>
</tr>
<tr>
<td>&lt;</td>
<td>80</td>
<td>80</td>
</tr>
<tr>
<td>&lt;=</td>
<td>80</td>
<td>80</td>
</tr>
<tr>
<td>not</td>
<td></td>
<td>70</td>
</tr>
<tr>
<td>and</td>
<td>65</td>
<td></td>
</tr>
<tr>
<td>or</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>,</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>;</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

`remove` and `kill` remove operator properties from an atom. `remove ("a", op)` removes only the operator properties of `a`. `kill ("a")` removes all properties of `a`, including the operator properties. Note that the name of the operator must be enclosed in quotation marks.

(%i1) infix ("##");
(%o1) ##

(%i2) "##" (a, b) := a\^b;
(%o2) a ## b := a\^b

(%i3) 5 ## 3;
(%o3) 125

(%i4) remove ("##", op);
(%o4) done
7.2 Arithmetic operators

The symbols +, *, /, and ^ represent addition, multiplication, division, and exponentiation, respectively. The names of these operators are "+", "*", "/", and "^", which may appear where the name of a function or operator is required.

The symbols + and - represent unary addition and negation, respectively, and the names of these operators are "+" and "-", respectively.

Subtraction \(a - b\) is represented within Maxima as addition, \(a + (-b)\). Expressions such as \(a + (-b)\) are displayed as subtraction. Maxima recognizes "-" only as the name of the unary negation operator, and not as the name of the binary subtraction operator.

Division \(a / b\) is represented within Maxima as multiplication, \(a * b^(-1)\). Expressions such as \(a * b^(-1)\) are displayed as division. Maxima recognizes "/" as the name of the division operator.

Addition and multiplication are n-ary, commutative operators. Division and exponentiation are binary, noncommutative operators.

Maxima sorts the operands of commutative operators to construct a canonical representation. For internal storage, the ordering is determined by orderlessp. For display, the ordering for addition is determined by ordergreatp, and for multiplication, it is the same as the internal ordering.

Arithmetic computations are carried out on literal numbers (integers, rationals, ordinary floats, and bigfloats). Except for exponentiation, all arithmetic operations on
numbers are simplified to numbers. Exponentiation is simplified to a number if either operand is an ordinary float or bigfloat or if the result is an exact integer or rational; otherwise an exponentiation may be simplified to \texttt{sqrt} or another exponentiation or left unchanged.

Floating-point contagion applies to arithmetic computations: if any operand is a bigfloat, the result is a bigfloat; otherwise, if any operand is an ordinary float, the result is an ordinary float; otherwise, the operands are rationals or integers and the result is a rational or integer.

Arithmetic computations are a simplification, not an evaluation. Thus arithmetic is carried out in quoted (but simplified) expressions.

Arithmetic operations are applied element-by-element to lists when the global flag \texttt{listarith} is \texttt{true}, and always applied element-by-element to matrices. When one operand is a list or matrix and another is an operand of some other type, the other operand is combined with each of the elements of the list or matrix.

Examples:

Addition and multiplication are n-ary, commutative operators. Maxima sorts the operands to construct a canonical representation. The names of these operators are \texttt{"+"} and \texttt{"*"}.

\begin{verbatim}
(%i1) c + g + d + a + b + e + f;
     g + f + e + d + c + b + a
(%o1) [op (%), args (%)];
     [+ , [g, f, e, d, c, b, a]]
(%i2) c * g * d * a * b * e * f;
     a b c d e f g
(%o2) [op (%), args (%)];
     [* , [a, b, c, d, e, f, g]]
(%i3) apply ("+", [a, 8, x, 2, 9, x, x, a]);
     3 x + 2 a + 19
(%o3) apply ("*", [a, 8, x, 2, 9, x, x, a]);
     144 a x
\end{verbatim}

Division and exponentiation are binary, noncommutative operators. The names of these operators are \texttt{"/"} and \texttt{"^"}.

\begin{verbatim}
(%i1) [a / b, a ^ b];
     a b
(%o1) [- , a ]
     b
(%i2) map (op, %), map (args, %)];
     [[/ , []], [[a, b], [a, b]]]
(%i3) apply ("/", [a, b]), apply ("^", [a, b]);
     a b
(%o3) [- , a ]
     b
\end{verbatim}

Subtraction and division are represented internally in terms of addition and multiplication, respectively.
Computations are carried out on literal numbers. Floating-point contagion applies.

\[
(\text{i1})\ 17 + b - (1/2)*29 + 11^{(2/4)}; \\
(\text{o1})\ b + \sqrt{11} + -
\]

Arithmetic computations are a simplification, not an evaluation.

\[
(\text{i1})\ \text{simp} : \text{false}; \\
(\text{o1})\ \text{false} \\
(\text{i2})\ '(17 + 29*11/7 - 5^3); \\
(\text{o2})\ 17 + ------ - 5
\]

Arithmetic is carried out element-by-element for lists (depending on listarith) and matrices.

\[
(\text{i1})\ \text{matrix} ([a, x], [h, u]) - \text{matrix} ([1, 2], [3, 4]); \\
(\text{o1})\ [a - 1 \ x - 2 ] \\
(\text{i2})\ 5 * \text{matrix} ([a, x], [h, u]); \\
(\text{o2})\ [ 5 a 5 x ] \\
(\text{i3})\ \text{listarith} : \text{false}; \\
(\text{o3})\ \text{false} \\
(\text{i4})\ [a, c, m, t] / [1, 7, 2, 9]; \\
(\text{o4})\ [a, c, m, t] \\
(\text{i5})\ [a, c, m, t] ^ x; \\
(\text{o5})\ [a, c, m, t] \ x
(\%i6) listarith : true;
(\%o6)         true
(\%i7) [a, c, m, t] / [1, 7, 2, 9];
            c m t
(\%o7)         [a, -, -, -]
            7 2 9
(\%i8) [a, c, m, t] ^ x;
            x x x x
(\%o8)         [a , c , m , t ]

**
Exponentiation operator. Maxima recognizes ** as the same operator as ~ in input, and it is displayed as ~ in 1-dimensional output, or by placing the exponent as a superscript in 2-dimensional output.

The fortran function displays the exponentiation operator as **, whether it was input as ** or ~.

Examples:
(\%i1) is (a**b = a^b);
(\%o1)         true
(\%i2) x**y + x^z;
            z y
(\%o2)         x + x
(\%i3) string (x**y + x^z);
(\%o3)         x^z+x^y
(\%i4) fortran (x**y + x^z);
            x**z+x**y
(\%o4)         done

^^
Noncommutative exponentiation operator. ^^ is the exponentiation operator corresponding to noncommutative multiplication ., just as the ordinary exponentiation operator ~ corresponds to commutative multiplication *.

Noncommutative exponentiation is displayed by ^^ in 1-dimensional output, and by placing the exponent as a superscript within angle brackets <> in 2-dimensional output.

Examples:
(\%i1) a . a . b . b + a * a * a * b * b;
            3 2 <2> <3>
(\%o1)         a b + a . b
(\%i2) string (a . a . b . b + a * a * a * b * b);
(\%o2)         a^3*b^2+a^2 . b^3

. 
The dot operator, for matrix (non-commutative) multiplication. When "." is used in this way, spaces should be left on both sides of it, e.g. A . B This distinguishes it plainly from a decimal point in a floating point number.
7.3 Relational operators

The symbols $<$, $\leq$, $\geq$, and $>$ represent less than, less than or equal, greater than or equal, and greater than, respectively. The names of these operators are "<", "\leq", "\geq", and ">", which may appear where the name of a function or operator is required.

These relational operators are all binary operators; constructs such as $a < b < c$ are not recognized by Maxima.

Relational expressions are evaluated to Boolean values by the functions `is` and `maybe`, and the programming constructs `if`, `while`, and `unless`. Relational expressions are not otherwise evaluated or simplified to Boolean values, although the arguments of relational expressions are evaluated (when evaluation is not otherwise prevented by quotation).

When a relational expression cannot be evaluated to `true` or `false`, the behavior of `is` and `if` are governed by the global flag `prederror`. When `prederror` is `true`, `is` and `if` trigger an error. When `prederror` is `false`, `is` returns `unknown`, and `if` returns a partially-evaluated conditional expression.

`maybe` always behaves as if `prederror` were `false`, and `while` and `unless` always behave as if `prederror` were `true`.

Relational operators do not distribute over lists or other aggregates.

See also `=`, `#`, `equal`, and `notequal`.

Examples:

Relational expressions are evaluated to Boolean values by some functions and programming constructs.

```
(%i1) [x, y, z] : [123, 456, 789];
(%o1) [123, 456, 789]
(%i2) is (x < y);
(%o2) true
(%i3) maybe (y > z);
(%o3) false
(%i4) if x >= z then 1 else 0;
(%o4) 0
(%i5) block ([S], S : 0, for i:1 while i <= 100 do S : S + i, return (S));
(%o5) 5050
```

Relational expressions are not otherwise evaluated or simplified to Boolean values, although the arguments of relational expressions are evaluated.

```
(%o1) [123, 456, 789]
```
(%i2) [x < y, y <= z, z >= y, y > z];
(%o2) [123 < 456, 456 <= 789, 789 >= 456, 456 > 789]
(%i3) map (is, %);
(%o3) [true, true, true, false]

7.4 Logical operators

and

The logical conjunction operator. and is an n-ary infix operator; its operands are Boolean expressions, and its result is a Boolean value.

and forces evaluation (like is) of one or more operands, and may force evaluation of all operands.

Operands are evaluated in the order in which they appear. and evaluates only as many of its operands as necessary to determine the result. If any operand is false, the result is false and no further operands are evaluated.

The global flag prederror governs the behavior of and when an evaluated operand cannot be determined to be true or false. and prints an error message when prederror is true. Otherwise, operands which do not evaluate to true or false are accepted, and the result is a Boolean expression.

and is not commutative: a and b might not be equal to b and a due to the treatment of indeterminate operands.

not

The logical negation operator. not is a prefix operator; its operand is a Boolean expression, and its result is a Boolean value.

not forces evaluation (like is) of its operand.

The global flag prederror governs the behavior of not when its operand cannot be determined to be true or false. not prints an error message when prederror is true. Otherwise, operands which do not evaluate to true or false are accepted, and the result is a Boolean expression.

or

The logical disjunction operator. or is an n-ary infix operator; its operands are Boolean expressions, and its result is a Boolean value.

or forces evaluation (like is) of one or more operands, and may force evaluation of all operands.

Operands are evaluated in the order in which they appear. or evaluates only as many of its operands as necessary to determine the result. If any operand is true, the result is true and no further operands are evaluated.

The global flag prederror governs the behavior of or when an evaluated operand cannot be determined to be true or false. or prints an error message when prederror is true. Otherwise, operands which do not evaluate to true or false are accepted, and the result is a Boolean expression.

or is not commutative: a or b might not be equal to b or a due to the treatment of indeterminate operands.
Chapter 7: Operators

7.5 Operators for Equations

#

[Operator]

Represents the negation of syntactic equality =.

Note that because of the rules for evaluation of predicate expressions (in particular because not expr causes evaluation of expr), not a = b is equivalent to is(a # b), instead of a # b.

Examples:

(%i1) a = b;
(%o1) a = b

(%i2) is (a = b);
(%o2) false

(%i3) a # b;
(%o3) a # b

(%i4) not a = b;
(%o4) true

(%i5) is (a # b);
(%o5) true

(%i6) is (not a = b);
(%o6) true

=  

[Operator]

The equation operator.

An expression a = b, by itself, represents an unevaluated equation, which might or might not hold. Unevaluated equations may appear as arguments to solve and algsys or some other functions.

The function is evaluates = to a Boolean value. is(a = b) evaluates a = b to true when a and b are identical. That is, a and b are atoms which are identical, or they are not atoms and their operators are identical and their arguments are identical. Otherwise, is(a = b) evaluates to false; it never evaluates to unknown. When is(a = b) is true, a and b are said to be syntactically equal, in contrast to equivalent expressions, for which is(equal(a, b)) is true. Expressions can be equivalent and not syntactically equal.

The negation of = is represented by #. As with =, an expression a # b, by itself, is not evaluated. is(a # b) evaluates a # b to true or false.

In addition to is, some other operators evaluate = and # to true or false, namely if, and, or, and not.

Note that because of the rules for evaluation of predicate expressions (in particular because not expr causes evaluation of expr), not a = b is equivalent to is(a # b), instead of a # b.

rhs and lhs return the right-hand and left-hand sides, respectively, of an equation or inequation.

See also equal and notequal.

Examples:
An expression \( a = b \), by itself, represents an unevaluated equation, which might or might not hold.

\[
\begin{align*}
\%\text{i1} & : \ a \cdot x - 5 \cdot y = 17; \\
\%\text{o1} & : \ ax - 5y = 17 \\
\%\text{i2} & : \ b \cdot x + 3 \cdot y = 29; \\
\%\text{o2} & : \ 3y + bx = 29 \\
\%\text{i3} & : \ \text{solve ([eq \_1, eq \_2], [x, y]);} \\
\%\text{o3} & : \ \left[ x = \frac{196}{5b+3a}, \ y = \frac{29a-17b}{5b+3a} \right] \\
\%\text{i4} & : \ \text{subst (\%o3, [eq \_1, eq \_2]);} \\
\%\text{o4} & : \ \left[ \frac{196a}{5b+3a} - \frac{3(29a-17b)}{5b+3a} = 17, \right. \\
& \left. \frac{196b}{5b+3a} + \frac{3(29a-17b)}{5b+3a} = 29 \right] \\
\%\text{i5} & : \ \text{ratsimp (\%o4);} \\
\%\text{o5} & : \ \left[ 17 = 17, 29 = 29 \right]
\end{align*}
\]

\( \text{is(a = b)} \) evaluates \( a = b \) to \text{true} when \( a \) and \( b \) are syntactically equal (that is, identical). Expressions can be equivalent and not syntactically equal.

\[
\begin{align*}
\%\text{i1} & : \ (x + 1) \cdot (x - 1); \\
\%\text{o1} & : \ (x - 1)(x + 1) \\
\%\text{i2} & : \ x^2 - 1; \\
\%\text{o2} & : \ x - 1 \\
\%\text{i3} & : \ \text{[is (a = b), is (a \# b)];} \\
\%\text{o3} & : \ \text{[false, true]} \\
\%\text{i4} & : \ \text{[is (equal (a, b)), is (notequal (a, b))];} \\
\%\text{o4} & : \ \text{[true, false]}
\end{align*}
\]

Some operators evaluate \( = \) and \# to \text{true} or \text{false}.

\[
\begin{align*}
\%\text{i1} & : \ \text{if expand ((x + y)^2) = x^2 + 2 \cdot x \cdot y + y^2 then FOO else BAR;} \\
\%\text{o1} & : \ \text{FOO} \\
\%\text{i2} & : \ \text{eq \_3 : 2 \cdot x = 3 \cdot x;} \\
\%\text{o2} & : \ 2x = 3x \\
\%\text{i3} & : \ \text{eq \_4 : \exp (2) = \%e^2;} \\
\%\text{o3} & : \ \text{2 \ 2} \\
\%\text{i4} & : \ \text{\%e = \%e} \\
\%\text{o4} & : \ \text{[false, true, true]}
\end{align*}
\]

Because \text{not expr} causes evaluation of \( \text{expr} \), \text{not a = b} is equivalent to \( \text{is(a \# b)} \).

\[
\begin{align*}
\%\text{i1} & : \ [2 \cdot x \# 3 \cdot x, \ \text{not (2 \cdot x = 3 \cdot x)}]; \\
\%\text{o1} & : \ [2x \# 3x, \text{true}] \\
\%\text{i2} & : \ \text{is (2 \cdot x \# 3 \cdot x)};
\end{align*}
\]
Chapter 7: Operators

(\%o2) \hspace{1cm} \text{true}

7.6 Assignment operators

: [Operator]

Assignment operator.

When the left-hand side is a simple variable (not subscripted), \( : \) evaluates its right-hand side and associates that value with the left-hand side.

When the left-hand side is a subscripted element of a list, matrix, declared Maxima array, or Lisp array, the right-hand side is assigned to that element. The subscript must name an existing element; such objects cannot be extended by naming nonexistent elements.

When the left-hand side is a subscripted element of a hashed array, the right-hand side is assigned to that element, if it already exists, or a new element is allocated, if it does not already exist.

When the left-hand side is a list of simple and/or subscripted variables, the right-hand side must evaluate to a list, and the elements of the right-hand side are assigned to the elements of the left-hand side, in parallel.

See also \texttt{kill} and \texttt{remvalue}, which undo the association between the left-hand side and its value.

Examples:

Assignment to a simple variable.

\begin{verbatim}
(%i1) a;
(%o1) a
(%i2) a : 123;
(%o2) 123
(%i3) a;
(%o3) 123
\end{verbatim}

Assignment to an element of a list.

\begin{verbatim}
(%i1) b : [1, 2, 3];
(%o1) [1, 2, 3]
(%i2) b[3] : 456;
(%o2) 456
(%i3) b;
(%o3) [1, 2, 456]
\end{verbatim}

Assignment to a variable that neither is the name of a list nor of an array creates a hashed array.

\begin{verbatim}
(%i1) c[99] : 789;
(%o1) 789
(%i2) c[99];
(%o2) 789
(%i3) c;
(%o3) c
(%i4) arrayinfo (c);
\end{verbatim}
Multiple assignment.

\begin{verbatim}
(%i1) [a, b, c] : [45, 67, 89];
(%o1) [45, 67, 89]
(%i2) a;  
(%o2) 45
(%i3) b;  
(%o3) 67
(%i4) c;  
(%o4) 89
\end{verbatim}

Multiple assignment is carried out in parallel. The values of \(a\) and \(b\) are exchanged in this example.

\begin{verbatim}
(%i1) [a, b] : [33, 55];
(%o1) [33, 55]
(%i2) [a, b] : [b, a];
(%o2) [55, 33]
(%i3) a;  
(%o3) 55
(%i4) b;  
(%o4) 33
\end{verbatim}

":[[Operator]

\(::\) is the same as : (which see) except that :: evaluates its left-hand side as well as its right-hand side.

Examples:

\begin{verbatim}
(%i1) x : 'foo;
(%o1) foo
(%i2) x :: 123;
(%o2) 123
(%i3) foo;
(%o3) 123
(%i4) x : '[a, b, c];
(%o4) [a, b, c]
(%i5) x :: [11, 22, 33];
(%o5) [11, 22, 33]
(%i6) a;  
(%o6) 11
(%i7) b;  
(%o7) 22
(%i8) c;  
(%o8) 33
\end{verbatim}
::= Macro function definition operator. ::= defines a function (called a "macro" for historical reasons) which quotes its arguments, and the expression which it returns (called the "macro expansion") is evaluated in the context from which the macro was called. A macro function is otherwise the same as an ordinary function.

**macroexpand** returns a macro expansion (without evaluating it). **macroexpand** (foo (x)) followed by ''% is equivalent to foo (x) when foo is a macro function.

::= puts the name of the new macro function onto the global list **macros**. **kill**, **remove**, and **remfunction** unbind macro function definitions and remove names from **macros**.

**fundef** or **dispfun** return a macro function definition or assign it to a label, respectively.

Macro functions commonly contain **buildq** and **splice** expressions to construct an expression, which is then evaluated.

**Examples**
A macro function quotes its arguments, so message (1) shows y - z, not the value of y - z. The macro expansion (the quoted expression '(print "(2) x is equal to", x)) is evaluated in the context from which the macro was called, printing message (2).

```lisp
(%i1) x: %pi$
(%i2) y: 1234$
(%i3) z: 1729 * w$
(%i4) printq1 (x) ::= block (print "(1) x is equal to", x),
    '(print "(2) x is equal to", x))$
(%i5) printq1 (y - z);
(1) x is equal to y - z
(2) x is equal to %pi
(%o5) %pi
```

An ordinary function evaluates its arguments, so message (1) shows the value of y - z. The return value is not evaluated, so message (2) is not printed until the explicit evaluation ''%.

```lisp
(%i1) x: %pi$
(%i2) y: 1234$
(%i3) z: 1729 * w$
(%i4) printe1 (x) ::= block (print "(1) x is equal to", x),
    '(print "(2) x is equal to", x))$
(%i5) printe1 (y - z);
(1) x is equal to 1234 - 1729 w
(2) x is equal to %pi
(%o5) print((2) x is equal to, x)
(%i6) ''%;
(2) x is equal to %pi
(%o6) %pi
```

**macroexpand** returns a macro expansion. **macroexpand** (foo (x)) followed by ''% is equivalent to foo (x) when foo is a macro function.

```lisp
(%i11) x: %pi$
```
The function definition operator.

\( f(x_1, \ldots, x_n) := expr \) defines a function named \( f \) with arguments \( x_1, \ldots, x_n \) and function body \( expr \). \( := \) never evaluates the function body (unless explicitly evaluated by quote-quote \( '' \)). The function body is evaluated every time the function is called.

\( f[x_1, \ldots, x_n] := expr \) defines a so-called memoizing function. Its function body is evaluated just once for each distinct value of its arguments, and that value is returned, without evaluating the function body, whenever the arguments have those values again. (A function of this kind is also known as a “array function”.)

\( f[x_1, \ldots, x_n](y_1, \ldots, y_m) := expr \) is a special case of a memoizing function. \( f[x_1, \ldots, x_n] \) is a memoizing function which returns a lambda expression with arguments \( y_1, \ldots, y_m \). The function body is evaluated once for each distinct value of \( x_1, \ldots, x_n \), and the body of the lambda expression is that value.

When the last or only function argument \( x_n \) is a list of one element, the function defined by \( := \) accepts a variable number of arguments. Actual arguments are assigned one-to-one to formal arguments \( x_1, \ldots, x_{(n - 1)} \), and any further actual arguments, if present, are assigned to \( x_n \) as a list.

All function definitions appear in the same namespace; defining a function \( f \) within another function \( g \) does not automatically limit the scope of \( f \) to \( g \). However, \( \text{local}(f) \) makes the definition of function \( f \) effective only within the block or other compound expression in which \( \text{local} \) appears.

If some formal argument \( x_k \) is a quoted symbol, the function defined by \( := \) does not evaluate the corresponding actual argument. Otherwise all actual arguments are evaluated.

See also \texttt{define} and \( ::= \).

Examples:

\( := \) never evaluates the function body (unless explicitly evaluated by quote-quote).

\[ %i1) expr : \cos(y) - \sin(x); \]
\[ %o1) \cos(y) - \sin(x) \]
\[ %i2) F1(x, y) := expr; \]
\[ %o2) F1(x, y) := expr \]
When the last or only function argument is a list of one element, the function defined by := accepts a variable number of arguments.

```lisp
(%i1) H ([L]) := apply ("+", L);
(%o1) H([L]) := apply("+", L)
(%i2) H (a, b, c);
```

`local` makes a local function definition.

```lisp
(%i1) foo (x) := 1 - x;
(%o1) foo(x) := 1 - x
(%i2) foo (100);  
```

```
7.7 User defined operators

infix

infix (op)
infix (op, lbp, rbp)
infix (op, lbp, rbp, lpos, rpos, pos)

 Declares op to be an infix operator. An infix operator is a function of two arguments, with the name of the function written between the arguments. For example, the subtraction operator - is an infix operator.

infix (op) declares op to be an infix operator with default binding powers (left and right both equal to 180) and parts of speech (left and right both equal to any).

infix (op, lbp, rbp) declares op to be an infix operator with stated left and right binding powers and default parts of speech (left and right both equal to any).

infix (op, lbp, rbp, lpos, rpos, pos) declares op to be an infix operator with stated left and right binding powers and parts of speech lpos, rpos, and pos for the left operand, the right operand, and the operator result, respectively.
```
"Part of speech", in reference to operator declarations, means expression type. Three types are recognized: `expr`, `clause`, and `any`, indicating an algebraic expression, a Boolean expression, or any kind of expression, respectively. Maxima can detect some syntax errors by comparing the declared part of speech to an actual expression.

The precedence of `op` with respect to other operators derives from the left and right binding powers of the operators in question. If the left and right binding powers of `op` are both greater the left and right binding powers of some other operator, then `op` takes precedence over the other operator. If the binding powers are not both greater or less, some more complicated relation holds.

The associativity of `op` depends on its binding powers. Greater left binding power (`lbp`) implies an instance of `op` is evaluated before other operators to its left in an expression, while greater right binding power (`rbp`) implies an instance of `op` is evaluated before other operators to its right in an expression. Thus greater `lbp` makes `op` right-associative, while greater `rbp` makes `op` left-associative. If `lbp` is equal to `rbp`, `op` is left-associative.

See also Section 7.1 [Introduction to operators], page 115.

Examples:

If the left and right binding powers of `op` are both greater the left and right binding powers of some other operator, then `op` takes precedence over the other operator.

```
(%i1) :lisp (get '$+ 'lbp)
100
(%i1) :lisp (get '$+ 'rbp)
100
(%i1) infix ("##", 101, 101);
  (a,b) + 3
(%i4) infix ("##", 99, 99);
  (a+1,b+2)
```

Greater `lbp` makes `op` right-associative, while greater `rbp` makes `op` left-associative.

```
(%i1) infix ("##", 100, 99);
  (foo,(bar,baz))
(%i4) infix ("##", 100, 101);
  ((foo,bar),baz)
```

Maxima can detect some syntax errors by comparing the declared part of speech to an actual expression.
(\%i1) infix ("##", 100, 99, expr, expr, expr);
(\%o1)  ##
(\%i2) if x ## y then 1 else 0;
Incorrect syntax: Found algebraic expression where logical expression expected
if x ## y then
  
(\%i2) infix ("##", 100, 99, expr, expr, clause);
(\%o2)  ##
(\%i3) if x ## y then 1 else 0;
(\%o3)  if x ## y then 1 else 0

\textbf{matchfix}

\textbf{matchfix} (ldelimiter, rdelimiter)
\textbf{matchfix} (ldelimiter, rdelimiter, arg\_pos, pos)
Declares a matchfix operator with left and right delimiters \texttt{ldelimiter} and \texttt{rdelimiter}. The delimiters are specified as strings.

A "matchfix" operator is a function of any number of arguments, such that the arguments occur between matching left and right delimiters. The delimiters may be any strings, so long as the parser can distinguish the delimiters from the operands and other expressions and operators. In practice this rules out unparsable delimiters such as \%, , $ and ;, and may require isolating the delimiters with white space. The right delimiter can be the same or different from the left delimiter.

A left delimiter can be associated with only one right delimiter; two different matchfix operators cannot have the same left delimiter.

An existing operator may be redeclared as a matchfix operator without changing its other properties. In particular, built-in operators such as addition + can be declared matchfix, but operator functions cannot be defined for built-in operators.

The command \textbf{matchfix} (ldelimiter, rdelimiter, arg\_pos, pos) declares the argument part-of-speech \texttt{arg\_pos} and result part-of-speech \texttt{pos}, and the delimiters \texttt{ldelimiter} and \texttt{rd delimiter}.

"Part of speech", in reference to operator declarations, means expression type. Three types are recognized: \texttt{expr}, \texttt{clause}, and \texttt{any}, indicating an algebraic expression, a Boolean expression, or any kind of expression, respectively. Maxima can detect some syntax errors by comparing the declared part of speech to an actual expression.

The function to carry out a matchfix operation is an ordinary user-defined function. The operator function is defined in the usual way with the function definition operator := or define. The arguments may be written between the delimiters, or with the left delimiter as a quoted string and the arguments following in parentheses. \textbf{dispfun} (ldelimiter) displays the function definition.

The only built-in matchfix operator is the list constructor [ ]. Parentheses ( ) and double-quotes " " act like matchfix operators, but are not treated as such by the Maxima parser.

\textbf{matchfix} evaluates its arguments. \textbf{matchfix} returns its first argument, \texttt{ldelimiter}.

Examples:
Delimiters may be almost any strings.

\(\%i1\) matchfix (@@, "-");  
\(\%o1\) @@

\(\%i2\) @@ a, b, c ~;
\(\%o2\) @@a, b, c~

\(\%i3\) matchfix (>>, "<<");  
\(\%o3\) >>

\(\%i4\) >> a, b, c <<;
\(\%o4\) >>a, b, c<<

\(\%i5\) matchfix ("foo", "oof");  
\(\%o5\) foo

\(\%i6\) foo a, b, c oof;
\(\%o6\) fooa, b, coof

\(\%i7\) >> w + foo x, y oof + z << / @@ p, q ~;
\(\%o7\) ----------------------
@@p, q~

Matchfix operators are ordinary user-defined functions.

\(\%i1\) matchfix ("!-", "-!");  
\(\%o1\) "!-

\(\%i2\) !- x, y -! := x/y - y/x;
\(\%o2\) !-x, y-! := - - -
         x y

\(\%i3\) define (!-x, y-!, x/y - y/x);
\(\%o3\) !-x, y-! := - - -
         y x

\(\%i4\) define ("!-" (x, y), x/y - y/x);
\(\%o4\) !-x, y-! := - - -
         y x

\(\%i5\) dispfun ("!-");
\(\%t5\) !-x, y-! := - - -
         y x

\(\%o5\) done  
\(\%i6\) !-3, 5-!;
\(\%o6\) 16

\(\%o6\) 15

\(\%i7\) "!-" (3, 5);
\(\%o7\) 16

\(\%o7\) 15
**nary**

nary (op)

nary (op, bp, arg_pos, pos)

An nary operator is used to denote a function of any number of arguments, each of which is separated by an occurrence of the operator, e.g. A+B or A+B+C. The nary("x") function is a syntax extension function to declare x to be an nary operator. Functions may be declared to be nary. If declare(j,nary); is done, this tells the simplifier to simplify, e.g. j(j(a,b),j(c,d)) to j(a, b, c, d).

See also Section 7.1 [Introduction to operators], page 115.

**nofix**

nofix (op)
nofix (op, pos)

nofix operators are used to denote functions of no arguments. The mere presence of such an operator in a command will cause the corresponding function to be evaluated. For example, when one types "exit;" to exit from a Maxima break, "exit" is behaving similar to a nofix operator. The function nofix("x") is a syntax extension function which declares x to be a nofix operator.

See also Section 7.1 [Introduction to operators], page 115.

**postfix**

postfix (op)

postfix (op, lbp, lpos, pos)

postfix operators like the prefix variety denote functions of a single argument, but in this case the argument immediately precedes an occurrence of the operator in the input string, e.g. 3!. The postfix("x") function is a syntax extension function to declare x to be a postfix operator.

See also Section 7.1 [Introduction to operators], page 115.

**prefix**

prefix (op)

prefix (op, rbp, rpos, pos)

A prefix operator is one which signifies a function of one argument, which argument immediately follows an occurrence of the operator. prefix("x") is a syntax extension function to declare x to be a prefix operator.

See also Section 7.1 [Introduction to operators], page 115.
8 Evaluation

8.1 Functions and Variables for Evaluation

The single quote operator ' prevents evaluation.

Applied to a symbol, the single quote prevents evaluation of the symbol.

Applied to a function call, the single quote prevents evaluation of the function call, although the arguments of the function are still evaluated (if evaluation is not otherwise prevented). The result is the noun form of the function call.

Applied to a parenthesized expression, the single quote prevents evaluation of all symbols and function calls in the expression. E.g., '(f(x)) means do not evaluate the expression f(x). 'f(x) (with the single quote applied to f instead of f(x)) means return the noun form of f applied to [x].

The single quote does not prevent simplification.

When the global flag noundisp is true, nouns display with a single quote. This switch is always true when displaying function definitions.

See also the quote-quote operator [quote-quote], page 137 and nouns.

Examples:

Applied to a symbol, the single quote prevents evaluation of the symbol.

(%i1) aa: 1024;
(%o1) 1024
(%i2) aa^2;
(%o2) 1048576
(%i3) 'aa^2;
(%o3) aa
(%i4) ''%;
(%o4) 1048576

Applied to a function call, the single quote prevents evaluation of the function call. The result is the noun form of the function call.

(%i1) x0: 5;
(%o1) 5
(%i2) x1: 7;
(%o2) 7
(%i3) integrate (x^2, x, x0, x1);
(%o3) ---
     3
(%i4) 'integrate (x^2, x, x0, x1);
Applied to a parenthesized expression, the single quote prevents evaluation of all symbols and function calls in the expression.

```
(%i1) aa: 1024;
(%o1) 1024
(%i2) bb: 19;
(%o2) 19
(%i3) sqrt(aa) + bb;
(%o3) 51
(%i4) '(sqrt(aa) + bb);
(%o4) bb + sqrt(aa)
(%i5) ''; 
(%o5) 51
```

The single quote does not prevent simplification.

```
(%i1) sin (17 * %pi) + cos (17 * %pi);
(%o1) - 1
(%i2) '(sin (17 * %pi) + cos (17 * %pi));
(%o2) - 1
```

Maxima considers floating point operations by its in-built mathematical functions to be a simplification.

```
(%i1) sin(1.0);
(%o1) 0.8414709848078965
(%i2) '(sin(1.0));
(%o2) 0.8414709848078965
```

When the global flag `noundisp` is true, nouns display with a single quote.

```
(%i1) x:%pi;
(%o1) %pi
(%i2) bfloat(x);
(%o2) 3.141592653589793b0
(%i3) sin(x);
(%o3) 0
(%i4) noundisp;
(%o4) false
(%i5) 'bfloat(x);
(%o5) bfloat(%pi)
```
The quote-quote operator `'` (two single quote marks) modifies evaluation in input expressions.

Applied to a general expression expr, quote-quote causes the value of expr to be substituted for expr in the input expression.

Applied to the operator of an expression, quote-quote changes the operator from a noun to a verb (if it is not already a verb).

The quote-quote operator is applied by the input parser; it is not stored as part of a parsed input expression. The quote-quote operator is always applied as soon as it is parsed, and cannot be quoted. Thus quote-quote causes evaluation when evaluation is otherwise suppressed, such as in function definitions, lambda expressions, and expressions quoted by single quote `'`.

Quote-quote is recognized by `batch` and `load`.

See also `ev`, the single-quote operator `[quote]`, page 135 and `nouns`.

Examples:

Applied to a general expression expr, quote-quote causes the value of expr to be substituted for expr in the input expression.

```
(%i1) expand ((a + b)^3);
     3   2    2   3
(%o1)         b   + 3 a b   + 3 a b + a
(%i2) [, '],
     3   3    2   2   3
(%o2) , expand((b + a) ), b   + 3 a b   + 3 a b + a
(%i3) [%i1, ''],
     3   3    2   2   3
(%o3) , expand((b + a) ), b   + 3 a b   + 3 a b + a
(%i4) [aa : cc, bb : dd, cc : 17, dd : 29];
```
Applied to the operator of an expression, quote-quote changes the operator from a noun to a verb (if it is not already a verb).

```
(%i1) declare (foo, noun); done
(%i2) foo (x) := x - 1729; ''foo(x) := x - 1729
(%i3) foo (100); foo(100)
(%i4) ''foo (100); - 1629
```

The quote-quote operator is applied by the input parser; it is not stored as part of a parsed input expression.

```
(%i1) [aa : bb, cc : dd, bb : 1234, dd : 5678]; [bb, dd, 1234, 5678]
(%i2) aa + cc; dd + bb
(%i3) display (_, op (_), args (_));
_ = cc + aa
    op(cc + aa) = +
    args(cc + aa) = [cc, aa]
```

```
(%o3) done
(%o4) ''(aa + cc);
```
Quote-quote causes evaluation when evaluation is otherwise suppressed, such as in function definitions, lambda expressions, and expressions quoted by single quote '.

\[
\begin{align*}
\text{%i11} & \text{ foo}_1\text{a} (x) := \text{''}(\text{integrate} \ (\log (x), \ x)); \\
\text{%o11} & \text{ foo}_1\text{a}(x) := x \log(x) - x \\
\text{%i12} & \text{ foo}_1\text{b} (x) := \text{integrate} \ (\log (x), \ x); \\
\text{%o12} & \text{ foo}_1\text{b}(x) := \text{integrate} (\log(x), \ x) \\
\text{%i13} & \text{ dispfun} \ (\text{foo}_1\text{a}, \ \text{foo}_1\text{b}); \\
\text{%t13} & \text{ foo}_1\text{a}(x) := x \log(x) - x \\
\text{%t14} & \text{ foo}_1\text{b}(x) := \text{integrate} (\log(x), \ x) \\
\text{%o14} & \text{ [foo}_1\text{a}, \ foo}_1\text{b]} \\
\text{%i15} & \text{ integrate} \ (\log (x), \ x); \\
\text{%o15} & \text{ x log(x) - x} \\
\text{%i16} & \text{ foo}_2\text{a} (x) := \text{''}%; \\
\text{%o16} & \text{ foo}_2\text{a}(x) := x \log(x) - x \\
\text{%i17} & \text{ foo}_2\text{b} (x) := %; \\
\text{%o17} & \text{ foo}_2\text{b}(x) := % \\
\text{%i18} & \text{ dispfun} \ (\text{foo}_2\text{a}, \ \text{foo}_2\text{b}); \\
\text{%t18} & \text{ foo}_2\text{a}(x) := x \log(x) - x \\
\text{%t19} & \text{ foo}_2\text{b}(x) := % \\
\text{%o19} & \text{ [foo}_2\text{a}, \ foo}_2\text{b]} \\
\text{%i20} & \text{ F} : \text{ lambda} \ ([u], \ \text{diff} \ (\sin \ (u), \ u)); \\
\text{%o20} & \text{ lambda} ([u], \ \text{diff} (\sin(u), \ u)) \\
\text{%i21} & \text{ G} : \text{ lambda} \ ([u], \ ''(\text{diff} \ (\sin \ (u), \ u))); \\
\text{%o21} & \text{ lambda} ([u], \ \cos(u)) \\
\text{%i22} & ''(\text{sum} \ (a[k], \ k, \ 1, \ 3) + \text{sum} \ (b[k], \ k, \ 1, \ 3)); \\
\text{%o22} & \text{ sum} (b \ , \ k, \ 1, \ 3) + \text{sum} (a \ , \ k, \ 1, \ 3) \\
\text{%i23} & ''( ''(\text{sum} \ (a[k], \ k, \ 1, \ 3)) + ''(\text{sum} \ (b[k], \ k, \ 1, \ 3))); \\
\text{%o23} & \text{ b + a + b + a + b + a} \\
& \text{ 3 3 2 2 1 1}
\end{align*}
\]
**ev** *(expr, arg_1, ..., arg_n)*  

Evaluates the expression *expr* in the environment specified by the arguments *arg_1*, ..., *arg_n*. The arguments are switches (Boolean flags), assignments, equations, and functions. *ev* returns the result (another expression) of the evaluation.

The evaluation is carried out in steps, as follows.

1. First the environment is set up by scanning the arguments which may be any or all of the following.
   - **simp** causes *expr* to be simplified regardless of the setting of the switch **simp** which inhibits simplification if **false**.
   - **noeval** suppresses the evaluation phase of *ev* (see step (4) below). This is useful in conjunction with the other switches and in causing *expr* to be resimplified without being reevaluated.
   - **nouns** causes the evaluation of noun forms (typically unevaluated functions such as **'integrate** or **'diff**) in *expr*.
   - **expand** causes expansion.
   - **expand** (*m*, *n*) causes expansion, setting the values of **maxposex** and **maxnegex** to *m* and *n* respectively.
   - **detout** causes any matrix inverses computed in *expr* to have their determinant kept outside of the inverse rather than dividing through each element.
   - **diff** causes all differentiations indicated in *expr* to be performed.
   - **derivlist** (*x*, *y*, *z*, ...) causes only differentiations with respect to the indicated variables. See also **derivlist**.
   - **risch** causes integrals in *expr* to be evaluated using the Risch algorithm. See **risch**. The standard integration routine is invoked when using the special symbol **nouns**.
   - **float** causes non-integral rational numbers to be converted to floating point.
   - **numer** causes some mathematical functions (including exponentiation) with numerical arguments to be evaluated in floating point. It causes variables in *expr* which have been given numervals to be replaced by their values. It also sets the **float** switch on.
   - **pred** causes predicates (expressions which evaluate to **true** or **false**) to be evaluated.
   - **eval** causes an extra post-evaluation of *expr* to occur. (See step (5) below.) **eval** may occur multiple times. For each instance of **eval**, the expression is evaluated again.
   - A where A is an atom declared to be an evaluation flag **evflag** causes A to be bound to **true** during the evaluation of *expr*.
   - V: **expression** (or alternately V=**expression**) causes V to be bound to the value of **expression** during the evaluation of *expr*. Note that if V is a Maxima option, then **expression** is used for its value during the evaluation of *expr*. If more than one argument to *ev* is of this type then the binding is done in parallel. If V is a non-atomic expression then a substitution rather than a binding is performed.
• F where F, a function name, has been declared to be an evaluation function `evfun` causes F to be applied to expr.

• Any other function names, e.g. `sum`, cause evaluation of occurrences of those names in expr as though they were verbs.

• In addition a function occurring in expr (say F(x)) may be defined locally for the purpose of this evaluation of expr by giving `F(x) := expression` as an argument to ev.

• If an atom not mentioned above or a subscripted variable or subscripted expression was given as an argument, it is evaluated and if the result is an equation or assignment then the indicated binding or substitution is performed. If the result is a list then the members of the list are treated as if they were additional arguments given to ev. This permits a list of equations to be given (e.g. `[X=1, Y=A**2]`) or a list of names of equations (e.g., `[%t1, %t2]` where `%t1` and `%t2` are equations) such as that returned by `solve`.

The arguments of ev may be given in any order with the exception of substitution equations which are handled in sequence, left to right, and evaluation functions which are composed, e.g., `ev (expr, ratsimp, realpart)` is handled as `realpart (ratsimp (expr))`.

The `simp`, `numer`, and `float` switches may also be set locally in a block, or globally in Maxima so that they will remain in effect until being reset.

If expr is a canonical rational expression (CRE), then the expression returned by ev is also a CRE, provided the `numer` and `float` switches are not both `true`.

2. During step (1), a list is made of the non-subscripted variables appearing on the left side of equations in the arguments or in the value of some arguments if the value is an equation. The variables (subscripted variables which do not have associated memoizing functions as well as non-subscripted variables) in the expression expr are replaced by their global values, except for those appearing in this list. Usually, expr is just a label or `%` (as in `%i2` in the example below), so this step simply retrieves the expression named by the label, so that ev may work on it.

3. If any substitutions are indicated by the arguments, they are carried out now.

4. The resulting expression is then re-evaluated (unless one of the arguments was `noeval`) and simplified according to the arguments. Note that any function calls in expr will be carried out after the variables in it are evaluated and that ev(F(x)) thus may behave like F(ev(x)).

5. For each instance of `eval` in the arguments, steps (3) and (4) are repeated.

See also `[quote-quote], page 137, at and subst`.

Examples:

(\%i1) \sin(x) + \cos(y) + (w+1)^2 + 'diff (\sin(w), w);
\d\cos(y) + \sin(x) + \frac{d^2 (\sin(w))}{dw^2} + (w + 1)

(\%i2) ev (\%, numer, expand, diff, x=2, y=1);
An alternate top level syntax has been provided for `ev`, whereby one may just type its arguments, without the `ev()` . That is, one may write simply

`expr, arg_1, ..., arg_n`

This is not permitted as part of another expression, e.g., in functions, blocks, etc.

Notice the parallel binding process in the following example.

\[
\begin{align*}
(\%i3) \quad \text{programmode: false;} & \quad \text{false} \\
(\%i4) \quad x+y, x: a+y, y: 2; & \quad y + a + 2 \\
(\%i5) \quad 2*x - 3*y = 3$ & \quad - 4 = - 4 \\
(\%i6) \quad -3*x + 2*y = -4$ & \quad x + 1/x > gamma (1/2); \\
(\%i7) \quad \text{solve ([%o5, %o6]);} & \quad [\%t7, %t8] \\
\text{Solution} & \\
& \frac{1}{5} \\
& \frac{6}{5} \\
& x = - \\
& y = - \\
& y = - \\
(\%o8) & \quad [[%t7, %t8]] \\
(\%o8) & \quad [\%t7, %t8] \\
(\%i9) \quad x + 1/x > gamma (1/2); & \quad x + 1/x > sqrt(%pi) \\
(\%i10) \quad %, numer, x=1/2; & \quad 2.5 > 1.772453850905516 \\
(\%o10) & \quad 2.5 > 1.772453850905516 \\
(\%i11) \quad %, pred; & \quad true \\
(\%o11) & \quad [\%t7, %t8] \\
\end{align*}
\]

`eval`  
As an argument in a call to `ev (expr)` , `eval` causes an extra evaluation of `expr` . See `ev` .

Example:

\[
\begin{align*}
(\%i1) \quad [a:b, b:c, c:d, d:e]; & \quad [b, c, d, e] \\
(\%o1) & \quad [b, c, d, e] \\
(\%i2) \quad a; & \quad b \\
(\%o2) & \quad b \\
(\%i3) \quad ev(a); & \quad c \\
(\%o3) & \quad c \\
(\%i4) \quad ev(a),eval; & \quad e \\
(\%o4) & \quad e \\
\end{align*}
\]
evflag

When a symbol \( x \) has the \texttt{evflag} property, the expressions \( \texttt{ev(expr, x)} \) and \( \texttt{expr} \), \( x \) (at the interactive prompt) are equivalent to \( \texttt{ev(expr, x = true)} \). That is, \( x \) is bound to \texttt{true} while \( \texttt{expr} \) is evaluated.

The expression \texttt{declare(x, evflag)} gives the \texttt{evflag} property to the variable \( x \).

The flags which have the \texttt{evflag} property by default are the following:

\begin{verbatim}
  algebraic  cauchy_sum  demoivre
  dots_product  \%emode  \%enumerator
  exponentialize  expt_isolate  factorflag
  float  half_angles  infeval
  isolate_wrt_times  keepfloat  letrat
  listarith  logabs  logarc
  logexpand  lognegint
  m1pbranch  numer_pbranch  programmode
  radexpand  ratalgdenom  ratfac
  ratsx  ratsimpexpons  simp
  simpproduct  simpsum  sumexpand
  trigexpand
\end{verbatim}

Examples:

\begin{verbatim}
(%i1) sin (1/2);  
                1
(%o1) sin(-)
                2
(%i2) sin (1/2), float;  
            0.479425538604203
(%o2) 0.479425538604203
(%i3) sin (1/2), float=true;  
            0.479425538604203
(%o3) 0.479425538604203
(%i4) simp : false;  
                false
(%o4) false
(%i5) 1 + 1;  
                1 + 1
(%o5) 1 + 1
(%i6) 1 + 1, simp;  
                2
(%o6) 2
(%i7) simp : true;  
               true
(%o7) true
(%i8) sum (1/k^2, k, 1, inf);  
                         inf
               ==== 1
(%o8) > -- 2
  \  /  k
  \ k = 1
\end{verbatim}
(%i9) sum (1/k^2, k, 1, inf), simpsum;
                2
(%o9) --
        6

(%i10) declare (aa, evflag);
(%o10) done

(%i11) if aa = true then YES else NO;
(%o11) NO

(%i12) if aa = true then YES else NO, aa;
(%o12) YES

---

[Property]

evfun

When a function \( F \) has the \texttt{evfun} property, the expressions \texttt{ev(expr, F)} and \texttt{expr, F} (at the interactive prompt) are equivalent to \( F(\texttt{ev(expr)}) \).

If two or more \texttt{evfun} functions \( F, G, \) etc., are specified, the functions are applied in the order that they are specified.

The expression \texttt{declare(F, evfun)} gives the \texttt{evfun} property to the function \( F \). The functions which have the \texttt{evfun} property by default are the following:

\begin{verbatim}
bfloat factor fullratsimp
logcontract polarform radcan
ratexpand ratsimp rectform
rootscontract trigexpand trigreduce
\end{verbatim}

Examples:

(%i1) x^3 - 1;
3
(%o1) x - 1

(%i2) x^3 - 1, factor;
2
(%o2) (x - 1) (x + x + 1)

(%i3) factor (x^3 - 1);
2
(%o3) (x - 1) (x + x + 1)

(%i4) cos(4 * x) / sin(x)^4;
\cos(4 \, x)
(%o4) \frac{\cos(4 \, x)}{\sin(x)^4}

(%i5) cos(4 * x) / sin(x)^4, trigexpand;
\sin(x)^4 \, \cos(x) - 6 \sin(x)^4 \, \cos(x)
(%o5) \frac{\sin(x)^4 \, \cos(x) - 6 \sin(x)^4 \, \cos(x)}{\sin(x)^4}

(%i6) cos(4 * x) / sin(x)^4, trigexpand, ratexpand;
\frac{\sin(x)^4 \, \cos(x)^2 - 6 \sin(x)^4 \, \cos(x)}{\sin(x)^4}
(%o6) \frac{\sin(x)^4 \, \cos(x)^2 - 6 \sin(x)^4 \, \cos(x)}{\sin(x)^4}
\[
\begin{align*}
6 \cos(x) \cos(x) \\
\frac{-}{2} + \frac{4}{\sin(x) \sin(x)} + 1
\end{align*}
\]

\[(\%o6)\]

\[(\%i7)\text{ ratexpand (trigexpand (cos(4 * x) / sin(x)**4))};
\]
\[
\begin{align*}
6 \cos(x) \cos(x) \\
\frac{-}{2} + \frac{4}{\sin(x) \sin(x)} + 1
\end{align*}
\]

\[(\%o7)\]

\[(\%i8)\text{ declare ([F, G], evfun)};
\]

\[(\%i9)\text{ (aa : bb, bb : cc, cc : dd)};
\]

\[(\%o9)\]

\[(\%i10)\text{ aa};
\]

\[(\%o10)\]

\[(\%i11)\text{ aa, F};
\]

\[(\%o11)\]

\[(\%i12)\text{ F (aa)};
\]

\[(\%o12)\]

\[(\%i13)\text{ F (ev (aa))};
\]

\[(\%o13)\]

\[(\%i14)\text{ aa, F, G};
\]

\[(\%o14)\]

\[(\%i15)\text{ G (F (ev (aa)))};
\]

\[(\%o15)\]

\[
\text{infeval}[\text{Option variable}]\]

Enables "infinite evaluation" mode. \text{ev} repeatedly evaluates an expression until it stops changing. To prevent a variable, say \(X\), from being evaluated away in this mode, simply include \(X='X\) as an argument to \text{ev}. Of course expressions such as \text{ev} (\(X, X=1, \text{infeval}\)) will generate an infinite loop.

\[
\text{noeval}[\text{Special symbol}]\]

\text{noeval} suppresses the evaluation phase of \text{ev}. This is useful in conjunction with other switches and in causing expressions to be resimplified without being reevaluated.

\[
\text{nouns}[\text{Special symbol}]\]

\text{nouns} is an \text{evflag}. When used as an option to the \text{ev} command, \text{nouns} converts all "noun" forms occurring in the expression being \text{ev}'d to "verbs", i.e., evaluates them. See also \text{noun}, \text{nounify}, \text{verb}, and \text{verbify}.

\[
\text{pred}[\text{Special symbol}]\]

As an argument in a call to \text{ev (expr)}, \text{pred} causes predicates (expressions which evaluate to \text{true} or \text{false}) to be evaluated. See \text{ev}.

Example:
\[(\%i11)\text{ 1<2};
\]
\[(\%o1)\text{ 1 < 2}\]
(%i2) 1<2, pred;
(%o2) true
9 Simplification

9.1 Introduction to Simplification

Maxima interacts with the user through a cycle of actions called the read-eval-print loop (REPL). This consists of three steps: reading and parsing, evaluating and simplifying, and outputting. Parsing converts a syntactically valid sequence of typed characters into an internal data structure. Evaluation replaces variable and function names with their values and simplification rewrites expressions to be easier for the user or other programs to understand. Output displays results in a variety of different formats and notations.

Evaluation and simplification sometimes appear to have similar functionality, and Maxima uses simplification in many cases where other systems use evaluation. For example, arithmetic both on numbers and on symbolic expressions is simplification, not evaluation: $2+2$ simplifies to $4$, $2*x*x$ simplifies to $2+2*x$, and $\sqrt{7}^4$ simplifies to $49$. Evaluation and simplification are interleaved. For example, \texttt{factor(integrate(x+1,x))} first calls the built-in function \texttt{integrate}, giving $x+x*x*2^{-1}$; that simplifies to $x+(1/2)*x^2$; this in turn is passed to the \texttt{factor} function, which returns $(x*(x+2))/2$.

Evaluation is what makes Maxima a programming language: it implements functions, subroutines, variables, values, loops, assignments and so on. Evaluation replaces built-in or user-defined function names by their definitions and variables by their values. This is largely the same as activities of a conventional programming language, but extended to work with symbolic mathematical data. The system has various optional "flags" which the user can set to control the details of evaluation. See Section 8.1 [Functions and Variables for Evaluation], page 135.

Simplification maintains the value of an expression while re-formulating its form to be smaller, simpler to understand, or to conform to a particular specification (like expanded). For example, $\sin(\pi/2)$ to $1$, and $x+x$ to $2*x$. There are many flags which control simplification. For example, with \texttt{triginverses: true, atan(tan(x))} does not simplify to x, but with \texttt{triginverses: all}, it does.

Simplification can be provided in three ways:

- The internal, built-in automated simplifier,
- User-written pattern-matching transformations, linked to the simplifier by using "\texttt{tellsimp}" or "\texttt{tellsimpafter}" and called automatically,
- User-written simplification routines adding using the \texttt{simplifying} subsystem.

The internal simplifier belongs to the heart of Maxima. It is a large and complicated collection of programs, and it has been refined over many years and by thousands of users. Nevertheless, especially if you are trying out novel ideas or unconventional notation, you may find it helpful to make small (or large) changes to the program yourself. For details see for example the paper at the end of https://people.eecs.berkeley.edu/~fateman/papers/intro5.txt.

Maxima internally represents expressions as "trees" with operators or "roots" like $+$, $*$, $=$ and operands ("leaves") which are variables like $x$, $y$, $z$, functions or sub-trees, like $x*y$. Each operator has a simplification program associated with it. $+$ (which also covers binary – since $a-b = a+(-1)*b$) and $*$ (which also covers $/$ since $a/b = a*b^(-1)$) have rather elaborate
simplification programs. These simplification programs (simpplus, simptimes, simpexpt, etc.)
are called whenever the simplifier encounters the respective arithmetic operators in an
expression tree to be analyzed.

The structure of the simplifier dates back to 1965, and many hands have worked on it
through the years. It is data-directed, or object-oriented in the sense that it dispatches to
the appropriate routine depending on the root of some sub-tree of the expression, recursively.
This general approach means that modifications to simplification are generally localized. In
many cases it is straightforward to add an operator and its simplification routine without
disturbing existing code.

Maxima also provides a variety of transformation routines that can change the form of an
expression, including factor (polynomial factorization), horner (reorganize a polynomial
using Horner’s rule), partfrac (rewrite a rational function as partial fractions), trigexpand
(apply the sum formulas for trigonometric functions), and so on.

Users can also write routines that change the form of an expression.

Besides this general simplifier operating on algebraic expression trees, there are several
other representations of expressions in Maxima which have separate methods. For example,
the rat function converts polynomials to vectors of coefficients to assist in rapid manip-
ulation of such forms. Other representations include Taylor series and the (rarely used)
Poisson series.

All operators introduced by the user initially have no simplification programs associated
with them. Maxima does not know anything about function "f" and so typing f(a,b) will
result in simplifying a,b, but not f. Even some built-in operators have no simplifications.
For example, = does not "simplify" – it is a place-holder with no simplification semantics
other than to simplify its two arguments, in this case referred to as the left and right sides.
Other parts of Maxima such as the solve program take special note of equations, that is,
trees with = as the root. (Note – in Maxima, the assignment operation is : . That is, q: 4
sets the value of the symbol q to 4. Function definition is done with := .)

The general simplifier returns results with an internal flag indicating the expression and
each sub-expression has been simplified. This does not guarantee that it is unique over
all possible equivalent expressions. That’s too hard (theoretically, not possible given the
generality of what can be expressed in Maxima). However, some aspects of the expression,
such as the ordering of terms in a sum or product, are made uniform. This is important for
the other programs to work properly.

A number of option variables control simplification. Indeed, simplification can be turned
off entirely using simp:false. However, many internal routines will not operate correctly
with simp:false. (About the only time it seems plausible to turn off the simplifier is in
the rare case that you want to over-ride a built-in simplification. In that case you might
temporarily disable the simplifier, put in the new transformation via tellsimp, and then
re-enable the simplifier by simp:true.)

It is more plausible for you to associate user-defined symbolic function names or operators
with properties (additive, lassociative, oddfun, antisymmetric, linear, outative,
commutative, multiplicative, rassociative, evenfun, nary and symmetric). These
options steer the simplifier processing in systematic directions.

For example, declare(f, oddfun) specifies that f is an odd function. Maxima will
simplify f(-x) to -f(x). In the case of an even function, that is declare(g, evenfun),
Maxima will simplify \( g(-x) \) to \( g(x) \). You can also associate a programming function with a name such as \( h(x):=x^2+1 \). In that case the evaluator will immediately replace \( h(3) \) by 10, and \( h(a+1) \) by \((a+1)^2+1\), so any properties of \( h \) will be ignored.

In addition to these directly related properties set up by the user, facts and properties from the actual context may have an impact on the simplifier’s behavior, too. See Section 11.1 [Introduction to Maxima’s Database], page 195.

Example: \( \sin(n\cdot\%pi) \) is simplified to zero, if \( n \) is an integer.

\[
\begin{align*}
\text{%i1} & \quad \sin(n\cdot\%pi) \\
\text{%o1} & \quad \sin(\%pi \ n) \\
\text{%i2} & \quad \text{declare}(n, \text{integer}); \\
\text{%o2} & \quad \text{done} \\
\text{%i3} & \quad \sin(n\cdot\%pi); \\
\text{%o3} & \quad 0
\end{align*}
\]

If automated simplification is not sufficient, you can consider a variety of built-in, but explicitly called simplification functions (\texttt{ratsimp}, \texttt{expand}, \texttt{factor}, \texttt{radcan} and others). There are also flags that will push simplification into one or another direction. Given \texttt{demoivre:true} the simplifier rewrites complex exponentials as trigonometric forms. Given \texttt{exponentialize:true} the simplifier tries to do the reverse: rewrite trigonometric forms as complex exponentials.

As everywhere in Maxima, by writing your own functions (be it in the Maxima user language or in the implementation language Lisp) and explicitly calling them at selected places in the program, you can respond to your individual simplification needs. Lisp gives you a handle on all the internal mechanisms, but you rarely need this full generality. "Tellisp" is designed to generate much of the Lisp internal interface into the simplifier automatically. See See Chapter 34 [Rules and Patterns], page 589.

Over the years (Maxima/Macsyma’s origins date back to about 1966!) users have contributed numerous application packages and tools to extend or alter its functional behavior. Various non-standard and "share" packages exist to modify or extend simplification as well. You are invited to look into this more experimental material where work is still in progress See Chapter 87 [simplification-pkg], page 1153.

The following appended material is optional on a first reading, and reading it is not necessary for productive use of Maxima. It is for the curious user who wants to understand what is going on, or the ambitious programmer who might wish to change the (open-source) code. Experimentation with redefining Maxima Lisp code is easily possible: to change the definition of a Lisp program (say the one that simplifies \texttt{cos()}, named \texttt{simp\%cos}), you simply load into Maxima a text file that will overwrite the \texttt{simp\%cos} function from the maxima package.

### 9.2 Functions and Variables for Simplification

**additive** \hspace{1cm} [Property]

If \texttt{declare(f,additive)} has been executed, then:

1. If \( f \) is univariate, whenever the simplifier encounters \( f \) applied to a sum, \( f \) will be distributed over that sum. I.e. \( f(y+x) \) will simplify to \( f(y)+f(x) \).
(2) If $f$ is a function of 2 or more arguments, additivity is defined as additivity in the first argument to $f$, as in the case of \texttt{sum} or \texttt{integrate}, i.e. $f(h(x)+g(x),x)$ will simplify to $f(h(x),x)+f(g(x),x)$. This simplification does not occur when $f$ is applied to expressions of the form $\texttt{sum(x[i],i,lower-limit,upper-limit)}$.

Example:

```maxima
(%i1) F3(a + b + c);
(%o1) F3(c + b + a)
(%i2) declare(F3, additive);
(%o2) done
(%i3) F3(a + b + c);
(%o3) F3(c) + F3(b) + F3(a)
```

\textbf{antisymmetric} \hfill [Property]

If \texttt{declare(h,antisymmetric)} is done, this tells the simplifier that $h$ is antisymmetric. E.g. $h(x,z,y)$ will simplify to $-h(x,y,z)$. That is, it will give $(-1)^n$ times the result given by \texttt{symmetric} or \texttt{commutative}, where $n$ is the number of interchanges of two arguments necessary to convert it to that form.

Examples:

```maxima
(%i1) S(b, a);
(%o1) S(b, a)
(%i2) declare(S, symmetric);
(%o2) done
(%i3) S(b, a);
(%o3) S(a, b)
(%i4) S(a, c, e, d, b);
(%o4) S(a, b, c, d, e)
(%i5) T(b, a);
(%o5) T(b, a)
(%i6) declare(T, antisymmetric);
(%o6) done
(%i7) T(b, a);
(%o7) - T(a, b)
(%i8) T(a, c, e, d, b);
(%o8) T(a, b, c, d, e)
```

\textbf{combine (expr)} \hfill [Function]

Simplifies the sum \texttt{expr} by combining terms with the same denominator into a single term.

See also: \texttt{rncombine}.

Example:

```maxima
(%i1) 1*f/2*b + 2*c/3*a + 3*f/4*b + c/5*b*a;
      5  b  f    a  b  c    2  a  c
(%o1) ----- + ----- + ----- + 4  5  3
```
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(\%i2) combine (%);

75 b f + 4 (3 a b c + 10 a c)

(\%o2) -----------------------------

60

commutative

If `declare(h, commutative)` is done, this tells the simplifier that \( h \) is a commutative function. E.g. \( h(x, z, y) \) will simplify to \( h(x, y, z) \). This is the same as symmetric.

Example:

(\%i1) S (b, a);

(\%o1) S(b, a)

(\%i2) S (a, b) + S (b, a);

(\%o2) S(b, a) + S(a, b)

(\%i3) declare (S, commutative);

(\%o3) done

(\%i4) S (b, a);

(\%o4) S(a, b)

(\%i5) S (a, b) + S (b, a);

(\%o5) 2 S(a, b)

(\%i6) S (a, c, e, d, b);

(\%o6) S(a, b, c, d, e)

demoivre (expr) [Function]

demoivre [Option variable]

The function `demoivre (expr)` converts one expression without setting the global variable `demoivre`.

When the variable `demoivre` is `true`, complex exponentials are converted into equivalent expressions in terms of circular functions: \( \text{exp} (a + b*%i) \) simplifies to \( \%e^a * (\cos(b) + %i*\sin(b)) \) if \( b \) is free of \( %i \). \( a \) and \( b \) are not expanded.

The default value of `demoivre` is `false`.

`exponentialize` converts circular and hyperbolic functions to exponential form. `demoivre` and `exponentialize` cannot both be true at the same time.

distrib (expr) [Function]

Distributes sums over products. It differs from `expand` in that it works at only the top level of an expression, i.e., it doesn’t recurse and it is faster than `expand`. It differs from `multthru` in that it expands all sums at that level.

Examples:

(\%i1) distrib ((a+b) * (c+d));

(\%o1) b d + a d + b c + a c

(\%i2) multthru ((a+b) * (c+d));

(\%o2) (b + a) d + (b + a) c

(\%i3) distrib (1/((a+b) * (c+d)));

(\%o3) -----------------------------

1

(b + a) (d + c)
(\%i4) expand (1/((a+b) * (c+d)), 1, 0);

\(1\)

\(\text{b d + a d + b c + a c}\)

\textbf{distribute\_over} \hspace{1cm} \text{[Option variable]}

Default value: \text{true}

distribute\_over \ controls the mapping of functions over bags like lists, matrices, and equations. At this time not all Maxima functions have this property. It is possible to look up this property with the command \textit{properties}.

The mapping of functions is switched off, when setting \text{distribute\_over} to the value \text{false}.

Examples:

The \text{sin} function maps over a list:

(\%i1) sin([x,1,1.0]);

(\%o1) [sin(x), sin(1), 0.8414709848078965]

\text{mod} is a function with two arguments which maps over lists. Mapping over nested lists is possible too:

(\%i1) mod([x,11,2*a],10);

(\%o1) [mod(x, 10), 1, 2 mod(a, 5)]

(\%i2) mod([[x,y,z],11,2*a],10);

(\%o2) [[mod(x, 10), mod(y, 10), mod(z, 10)], 1, 2 mod(a, 5)]

Mapping of the \text{floor} function over a matrix and an equation:

(\%i1) floor(matrix([a,b],[c,d]));

(\%o1)

\[
\begin{array}{cc}
\text{floor(a)} & \text{floor(b)} \\
\text{floor(c)} & \text{floor(d)} \\
\end{array}
\]

(\%i2) floor(a=b);

(\%o2) floor(a) = floor(b)

Functions with more than one argument map over any of the arguments or all arguments:

(\%i1) expintegral\_e([1,2],[x,y]);

(\%o1) [[\text{expintegral\_e}(1, x), \text{expintegral\_e}(1, y)],

[\text{expintegral\_e}(2, x), \text{expintegral\_e}(2, y)]]

Check if a function has the property distribute\_over:

(\%i1) properties(abs);

(\%o1) \text{[integral, rule, distributes over bags, noun, gradef, system function]}

The mapping of functions is switched off, when setting \text{distribute\_over} to the value \text{false}.

(\%i1) distribute\_over;

(\%o1) \text{true}

(\%i2) sin([x,1,1.0]);

(\%o2) [sin(x), sin(1), 0.8414709848078965]
(%i3) distribute_over : not distribute_over;
(%o3) false
(%i4) sin([x,1,1.0]);
(%o4) sin([x, 1, 1.0])

domain [Option variable]
Default value: real
When domain is set to complex, sqrt (x^2) will remain sqrt (x^2) instead of returning abs(x).

evenfun [Property]
oddfun [Property]
declare(f, evenfun) or declare(f, oddfun) tells Maxima to recognize the function f as an even or odd function.
Examples:
(%i1) o (- x) + o (x);
(%o1) o(x) + o(- x)
(%i2) declare (o, oddfun);
(%o2) done
(%i3) o (- x) + o (x);
(%o3) 0
(%i4) e (- x) - e (x);
(%o4) e(- x) - e(x)
(%i5) declare (e, evenfun);
(%o5) done
(%i6) e (- x) - e (x);
(%o6) 0

expand [Function]
expand (expr)
expand (expr, p, n)
Expand expression expr. Products of sums and exponentiated sums are multiplied out, numerators of rational expressions which are sums are split into their respective terms, and multiplication (commutative and non-commutative) are distributed over addition at all levels of expr.
For polynomials one should usually use ratexpand which uses a more efficient algorithm.
maxnegex and maxposex control the maximum negative and positive exponents, respectively, which will expand.

expand (expr, p, n) expands expr, using p for maxposex and n for maxnegex. This is useful in order to expand part but not all of an expression.
expon - the exponent of the largest negative power which is automatically expanded (independent of calls to expand). For example if expon is 4 then (x+1)^(-5) will not be automatically expanded.
expop - the highest positive exponent which is automatically expanded. Thus (x+1)^3, when typed, will be automatically expanded only if expop is greater than
or equal to 3. If it is desired to have \((x+1)^n\) expanded where \(n\) is greater than \texttt{expop} then executing \texttt{expand \((x+1)^n\)} will work only if \texttt{maxposex} is not less than \(n\).

\texttt{expand(expr, 0, 0)} causes a resimplification of expr. expr is not reevaluated. In distinction from \texttt{ev(expr, noeval)} a special representation (e.g. a CRE form) is removed. See also \texttt{ev}.

The \texttt{expand} flag used with \texttt{ev} causes expansion.

The file \texttt{share/simplification/facexp.mac} contains several related functions (in particular \texttt{facsum}, \texttt{factorfacsum} and \texttt{collectterms}, which are autoloaded) and variables (\texttt{nextlayerfactor} and \texttt{facsum_combine}) that provide the user with the ability to structure expressions by controlled expansion. Brief function descriptions are available in \texttt{simplification/facexp.usg}. A demo is available by doing \texttt{demo("facexp")}.

Examples:

```
(%i1) expr: (x+1)^2*(y+1)^3;
            2 3
(%o1) (x + 1) (y + 1)

(%i2) expand(expr);
            2 3 3 2 2 2 2 2
(%o2) x y + 2 x y + y + 3 x y + 6 x y + 3 y + 3 x y
               2
              + 6 x y + 3 y + x + 2 x + 1

(%i3) expand(expr,2);
            2
(%o3) x (y + 1) + 2 x (y + 1) + (y + 1)

(%i4) expr: (x+1)^-2*(y+1)^3;
            3
(%o4) ------------ + ------------ + ------------ + ------------
                 2 2 2 2
x + 2 x + 1 x + 2 x + 1 x + 2 x + 1 x + 2 x + 1

(%i5) expand(expr);
            3
(%o5) ------------ + ------------ + ------------ + ------------
                 2 2 2 2
y + 3 y + 3 y + 1
               2
               x + 2 x + 1

(%i6) expand(expr,2,2);
            3
(%o6) ------------ + ------------
                 2 2
x + 2 x + 1

Resimplify an expression without expansion:

```
(%i1) expr: (1+x)^2*sin(x);
            2
(%o1) (x + 1) sin(x)
```
(\%i2) exponentialize: true;
(\%o2) true
(\%i3) expand(expr, 0, 0);
(\%o3) \frac{2i x - i x}{2i (x + 1) (e^{x} - e^{-x})}

expandwrt (expr, x_1, \ldots, x_n)
[Function]
Expands expression expr with respect to the variables x_1, \ldots, x_n. All products involving the variables appear explicitly. The form returned will be free of products of sums of expressions that are not free of the variables. x_1, \ldots, x_n may be variables, operators, or expressions.

By default, denominators are not expanded, but this can be controlled by means of the switch expandwrt_denom.
This function is autoloaded from simplification/stopex.mac.

expandwrt_denom
[Option variable]
Default value: false
expandwrt_denom controls the treatment of rational expressions by expandwrt. If true, then both the numerator and denominator of the expression will be expanded according to the arguments of expandwrt, but if expandwrt_denom is false, then only the numerator will be expanded in that way.

expandwrt_factored (expr, x_1, \ldots, x_n)
[Function]
is similar to expandwrt, but treats expressions that are products somewhat differently. expandwrt_factored expands only on those factors of expr that contain the variables x_1, \ldots, x_n.

This function is autoloaded from simplification/stopex.mac.

expon
[Option variable]
Default value: 0
expon is the exponent of the largest negative power which is automatically expanded (independent of calls to expand). For example, if expon is 4 then (x+1)^(-5) will not be automatically expanded.

exponentialize (expr)
exponentialize
[Function]
The function exponentialize (expr) converts circular and hyperbolic functions in expr to exponentials, without setting the global variable exponentialize.

When the variable exponentialize is true, all circular and hyperbolic functions are converted to exponential form. The default value is false.
demoivre converts complex exponentials into circular functions. exponentialize and demoivre cannot both be true at the same time.

expop
[Option variable]
Default value: 0
expop is the highest positive exponent which is automatically expanded. Thus \((x + 1)^3\), when typed, will be automatically expanded only if expop is greater than or equal to 3. If it is desired to have \((x + 1)^n\) expanded where \(n\) is greater than expop then executing `expand ((x + 1)^n)` will work only if maxposex is not less than \(n\).

lassociative

declare (g, lassociative) tells the Maxima simplifier that \(g\) is left-associative. E.g., \(g (g (a, b), g (c, d))\) will simplify to \(g (g (a, b), c), d)\).

linear

One of Maxima’s operator properties. For univariate \(f\) so declared, "expansion" \(f(x + y)\) yields \(f(x) + f(y)\), \(f(a*x)\) yields \(a*f(x)\) takes place where \(a\) is a "constant". For functions of two or more arguments, "linearity" is defined to be as in the case of `sum` or `integrate`, i.e., \(f (a*x + b, x)\) yields \(a*f(x,x) + b*f(1,x)\) for \(a\) and \(b\) free of \(x\).

Example:

```
(%i1) declare (f, linear);
(%o1) done
(%i2) f(x+y);
(%o2) f(y) + f(x)
(%i3) declare (a, constant);
(%o3) done
(%i4) f(a*x);
(%o4) a f(x)
```

linear is equivalent to `additive` and `outative`. See also `opproperties`.

Example:

```
(%i1) 'sum (F(k) + G(k), k, 1, inf);
inf
====
\  
(%o1) > (G(k) + F(k))
/ 
====
 k = 1
(%i2) declare (nounify (sum), linear);
(%o2) done
(%i3) 'sum (F(k) + G(k), k, 1, inf);
inf        inf
====      ====
\  \ 
(%o3) > G(k) + > F(k)
/  / 
====  ==== 
 k = 1  k = 1
```

maxnegex

[Option variable]

Default value: 1000
maxnegex is the largest negative exponent which will be expanded by the \texttt{expand} command, see also \texttt{maxposex}.

\textbf{maxposex} \hspace{4cm} \textbf{[Option variable]}

Default value: 1000

\texttt{maxposex} is the largest exponent which will be expanded with the \texttt{expand} command, see also \texttt{maxnegex}.

\textbf{multiplicative} \hspace{4cm} \textbf{[Property]}

\texttt{declare(f, multiplicative)} tells the Maxima simplifier that \texttt{f} is multiplicative.

1. If \texttt{f} is univariate, whenever the simplifier encounters \texttt{f} applied to a product, \texttt{f} distributes over that product. E.g., \texttt{f(x*y)} simplifies to \texttt{f(x)*f(y)}. This simplification is not applied to expressions of the form \texttt{f(\texttt{\textbackslash product(...))}}.

2. If \texttt{f} is a function of 2 or more arguments, multiplicativity is defined as multiplicativity in the first argument to \texttt{f}, e.g., \texttt{f(g(x)*h(x), x)} simplifies to \texttt{f(g(x),x) * f(h(x), x)}.

\texttt{declare(nounify(product), multiplicative)} tells Maxima to simplify symbolic products.

Example:

\begin{verbatim}
(%i1) F2 (a * b * c);
     F2(a b c)
(%o1) F2(a b c)
(%i2) declare (F2, multiplicative);
     done
(%i3) F2 (a * b * c);
     F2(a) F2(b) F2(c)
\end{verbatim}

\texttt{declare(nounify(product), multiplicative)} tells Maxima to simplify symbolic products.

\begin{verbatim}
(%i1) product (a[i] * b[i], i, 1, n);
     n
    /===\  
   ! !
   " a b
   ! ! i i
   i = 1
(%o1) ! ! a b
   ! ! i i
   i = 1
(%i2) declare (nounify (product), multiplicative);
     done
(%i3) product (a[i] * b[i], i, 1, n);
     n    n
    /===\  /===\ 
   ! !   ! !
   " a b
   ! ! i i
   i = 1  i = 1
\end{verbatim}
multthru

multthru (expr)
multthru (expr_1, expr_2)

Multiplies a factor (which should be a sum) of expr by the other factors of expr. That is, expr is $f_1 f_2 \ldots f_n$ where at least one factor, say $f_i$, is a sum of terms. Each term in this sum is multiplied by the other factors in the product. (Namely all the factors except $f_i$). multthru does not expand exponentiated sums. This function is the fastest way to distribute products (commutative or noncommutative) over sums. Since quotients are represented as products multthru can be used to divide sums by products as well.

multthru (expr_1, expr_2) multiplies each term in expr_2 (which should be a sum or an equation) by expr_1. If expr_1 is not itself a sum then this form is equivalent to multthru (expr_1*expr_2).

(%i1) x/(x-y)^2 - 1/(x-y) - f(x)/(x-y)^3;
    1 x f(x)
 (%o1) - ----- + -------- - --------
        x - y       2          3
          (x - y)     (x - y)

(%i2) multthru ((x-y)^3, %);
    2
 (%o2) - (x - y) + x (x - y) - f(x)

(%i3) ratexpand (%);
    2
 (%o3) - y + x y - f(x)

(%i4) ((a+b)^10*s^2 + 2*a*b*s + (a*b)^2)/(a*b*s^2);
          2 2 2 2
(b + a) s + 2 a b s + a b
 %o4) ------------------------------
          2
          a b s

(%i5) multthru (%); /* note that this does not expand (b+a)^10 */
 10
   2   a b (b + a)
 (%o5) - + --- + ---------
        s 2    a b

(%i6) multthru (a.(b+c.(d+e)+f));
    10
 (%o6) a . f + a . c . (e + d) + a . b

(%i7) expand (a.(b+c.(d+e)+f));
    10
 (%o7) a . f + a . c . e + a . c . d + a . b

nary

declare(f, nary) tells Maxima to recognize the function f as an n-ary function.
The nary declaration is not the same as calling the [function_nary], page 134 function. The sole effect of declare(f, nary) is to instruct the Maxima simplifier to flatten nested expressions, for example, to simplify $\text{foo}(x, \text{foo}(y, z))$ to $\text{foo}(x, y, z)$. See also declare.
Example:

(\texttt{\%i1}) \ H \ (\ H \ (a, \ b), \ \ H \ (c, \ \ H \ (d, \ e)))
(\texttt{\%o1}) \ H(H(a, \ b), \ H(c, \ H(d, \ e)))
(\texttt{\%i2}) \ \texttt{declare} \ (H, \ \texttt{nary})
(\texttt{\%o2}) \ \\texttt{done}
(\texttt{\%i3}) \ H \ (H \ (a, \ b), \ H \ (c, \ H \ (d, \ e)))
(\texttt{\%o3}) \ \ H(a, \ b, \ c, \ d, \ e)

\texttt{negdistrib} \quad \quad \quad \quad \quad \quad \quad \quad [\text{Option variable}]

Default value: true

When \texttt{negdistrib} is true, -1 distributes over an expression. E.g., -(x + y) becomes - y - x. Setting it to \texttt{false} will allow - (x + y) to be displayed like that. This is sometimes useful but be very careful: like the \texttt{simp} flag, this is one flag you do not want to set to \texttt{false} as a matter of course or necessarily for other than local use in your Maxima.

Example:

(\texttt{\%i1}) \ \texttt{negdistrib};
(\texttt{\%o1}) \ true
(\texttt{\%i2}) \ -(x+y);
(\texttt{\%o2}) \ (- y) - x
(\texttt{\%i3}) \ \texttt{negdistrib} : \ \texttt{not} \ \texttt{negdistrib} ;
(\texttt{\%o3}) \ \texttt{false}
(\texttt{\%i4}) \ -(x+y);
(\texttt{\%o4}) \ - \ (y + x)

\texttt{opproperties} \quad \quad \quad \quad \quad \quad \quad \quad [\text{System variable}]

\texttt{opproperties} is the list of the special operator properties recognized by the Maxima simplifier.

Items are added to the \texttt{opproperties} list by the function \texttt{define_opproperty}.

Example:

(\texttt{\%i1}) \ \texttt{opproperties};
(\texttt{\%o1}) \ [\texttt{linear}, \ \texttt{additive}, \ \texttt{multiplicative}, \ \texttt{outative}, \ \texttt{evenfun},
\texttt{oddfun}, \ \texttt{commutative}, \ \texttt{symmetric}, \ \texttt{antisymmetric}, \ \texttt{nary},
\texttt{lassociative}, \ \texttt{rassociative}]

\texttt{define_opproperty} \ \ (\texttt{property\_name}, \ \texttt{simplifier\_fn}) \quad \quad \quad \quad \quad [\text{Function}]

Declares the symbol \texttt{property\_name} to be an operator property, which is simplified by \texttt{simplifier\_fn}, which may be the name of a Maxima or Lisp function or a lambda expression. After \texttt{define_opproperty} is called, functions and operators may be declared to have the \texttt{property\_name} property, and \texttt{simplifier\_fn} is called to simplify them.

\texttt{simplifier\_fn} must be a function of one argument, which is an expression in which the main operator is declared to have the \texttt{property\_name} property.

\texttt{simplifier\_fn} is called with the global flag \texttt{simp} disabled. Therefore \texttt{simplifier\_fn} must be able to carry out its simplification without making use of the general simplifier. \texttt{define_opproperty} appends \texttt{property\_name} to the global list \texttt{opproperties}. 
define_opproperty returns done.

Example:
Declare a new property, identity, which is simplified by simplify_identity. Declare that \( f \) and \( g \) have the new property.

\[
\begin{align*}
\texttt{(i1)} & \quad \text{define_opproperty (identity, simplify_identity);} \\
\texttt{(o1)} & \quad \text{done} \\
\texttt{(i2)} & \quad \text{simplify_identity(e) := first(e);} \\
\texttt{(o2)} & \quad \text{simplify_identity(e) := first(e)} \\
\texttt{(i3)} & \quad \text{declare ([f, g], identity);} \\
\texttt{(o3)} & \quad \text{done} \\
\texttt{(i4)} & \quad f(10 + t); \\
\texttt{(o4)} & \quad t + 10 \\
\texttt{(i5)} & \quad g(3*u) - f(2*u); \\
\texttt{(o5)} & \quad u
\end{align*}
\]

outative

\texttt{declare(f, outative)} tells the Maxima simplifier that constant factors in the argument of \( f \) can be pulled out.

1. If \( f \) is univariate, whenever the simplifier encounters \( f \) applied to a product, that product will be partitioned into factors that are constant and factors that are not and the constant factors will be pulled out. E.g., \( f(a*x) \) will simplify to \( a*f(x) \) where \( a \) is a constant. Non-atomic constant factors will not be pulled out.

2. If \( f \) is a function of 2 or more arguments, outativity is defined as in the case of \texttt{sum} or \texttt{integrate}, i.e., \( f \ (a*g(x), x) \) will simplify to \( a \ast f(g(x), x) \) for \( a \) free of \( x \).

\texttt{sum}, \texttt{integrate}, and \texttt{limit} are all outative.

Example:
\[
\begin{align*}
\texttt{(i1)} & \quad F1 \ (100 \ast x); \\
\texttt{(o1)} & \quad F1(100 \ x) \\
\texttt{(i2)} & \quad \text{declare (F1, outative);} \\
\texttt{(o2)} & \quad \text{done} \\
\texttt{(i3)} & \quad F1 \ (100 \ast x); \\
\texttt{(o3)} & \quad 100 \ F1(x) \\
\texttt{(i4)} & \quad \text{declare (zz, constant);} \\
\texttt{(o4)} & \quad \text{done} \\
\texttt{(i5)} & \quad F1 \ (zz \ast y); \\
\texttt{(o5)} & \quad zz \ F1(y)
\end{align*}
\]

\texttt{radcan (expr)}

Simplifies \( expr \), which can contain logs, exponentials, and radicals, by converting it into a form which is canonical over a large class of expressions and a given ordering of variables; that is, all functionally equivalent forms are mapped into a unique form. For a somewhat larger class of expressions, \texttt{radcan} produces a regular form. Two equivalent expressions in this class do not necessarily have the same appearance, but their difference can be simplified by \texttt{radcan} to zero.
For some expressions \texttt{radcan} is quite time consuming. This is the cost of exploring certain relationships among the components of the expression for simplifications based on factoring and partial-fraction expansions of exponents.

Examples:

\begin{verbatim}
(%i1) radcan((log(x+x^2)-log(x))^(a/2)/log(1+x));
a/2
(%o1) log(x + 1)
(%i2) radcan((log(1+2*a^x+a^x)/log(1+a^x)))
2
(%o2) 2
(%i3) radcan((%e^x-1)/(1+%e^(x/2)));
x/2
(%o3) %e - 1
\end{verbatim}

\textbf{radexpand} \\
Default value: \texttt{true}

\texttt{radexpand} controls some simplifications of radicals.

When \texttt{radexpand} is \texttt{all}, causes nth roots of factors of a product which are powers of n to be pulled outside of the radical. E.g. if \texttt{radexpand} is \texttt{all}, \texttt{sqrt (16*x^2)} simplifies to \texttt{4*x}.

More particularly, consider \texttt{sqrt (x^2)}.

- If \texttt{radexpand} is \texttt{all} or \texttt{assume (x > 0)} has been executed, \texttt{sqrt(x^2)} simplifies to \texttt{x}.
- If \texttt{radexpand} is \texttt{true} and \texttt{domain} is \texttt{real} (its default), \texttt{sqrt(x^2)} simplifies to \texttt{abs(x)}.
- If \texttt{radexpand} is \texttt{false}, or \texttt{radexpand} is \texttt{true} and \texttt{domain} is \texttt{complex}, \texttt{sqrt(x^2)} is not simplified.

Note that \texttt{domain} only matters when \texttt{radexpand} is \texttt{true}.

\textbf{rassociative} \\
\texttt{declare (g, rassociative)} tells the Maxima simplifier that \texttt{g} is right-associative. E.g., \texttt{g(g(a, b), g(c, d))} simplifies to \texttt{g(a, g(b, g(c, d)))}.

\textbf{scsimp (expr, rule_1, \ldots, rule_n)} \\
Sequential Comparative Simplification (method due to Stoute). \texttt{scsimp} attempts to simplify \texttt{expr} according to the rules \texttt{rule_1, \ldots, rule_n}. If a smaller expression is obtained, the process repeats. Otherwise after all simplifications are tried, it returns the original answer.

\texttt{example (scsimp)} displays some examples.

\textbf{simp} \\
Default value: \texttt{true}

\texttt{simp} enables simplification. This is the default. \texttt{simp} is also an \texttt{evflag}, which is recognized by the function \texttt{ev}. See \texttt{ev}.

When \texttt{simp} is used as an \texttt{evflag} with a value \texttt{false}, the simplification is suppressed only during the evaluation phase of an expression. The flag does not suppress the simplification which follows the evaluation phase.
Many Maxima functions and operations require simplification to be enabled to work normally. When simplification is disabled, many results will be incomplete, and in addition there may be incorrect results or program errors.

Examples:
The simplification is switched off globally. The expression $\sin(1.0)$ is not simplified to its numerical value. The simp-flag switches the simplification on.

\begin{verbatim}
(%i1) simp:false;
(%o1) false
(%i2) sin(1.0);
(%o2) sin(1.0)
(%i3) sin(1.0),simp;
(%o3) 0.8414709848078965
\end{verbatim}

The simplification is switched on again. The simp-flag cannot suppress the simplification completely. The output shows a simplified expression, but the variable $x$ has an unsimplified expression as a value, because the assignment has occurred during the evaluation phase of the expression.

\begin{verbatim}
(%i1) simp:true;
(%o1) true
(%i2) x:sin(1.0),simp:false;
(%o2) 0.8414709848078965
(%i3) :lisp $x
((%SIN) 1.0)
\end{verbatim}

\textbf{symmetric}  
\textit{declare} (h, symmetric) tells the Maxima simplifier that $h$ is a symmetric function. E.g., $h(x, z, y)$ simplifies to $h(x, y, z)$.

\textbf{commutative} is synonymous with symmetric.

\textbf{xthru (expr)}  
\textit{Combines all terms of expr (which should be a sum) over a common denominator without expanding products and exponentiated sums as ratsimp does. xthru cancels common factors in the numerator and denominator of rational expressions but only if the factors are explicit.}

Sometimes it is better to use \textbf{xthru} before ratsimping an expression in order to cause explicit factors of the gcd of the numerator and denominator to be canceled thus simplifying the expression to be ratsimped.

Examples:

\begin{verbatim}
(%i1) ((x+2)^20 - 2*y)/(x+y)^20 + (x+y)^(-19) - x/(x+y)^20;
\end{verbatim}

\begin{verbatim}
20
1 (x + 2) - 2 y x
--------- + --------------- - ---------
19 20 20
(y + x) (y + x) (y + x)
\end{verbatim}
(%i2) xthru (%);

(%o2)
\[
\frac{20}{20} \frac{(x + 2) - y}{(y + x)}
\]
10 Mathematical Functions

10.1 Functions for Numbers

\texttt{abs (z)} \hspace{1cm} \textbf{[Function]}

The \texttt{abs} function represents the mathematical absolute value function and works for both numerical and symbolic values. If the argument, \( z \), is a real or complex number, \texttt{abs} returns the absolute value of \( z \). If possible, symbolic expressions using the absolute value function are also simplified.

Maxima can differentiate, integrate and calculate limits for expressions containing \texttt{abs}. The \texttt{abs_integrate} package further extends Maxima's ability to calculate integrals involving the \texttt{abs} function. See \texttt{(%i12)} in the examples below.

When applied to a list or matrix, \texttt{abs} automatically distributes over the terms. Similarly, it distributes over both sides of an equation. To alter this behaviour, see the variable \texttt{distribute_over}.

See also \texttt{cabs}.

Examples:

Calculation of \texttt{abs} for real and complex numbers, including numerical constants and various infinities. The first example shows how \texttt{abs} distributes over the elements of a list.

\[
\begin{align*}
\texttt{(\%i1)} & \quad \texttt{abs([-4, 0, 1, 1+%i])}; \\
& \quad \texttt{[4, 0, 1, sqrt(2)]} \\
\texttt{(\%i2)} & \quad \texttt{abs((1+%i)*(1-%i))}; \\
& \quad \texttt{2} \\
\texttt{(\%i3)} & \quad \texttt{abs(%e+%i)}; \\
& \quad \texttt{sqrt(%e + 1)} \\
\texttt{(\%i4)} & \quad \texttt{abs([inf, infinity, minf])}; \\
& \quad \texttt{[inf, inf, inf]}
\end{align*}
\]

Simplification of expressions containing \texttt{abs}:

\[
\begin{align*}
\texttt{(\%i5)} & \quad \texttt{abs(x^2)}; \\
& \quad \texttt{x^2} \\
\texttt{(\%i6)} & \quad \texttt{abs(x^3)}; \\
& \quad \texttt{2 \cdot x \cdot abs(x)} \\
\texttt{(\%i7)} & \quad \texttt{abs(abs(x))}; \\
& \quad \texttt{abs(x)} \\
\texttt{(\%i8)} & \quad \texttt{abs(conjugate(x))}; \\
& \quad \texttt{abs(x)}
\end{align*}
\]
Integrating and differentiating with the \texttt{abs} function. Note that more integrals involving the \texttt{abs} function can be performed, if the \texttt{abs_integrate} package is loaded. The last example shows the Laplace transform of \texttt{abs}: see \texttt{laplace}.

\begin{verbatim}
(%i9) diff(x*abs(x),x),expand;  
2 abs(x)
(%o9) -----------
x abs(x)

(%i10) integrate(abs(x),x);

(%o10) --------
     2

(%i11) integrate(x*abs(x),x);

(%i11)

(%o11)

(%i12) load("abs_integrate")$
(%i13) integrate(x*abs(x),x);

(%o13)

(%i14) integrate(abs(x),x,-2,%pi);

(%o14)

(%i15) laplace(abs(x),x,s);

(%o15)

\texttt{ceiling} (x)

\texttt{ceiling} (x) \hspace{1cm} \text{[Function]}

When \(x\) is a real number, return the least integer that is greater than or equal to \(x\). If \(x\) is a constant expression (\(10 \times \%pi\), for example), \texttt{ceiling} evaluates \(x\) using big floating point numbers, and applies \texttt{ceiling} to the resulting big float. Because \texttt{ceiling} uses floating point evaluation, it's possible, although unlikely, that \texttt{ceiling} could return an erroneous value for constant inputs. To guard against errors, the floating point evaluation is done using three values for \texttt{fpprec}.

For non-constant inputs, \texttt{ceiling} tries to return a simplified value. Here are examples of the simplifications that \texttt{ceiling} knows about:
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(%i1) ceiling (ceiling (x));
(%o1) ceiling(x)
(%i2) ceiling (floor (x));
(%o2) floor(x)
(%i3) declare (n, integer)$
(%i4) [ceiling (n), ceiling (abs (n)), ceiling (max (n, 6))];
(%o4) [n, abs(n), max(6, n)]
(%i5) assume (x > 0, x < 1)$
(%i6) ceiling (x);
(%o6) 1
(%i7) tex (ceiling (a));
$$\left \lceil a \right \rceil$$
(%o7) false

The ceiling function distributes over lists, matrices and equations. See distribute_over.

Finally, for all inputs that are manifestly complex, ceiling returns a noun form.

If the range of a function is a subset of the integers, it can be declared to be integervalued. Both the ceiling and floor functions can use this information; for example:

(%i1) declare (f, integervalued)$
(%i2) floor (f(x));
(%o2) f(x)
(%i3) ceiling (f(x) - 1);
(%o3) f(x) - 1

Example use:

(%i1) unitfrac(r) := block([uf : [], q],
   if not(ratnump(r)) then
     error("unitfrac: argument must be a rational number"),
   while r # 0 do (
     uf : cons(q : 1/ceiling(1/r), uf),
     r : r - q),
   reverse(uf));
(%i2) unitfrac (9/10);

1 1 1
(%o2) [-, -, --]
2 3 15
apply (+, %)

unitfrac (-9/10);

unitfrac (36/37);

entier (x)
Returns the largest integer less than or equal to x where x is numeric. fix (as in fixnum) is a synonym for this, so fix(x) is precisely the same.

floor (x)
When x is a real number, return the largest integer that is less than or equal to x. If x is a constant expression (10 * %pi, for example), floor evaluates x using big floating point numbers, and applies floor to the resulting big float. Because floor uses floating point evaluation, it’s possible, although unlikely, that floor could return an erroneous value for constant inputs. To guard against errors, the floating point evaluation is done using three values for fpprec. For non-constant inputs, floor tries to return a simplified value. Here are examples of the simplifications that floor knows about:

(>%1) floor (ceiling (x));
(>%2) floor (floor (x));
(>%3) declare (n, integer)$
(>%4) [floor (n), floor (abs (n)), floor (min (n, 6))];
(>%5) assume (x > 0, x < 1)$
(>%6) floor (x);
(>%7) tex (floor (a));
The `floor` function distributes over lists, matrices and equations. See distribute_over.

Finally, for all inputs that are manifestly complex, `floor` returns a noun form.

If the range of a function is a subset of the integers, it can be declared to be `integervalued`. Both the `ceiling` and `floor` functions can use this information; for example:

```
(%i1) declare (f, integervalued)$
(%i2) floor (f(x));
  f(x)
(%o2)
(%i3) ceiling (f(x) - 1);
  f(x) - 1
```

fix (x)  [Function]
A synonym for `entier (x)`.

hstep (x)  [Function]
The Heaviside unit step function, equal to 0 if x is negative, equal to 1 if x is positive and equal to 1/2 if x is equal to zero.

If you want a unit step function that takes on the value of 0 at x equal to zero, use unit_step.

lmax (L)  [Function]
When L is a list or a set, return `apply ('max, args (L))`. When L is not a list or a set, signal an error. See also lmin and max.

lmin (L)  [Function]
When L is a list or a set, return `apply ('min, args (L))`. When L is not a list or a set, signal an error. See also lmax and min.

max (x_1, . . ., x_n)  [Function]
Return a simplified value for the numerical maximum of the expressions x_1 through x_n. For an empty argument list, max yields minf.

The option variable `maxmin_effort` controls which simplification methods are applied. Using the default value of `twelve` for `maxmin_effort`, max uses all available simplification methods. To to inhibit all simplifications, set `maxmin_effort` to zero.

When `maxmin_effort` is one or more, for an explicit list of real numbers, max returns a number.

Unless max needs to simplify a lengthy list of expressions, we suggest using the default value of `maxmin_effort`. Setting `maxmin_effort` to zero (no simplifications), will cause problems for some Maxima functions; accordingly, generally `maxmin_effort` should be nonzero.

See also min, lmax., and lmin.

Examples:
In the first example, setting `maxmin_effort` to zero suppresses simplifications.

```
(%i11) block([maxmin_effort : 0], max(1,2,x,x, max(a,b)));
(%o11) max(1,2,max(a,b),x,x)
```
When `maxmin_effort` is two or more, `max` compares pairs of members:

```lisp
(%i1) block([maxmin_effort : 1], max(x, x+1, x+3));
(%o1) max(x, x+1, x+3)
(%i2) block([maxmin_effort : 2], max(x, x+1, x+3));
(%o2) x+3
```

Finally, when `maxmin_effort` is three or more, `max` compares triples members and excludes those that are in between; for example

```lisp
(%i1) block([maxmin_effort : 4], max(x, 2*x, 3*x, 4*x));
(%o1) max(x, 4*x)
```

### `min` Function

Return a simplified value for the numerical minimum of the expressions \(x_1\) through \(x_n\). For an empty argument list, `min` yields `inf`.

The option variable `maxmin_effort` controls which simplification methods are applied. Using the default value of `twelve` for `maxmin_effort`, `max` uses all available simplification methods. To to inhibit all simplifications, set `maxmin_effort` to zero.

When `maxmin_effort` is one or more, for an explicit list of real numbers, `min` returns a number. Unless `min` needs to simplify a lengthy list of expressions, we suggest using the default value of `maxmin_effort`. Setting `maxmin_effort` to zero (no simplifications), will cause problems for some Maxima functions; accordingly, generally `maxmin_effort` should be nonzero.

See also `max`, `lmax`, and `lmin`.

### Examples:

In the first example, setting `maxmin_effort` to zero suppresses simplifications.

```lisp
(%i1) block([maxmin_effort : 0], min(1, 2, x, x, min(a, b)));
(%o1) min(1, 2, a, b, x)
(%i2) block([maxmin_effort : 1], min(1, 2, x, x, min(a, b)));
(%o2) min(1, a, b, x)
```

When `maxmin_effort` is two or more, `min` compares pairs of members:

```lisp
(%i1) block([maxmin_effort : 1], min(x, x+1, x+3));
(%o1) min(x, x+1, x+3)
(%i2) block([maxmin_effort : 2], min(x, x+1, x+3));
(%o2) x
```

Finally, when `maxmin_effort` is three or more, `min` compares triples members and excludes those that are in between; for example

```lisp
(%i1) block([maxmin_effort : 4], min(x, 2*x, 3*x, 4*x));
(%o1) min(x, 4*x)
```
round(x)  [Function]
When \( x \) is a real number, returns the closest integer to \( x \). Multiples of 1/2 are rounded to the nearest even integer. Evaluation of \( x \) is similar to floor and ceiling.

The round function distributes over lists, matrices and equations. See distribute_over.

signum(x)  [Function]
For either real or complex numbers \( x \), the signum function returns 0 if \( x \) is zero; for a nonzero numeric input \( x \), the signum function returns \( x/\text{abs}(x) \).

For non-numeric inputs, Maxima attempts to determine the sign of the input. When the sign is negative, zero, or positive, signum returns -1, 0, 1, respectively. For all other values for the sign, signum a simplified but equivalent form. The simplifications include reflection (\( \text{signum}(-x) \) gives \( -\text{signum}(x) \)) and multiplicative identity (\( \text{signum}(x*y) \) gives \( \text{signum}(x) \* \text{signum}(y) \)).

The signum function distributes over a list, a matrix, or an equation. See sign and distribute_over.

truncate(x)  [Function]
When \( x \) is a real number, return the closest integer to \( x \) not greater in absolute value than \( x \). Evaluation of \( x \) is similar to floor and ceiling.

The truncate function distributes over lists, matrices and equations. See distribute_over.

10.2 Functions for Complex Numbers
cabs(expr)  [Function]
Calculates the absolute value of an expression representing a complex number. Unlike the function abs, the cabs function always decomposes its argument into a real and an imaginary part. If \( x \) and \( y \) represent real variables or expressions, the cabs function calculates the absolute value of \( x + \%i*y \) as

\[
\begin{align*}
\%i1) & \quad \text{cabs (1);} \quad 1 \\
\%o1) & \\
\%i2) & \quad \text{cabs (1 + \%i);} \quad \sqrt{2} \\
\%o2) & \\
\%i3) & \quad \text{cabs (exp (\%i));} \quad 1 \\
\%o3) & \\
\%i4) & \quad \text{cabs (exp (\%pi * \%i));} \quad 1 \\
\%o4) & \\
\%i5) & \quad \text{cabs (exp (3/2 * \%pi * \%i));} \quad 1 \\
\%o5) & \\
\%i6) & \quad \text{cabs (17 * exp (2 * \%i));} \quad 17 \\
\%o6) & 
\end{align*}
\]

If cabs returns a noun form this most commonly is caused by some properties of the variables involved not being known:

\[
\begin{align*}
\%i1) & \quad \text{cabs (a+\%i*b);} \\
\%o1) & \quad \sqrt{b^2 + a^2}
\end{align*}
\]
The `cabs` function can use known properties like symmetry properties of complex functions to help it calculate the absolute value of an expression. If such identities exist, they can be advertised to `cabs` using function properties. The symmetries that `cabs` understands are: mirror symmetry, conjugate function and complex characteristic.

`cabs` is a verb function and is not suitable for symbolic calculations. For such calculations (including integration, differentiation and taking limits of expressions containing absolute values), use `abs`.

The result of `cabs` can include the absolute value function, `abs`, and the arc tangent, `atan2`.

When applied to a list or matrix, `cabs` automatically distributes over the terms. Similarly, it distributes over both sides of an equation.

For further ways to compute with complex numbers, see the functions `rectform`, `realpart`, `imagpart`, `carg`, `conjugate` and `polarform`.

Examples:

Examples with `sqrt` and `sin`.

\[
\text{(i1) } \text{cabs(sqrt(1+i*x));} \\
\text{(o1)} \quad (x + 1)^{1/4} \\
\text{(i2) } \text{cabs(sin(x+i*y));} \\
\text{(o2)} \quad \sqrt{(\cos(x) \sinh(y) + \sin(x) \cosh(y))} \\
\]

The error function, `erf`, has mirror symmetry, which is used here in the calculation of the absolute value with a complex argument:

\[
\text{(i3) } \text{cabs(erf(x+i*y));} \\
\text{(o3)} \quad \sqrt{\frac{(\text{erf}(i y + x) - \text{erf}(i y - x))^2}{4}} \\
\]

Maxima knows complex identities for the Bessel functions, which allow it to compute the absolute value for complex arguments. Here is an example for `bessel_j`. 

\[
\text{(i2)} \quad \text{declare(a,real,b,real);} \\
\text{(o2)} \quad \text{done} \\
\text{(i3)} \quad \text{cabs (a+%i*b);} \\
\text{(o3)} \quad \sqrt{b^2 + a^2} \\
\text{(i4)} \quad \text{assume(a>0,b>0);} \\
\text{(o4)} \quad [a > 0, b > 0] \\
\text{(i5)} \quad \text{cabs (a+%i*b);} \\
\text{(o5)} \quad \sqrt{b^2 + a^2} \\
\]
\[
(%i4) \quad \text{cabs(bessel_j(1, \%i))};
\]
\[
(%o4) \quad \text{abs(bessel_j(1, \%i))}
\]

\section*{carg \((z)\)}

Returns the complex argument of \(z\). The complex argument is an angle \(\theta\) in \((-\pi, \pi]\) such that \(r \exp(\theta \%i) = z\) where \(r\) is the magnitude of \(z\).

\textit{carg} is a computational function, not a simplifying function.

See also \texttt{abs} (complex magnitude), \texttt{polarform}, \texttt{rectform}, \texttt{realpart}, and \texttt{imagpart}.

\textbf{Examples:}

\[
(%i1) \quad \text{carg (1)};
\]
\[
(%o1) \quad 0
\]
\[
(%i2) \quad \text{carg (1 + \%i)};
\]
\[
(%o2) \quad ---
\]
\[
(%i3) \quad \text{carg (exp (\%i))};
\]
\[
(%o3) \quad \text{atan(------)}
\]
\[
(%i4) \quad \text{carg (exp (\%pi * \%i))};
\]
\[
(%o4) \quad \%pi
\]
\[
(%i5) \quad \text{carg (exp (3/2 * \%pi * \%i))};
\]
\[
(%o5) \quad - ---
\]
\[
(%i6) \quad \text{carg (17 * exp (2 * \%i))};
\]
\[
(%o6) \quad \text{atan(------) + \%pi}
\]

If \texttt{carg} returns a noun form this most commonly is caused by some properties of the variables involved not being known:

\[
(%i1) \quad \text{carg (a+\%i*b)};
\]
\[
(%o1) \quad \text{atan2(b, a)}
\]
\[
(%i2) \quad \text{declare(a, real, b, real)};
\]
\[
(%o2) \quad \text{done}
\]
\[
(%i3) \quad \text{carg (a+\%i*b)};
\]
\[
(%o3) \quad \text{atan2(b, a)}
\]
\[
(%i4) \quad \text{assume(a>0, b>0)};
\]
\[
(%o4) \quad [a > 0, b > 0]
\]
\[
(%i5) \quad \text{carg (a+\%i*b)};
\]
\[
(%o5) \quad \text{atan(-)}
\]

\section*{conjugate \((x)\)}

Returns the complex conjugate of \(x\).
(%i1) declare ([aa, bb], real, cc, complex, ii, imaginary);
(%o1) done
(%i2) conjugate (aa + bb*%i);
(%o2) aa - %i bb
(%i3) conjugate (cc);
(%o3) conjugate(cc)
(%i4) conjugate (ii);
(%o4) - ii
(%i5) conjugate (xx + yy);
(%o5) yy + xx

imagpart (expr)
[Function]
Returns the imaginary part of the expression expr.
imagpart is a computational function, not a simplifying function.
See also abs, carg, polarform, rectform, and realpart.
Example:
(%i1) imagpart (a+b*%i);
(%o1) b
(%i2) imagpart (1+sqrt(2)*%i);
(%o2) sqrt(2)
(%i3) imagpart (1);
(%o3) 0
(%i4) imagpart (sqrt(2)*%i);
(%o4) sqrt(2)

polarform (expr)
[Function]
Returns an expression $r \, e^{\%i \, \theta}$ equivalent to expr, such that $r$ and $\theta$
are purely real.
Example:
(%i1) polarform(a+b*%i);
   2  2   %i atan2(b, a)
   sqrt(b + a) \, e
(%o1)                        
(%i2) polarform(1+%i);
   %i \, %pi
-----
   4
   sqrt(2) \, e
(%o2)                                                  
(%i3) polarform(1+2*%i);
   %i \, atan(2)
   sqrt(5) \, e
(%o3)                                                

realpart (expr)
[Function]
Returns the real part of expr. realpart and imagpart will work on expressions
involving trigonometric and hyperbolic functions, as well as square root, logarithm,
and exponentiation.
Example:

```lisp
(%i1) realpart (a+b*%i);
   \( \text{o1} \) a
(%i2) realpart (1+sqrt(2)*%i);
   \( \text{o2} \) 1
(%i3) realpart (sqrt(2)*%i);
   \( \text{o3} \) 0
(%i4) realpart (1);
   \( \text{o4} \) 1
```

rectform (expr)  
Returns an expression \( a + b \, \text{i} \) equivalent to expr, such that \( a \) and \( b \) are purely real.

Example:

```lisp
(%i1) rectform(sqrt(2)*%e^(%i*%pi/4));
   \( \text{e1} \) \text{i} + 1
(%i2) rectform(sqrt(b^2+a^2)*%e^(%i*atan2(b, a)));
   \( \text{e2} \) \text{i} \, b + a
(%i3) rectform(sqrt(5)*%e^(%i*atan(2)));
   \( \text{e3} \) 2 \text{i} + 1
```

10.3 Combinatorial Functions

!!  
The double factorial operator.

For an integer, float, or rational number \( n \), \( n!! \) evaluates to the product \( n \, (n-2) \, (n-4) \, \ldots \, (n - 2 \, (k-1)) \) where \( k \) is equal to \text{entier} (\( n/2 \)), that is, the largest integer less than or equal to \( n/2 \). Note that this definition does not coincide with other published definitions for arguments which are not integers.

For an even (or odd) integer \( n \), \( n!! \) evaluates to the product of all the consecutive even (or odd) integers from 2 (or 1) through \( n \) inclusive.

For an argument \( n \) which is not an integer, float, or rational, \( n!! \) yields a noun form \text{genfact} (\( n, n/2, 2 \)).

binomial (x, y)  
The binomial coefficient \( \frac{x!}{y! \, (x - y)!} \). If \( x \) and \( y \) are integers, then the numerical value of the binomial coefficient is computed. If \( y \), or \( x - y \), is an integer, the binomial coefficient is expressed as a polynomial.

Examples:

```lisp
(%i1) binomial (11, 7);
   \( \text{o1} \) 330
(%i2) 11! / 7! / (11 - 7) !;
   \( \text{o2} \) 330
(%i3) binomial (x, 7);
   \( \text{o3} \) \( x - 6 \) \( x - 5 \) \( x - 4 \) \( x - 3 \) \( x - 2 \) \( x - 1 \) \( x \)
```

\( \text{o3} \) \( \frac{5040}{\text{---}} \)
(%i4) binomial (x + 7, x);
(x + 1) (x + 2) (x + 3) (x + 4) (x + 5) (x + 6) (x + 7)
(%o4) -------------------------------------------------------
                5040
(%i5) binomial (11, y);
(%o5) binomial(11, y)

factcomb (expr) [Function]
Tries to combine the coefficients of factorials in expr with the factorials themselves
by converting, for example, (n + 1)*n! into (n + 1)!

sumsplitfact if set to false will cause minfactorial to be applied after a factcomb.
Example:

(%i1) sumsplitfact;
(%o1) true
(%i2) (n + 1)*(n + 1)*n!;
(%o2) (n + 1) n!
(%i3) factcomb (%);
(%o3) (n + 2)! - (n + 1)!
(%i4) sumsplitfact: not sumsplitfact;
(%o4) false
(%i5) (n + 1)*(n + 1)*n!;
(%o5) (n + 1) n!
(%i6) factcomb (%);
(%o6) n (n + 1)! + (n + 1)!

factorial [Function]

! [Operator]

Represents the factorial function. Maxima treats factorial (x) the same as x!.

For any complex number x, except for negative integers, x! is defined as gamma(x+1).

For an integer x, x! simplifies to the product of the integers from 1 to x inclusive.
0! simplifies to 1. For a real or complex number in float or bigfloat precision x, x!
simplifies to the value of gamma (x+1). For x equal to n/2 where n is an odd integer,
x! simplifies to a rational factor times sqrt (%pi) (since gamma (1/2) is equal to
sqrt (%pi)).

The option variables factlim and gammalim control the numerical evaluation of facto-
rials for integer and rational arguments. The functions minfactorial and factcomb
simplifies expressions containing factorials.

The functions gamma, bffac, and cbffac are varieties of the gamma function. bffac
and cbffac are called internally by gamma to evaluate the gamma function for real
and complex numbers in bigfloat precision.

makegamma substitutes gamma for factorials and related functions.

Maxima knows the derivative of the factorial function and the limits for specific values
like negative integers.
The option variable `factorial_expand` controls the simplification of expressions like 
$(n+x)!$, where $n$ is an integer.

See also `binomial`.

The factorial of an integer is simplified to an exact number unless the operand is greater than `factlim`. The factorial for real and complex numbers is evaluated in float or bigfloat precision.

```plaintext
(%i1) factlim : 10;
(%o1) 10
(%i2) [0!, (7/2)!, 8!, 20!];
   105 sqrt(%pi)
(%o2) [1, ------------, 40320, 20!]
    16
(%i3) [4,77!, (1.0+%i)!];
(%o3) [4, 77!, 0.3430658398165453 %i + 0.6529654964201667]
(%i4) [2.86b0!, (1.0b0+%i)!];
(%o4) [5.046635586910012b0, 3.430658398165454b-1 %i
    + 6.529654964201667b-1]
```

The factorial of a known constant, or general expression is not simplified. Even so it may be possible to simplify the factorial after evaluating the operand.

```plaintext
(%i1) [(%i + 1)!, %pi!, %e!, (cos(1) + sin(1))!];
(%o1) [(%i + 1)!, %pi!, %e!, (cos(1) + sin(1))!]
(%i2) ev (%, numer, %enumer);
(%o2) [0.3430658398165453 %i + 0.6529654964201667, 7.188082728976314, 4.260820476357003, 1.227580202486819]
```

Factorials are simplified, not evaluated. Thus $x!$ may be replaced even in a quoted expression.

```plaintext
(%i1) '([0!, (7/2)!, 4.77!, 8!, 20!]);
   105 sqrt(%pi)
(%o1) [1, ------------, 81.44668037931197, 40320, 2432902008176640000]
    16
```

Maxima knows the derivative of the factorial function.

```plaintext
(%i1) diff(x!,x);
(%o1) x! psi (x + 1)
```

The option variable `factorial_expand` controls expansion and simplification of expressions with the factorial function.

```plaintext
(%i1) (n+1)!/n!,factorial_expand:true;
(%o1) n + 1
```

`factlim` [Option variable]

Default value: 100000

`factlim` specifies the highest factorial which is automatically expanded. If it is -1 then all integers are expanded.
factorial_expand

Default value: false

The option variable factorial_expand controls the simplification of expressions like \((x+n)!\), where \(n\) is an integer. See factorial for an example.

genfact \((x, y, z)\)

Returns the generalized factorial, defined as \(x \cdot (x-z) \cdot (x-2 \cdot z) \ldots (x-(y-1) \cdot z)\). Thus, when \(x\) is an integer, \(\text{genfact} \ (x, x, 1) = x!\) and \(\text{genfact} \ (x, x/2, 2) = x!!\).

minfactorial \((\text{expr})\)

Examines \(\text{expr}\) for occurrences of two factorials which differ by an integer. minfactorial then turns one into a polynomial times the other.

\[
\begin{align*}
(\%i1) \quad & n!/(n+2)!; \\
(\%o1) \quad & \frac{n!}{(n+2)!} \\
(\%i2) \quad & \text{minfactorial} \ (\%); \\
(\%o2) \quad & \frac{1}{(n+1)(n+2)}
\end{align*}
\]

sumsplitfact

Default value: true

When sumsplitfact is false, minfactorial is applied after a factcomb.

\[
\begin{align*}
(\%i1) \quad & \text{sumsplitfact}; \\
(\%o1) \quad & \text{true} \\
(\%i2) \quad & n!/(n+2)!; \\
(\%o2) \quad & \frac{n!}{(n+2)!} \\
(\%i3) \quad & \text{factcomb} \ (\%); \\
(\%o3) \quad & \frac{n!}{(n+2)!} \\
(\%i4) \quad & \text{sumsplitfact: not sumsplitfact} ; \\
(\%o4) \quad & \text{false} \\
(\%i5) \quad & n!/(n+2)!; \\
(\%o5) \quad & \frac{n!}{(n+2)!} \\
(\%i6) \quad & \text{factcomb} \ (\%); \\
(\%o6) \quad & \frac{1}{(n+1)(n+2)}
\end{align*}
\]

%
10.4 Root, Exponential and Logarithmic Functions

%e_to_numlog [Option variable]
Default value: false
When true, \(r\) some rational number, and \(x\) some expression, \(\%e^{(r \times \log(x))}\) will be simplified into \(x^r\). It should be noted that the \texttt{radcan} command also does this transformation, and more complicated transformations of this ilk as well. The \texttt{logcontract} command "contracts" expressions containing \texttt{log}.

%emode [Option variable]
Default value: true
When \%emode is true, \(\%e^{(\%pi \times \%i \times x)}\) is simplified as follows.
\(\%e^{(\%pi \times \%i \times x)}\) simplifies to \(\cos(\%pi x) + \%i \sin(\%pi x)\) if \(x\) is a floating point number, an integer, or a multiple of \(1/2, 1/3, 1/4,\) or \(1/6\), and then further simplified.
For other numerical \(x\), \(\%e^{(\%pi \times \%i \times x)}\) simplifies to \(\%e^{(\%pi \times \%i \times y)}\) where \(y = x - 2k\) for some integer \(k\) such that \(\text{abs}(y) < 1\).
When \%emode is false, no special simplification of \(\%e^{(\%pi \times \%i \times x)}\) is carried out.

(%i1) %emode;
(%o1) true
(%i2) %e^(%pi*%i*1);
(%o2) - 1
(%i3) %e^(%pi*%i*216/144);
(%o3) - %i
(%i4) %e^(%pi*%i*192/144);
          sqrt(3) %i    1
     ( - -------) - -
            2    2
(%i5) %e^(%pi*%i*180/144);
     %i    1
     (- -------) - -------
          sqrt(2)    sqrt(2)
(%i6) %e^(%pi*%i*120/144);
     %i    sqrt(3)
     -- - -------
          2    2
(%i7) %e^(%pi*%i*121/144);
     121 %i %pi
     ----------
           144
(%o7) %e

%enumer [Option variable]
Default value: false
When \%enumer is true, \%e is replaced by its numeric value 2.718... whenever \texttt{numer} is true.
When \%enumer is false, this substitution is carried out only if the exponent in \%e^-x evaluates to a number.

See also ev and numer.

\begin{verbatim}
(%i1) %enumer;
(%o1) false
(%i2) numer;
(%o2) false
(%i3) 2*%e;
(%o3) 2 %e
(%i4) %enumer: not %enumer;
(%o4) true
(%i5) 2*%e;
(%o5) 2 %e
(%i6) numer: not numer;
(%o6) true
(%i7) 2*%e;
(%o7) 5.43656365691809
(%i8) 2*%e^-1;
(%o8) 5.43656365691809
(%i9) 2*%e^-x;
(%o9) 2 2.718281828459045
\end{verbatim}

\textbf{exp (x)} [Function]

Represents the exponential function. Instances of exp (x) in input are simplified to \%e^-x; exp does not appear in simplified expressions.

demoivre if true causes \%e^-(a + b \%i) to simplify to \%e^-(a (\cos(b) + \%i \sin(b))) if b is free of \%i. See demoivre.

\%emode, when true, causes \%e^-(\%pi \%i x) to be simplified. See \%emode.

\%enumer, when true causes \%e to be replaced by 2.718... whenever numer is true. See \%enumer.

\begin{verbatim}
(%i1) demoivre;
(%o1) false
(%i2) %e^-(a + b*%i);
(%o2) %e^-(a + b*%i)
(%i3) demoivre: not demoivre;
(%o3) true
(%i4) %e^-(a + b*%i);
(%o4) %e^-(a + b*%i)
\end{verbatim}

\textbf{li [s] (z)} [Function]

Represents the polylogarithm function of order s and argument z, defined by the infinite series
\[ \text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \]

\text{li} [1] is \(- \log (1 - z)\). \text{li} [2] and \text{li} [3] are the dilogarithm and trilogarithm functions, respectively.

When the order is 1, the polylogarithm simplifies to \(- \log (1 - z)\), which in turn simplifies to a numerical value if \(z\) is a real or complex floating point number or the \textbf{numer} evaluation flag is present.

When the order is 2 or 3, the polylogarithm simplifies to a numerical value if \(z\) is a real floating point number or the \textbf{numer} evaluation flag is present.

Examples:

```plaintext
(%i1) assume (x > 0);
(%o1) [x > 0]
(%i2) integrate ((log (1 - t)) / t, t, 0, x);
   - li (x)
   2
(%o2) 2
(%i3) li [2] (7);
(%o3) li (7)
2
(%i4) li [2] (7), numer;
1.248273182099423 - 6.113257028817991 %i
(%o4) 1.248273182099423 - 6.113257028817991 %i
(%i5) li [3] (7);
(%o5) li (7)
3
(%i6) li [3] (7), numer;
5.319257992145674 - 5.9479244808033 %i
(%i7) L : makelist (i / 4.0, i, 0, 8);
(%o7) [0.0, 0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0]
(%i8) map (lambda ([x], li [2] (x)), L);
(%o8) [0.0, 0.2676526390827326, 0.5822405264650125, 0.97846932930306, 1.644934066848226, 2.19017701141645 - 0.7010261415046585 %i, 2.7439527027248 - 1.7580848210787 %i, 2.48686765338205 - 1.7580848210787 %i, 2.67401100272339 - 2.177586090303601 %i]
(%i9) map (lambda ([x], li [3] (x)), L);
(%o9) [0.0, 0.2584613953442624, 0.537213192678042, 0.8444258046482203, 1.202069, 1.642866878950322, 1.07821473130035025 %i, 2.060877505514697, 2.582419849982037 %i, 2.433418896388322, 2.491926018322965 %i, 2.762071904015935, 2.7546938285978846 %i]

\textbf{log} (x)

\textbf{[Function]}

Represents the natural (base \(e\)) logarithm of \(x\).
Maxima does not have a built-in function for the base 10 logarithm or other bases. \( \log_{10}(x) := \frac{\log(x)}{\log(10)} \) is a useful definition.

Simplification and evaluation of logarithms is governed by several global flags:

- **logexpand**
  - causes \( \log(a^b) \) to become \( b \log(a) \). If it is set to **all**, \( \log(a^b) \) will also simplify to \( \log(a)+\log(b) \). If it is set to **super**, then \( \log(a/b) \) will also simplify to \( \log(a)-\log(b) \) for rational numbers \( a/b, a \neq 1 \). (\( \log(1/b) \), for \( b \) integer, always simplifies.) If it is set to **false**, all of these simplifications will be turned off.

- **logsimp**
  - if **false** then no simplification of \( \%e \) to a power containing \( \log \)'s is done.

- **lognegint**
  - if **true** implements the rule \( \log(-n) \rightarrow \log(n)+\%i*\%pi \) for \( n \) a positive integer.

- **\%e_to_numlog**
  - when **true**, \( r \) some rational number, and \( x \) some expression, the expression \( \%e^{r*log(x)} \) will be simplified into \( x^r \). It should be noted that the **radcan** command also does this transformation, and more complicated transformations of this as well. The **logcontract** command "contracts" expressions containing \( \log \).

- **logabs**
  - Default value: **false**
  - When doing indefinite integration where logs are generated, e.g. \texttt{integrate(1/x,x)}, the answer is given in terms of \( \log(\text{abs}(\ldots)) \) if **logabs** is **true**, but in terms of \( \log(\ldots) \) if **logabs** is **false**. For definite integration, the **logabs:true** setting is used, because here "evaluation" of the indefinite integral at the endpoints is often needed.

- **logarc** (\texttt{expr})
  - The function **logarc(expr)** carries out the replacement of inverse circular and hyperbolic functions with equivalent logarithmic functions for an expression \( \text{expr} \) without setting the global variable **logarc**.

- **logarc**
  - When the global variable **logarc** is **true**, inverse circular and hyperbolic functions are replaced by equivalent logarithmic functions. The default value of **logarc** is **false**.

- **logconcoeffp**
  - Default value: **false**
  - Controls which coefficients are contracted when using **logcontract**. It may be set to the name of a predicate function of one argument. E.g. if you like to generate SQRTs, you can do \texttt{logconcoeffp:'logconfun$ logconfun(m):=featurep(m,integer) or ratnump(m)$} . Then \texttt{logcontract(1/2*log(x))} will give \( \log(\sqrt{x}) \).
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logcontract (expr) [Function]
Recursively scans the expression expr, transforming subexpressions of the form
\[ a_1 \log(b_1) + a_2 \log(b_2) + c \] into
\[ \log(\text{ratsimp}(b_1^{a_1} \cdot b_2^{a_2})) + c \]

(%i1) 2*(a*log(x) + 2*a*log(y))$
(%i2) logcontract(%);

\[ 2 \quad 4 \quad a \quad \log(x \quad y ) \]
The declaration \texttt{declare(n,integer)} causes \texttt{logcontract(2*a*n*log(x))} to simplify to \texttt{a*log(x^(2*n))}. The coefficients that "contract" in this manner are those such as the 2 and the \texttt{n} here which satisfy \texttt{featurep(coeff,integer)}. The user can control which coefficients are contracted by setting the option \texttt{logconcoeffp} to the name of a predicate function of one argument. E.g. if you like to generate SQRTs, you can do \texttt{logconcoeffp:'logconfun$ logconfun(m):=featurep(m,integer) or ratnump(m)$}. Then \texttt{logcontract(1/2*log(x))} will give \texttt{log(sqrt(x))}.

logexpand [Option variable]
Default value: \texttt{true}
If \texttt{true}, that is the default value, causes \texttt{log(a^b)} to become \texttt{b*log(a)}. If it is set to \texttt{all}, \texttt{log(a*b)} will also simplify to \texttt{log(a)+log(b)}. If it is set to \texttt{super}, then \texttt{log(a/b)} will also simplify to \texttt{log(a)-log(b)} for rational numbers \texttt{a/b}, \texttt{a#1}. (\texttt{log(1/b)}, for integer \texttt{b}, always simplifies.) If it is set to \texttt{false}, all of these simplifications will be turned off.
When \texttt{logexpand} is set to \texttt{all} or \texttt{super}, the logarithm of a product expression simplifies to a summation of logarithms.
Examples:
When \texttt{logexpand} is \texttt{true}, \texttt{log(a^b)} simplifies to \texttt{b*log(a)}.

(%i1) log(n^2), logexpand=true;
(%o1) 2 \log(n)
When \texttt{logexpand} is \texttt{all}, \texttt{log(a*b)} simplifies to \texttt{log(a)+log(b)}.

(%i1) log(10*x), logexpand=all;
(%o1) \log(x) + \log(10)
When \texttt{logexpand} is \texttt{super}, \texttt{log(a/b)} simplifies to \texttt{log(a)-log(b)} for rational numbers \texttt{a/b} with \texttt{a#1}.

(%i1) log(a/(n + 1)), logexpand=super;
(%o1) \log(a) - \log(n + 1)
When \texttt{logexpand} is set to \texttt{all} or \texttt{super}, the logarithm of a product expression simplifies to a summation of logarithms.

(%i1) my_product : product (X(i), i, 1, n);

(%o1) \prod_{i=1}^{n} X(i)
When `logexpand` is `false`, these simplifications are disabled.

```
(%i1) logexpand : false $
(%i2) log(n^2);
   2
(%o2) log(n )
(%i3) log(10*x);
   log(10 x)
(%i4) log(a/(n + 1));
      a
(%o4) log(------)
       n + 1
(%i5) log ('product (X(i), i, 1, n));
      n
(%o5) log( ! ! X(i))
      ! !
       i = 1
```

`lognegint`  
Default value: `false`  
If `true` implements the rule \( \log(-n) \rightarrow \log(n) + i\pi \) for \( n \) a positive integer.

`logsimp`  
Default value: `true`  
If `false` then no simplification of \( e \) to a power containing \( \log \)'s is done.

`plog(x)`  
Represents the principal branch of the complex-valued natural logarithm with \(-\pi < \text{carg}(x) \leq +\pi\).
sqrt (x)

The square root of \( x \). It is represented internally by \( x^{1/2} \). See also \texttt{rootscontract} and \texttt{radexpand}.

[Function]

\texttt{sqrt} (x)
10.5 Trigonometric Functions

10.5.1 Introduction to Trigonometric

Maxima has many trigonometric functions defined. Not all trigonometric identities are programmed, but it is possible for the user to add many of them using the pattern matching capabilities of the system. The trigonometric functions defined in Maxima are: \( \text{acos}, \ \text{acosh}, \ \text{acot}, \ \text{acoth}, \ \text{acsc}, \ \text{acsch}, \ \text{asec}, \ \text{asech}, \ \text{asin}, \ \text{asinh}, \ \text{atan}, \ \text{atanh}, \ \text{cos}, \ \text{cosh}, \ \text{cot}, \ \text{coth}, \ \text{csc}, \ \text{csch}, \ \text{sec}, \ \text{sech}, \ \text{sin}, \ \text{sinh}, \ \text{tan}, \ \text{and tanh}. \) There are a number of commands especially for handling trigonometric functions, see \texttt{trigexpand}, \texttt{trigreduce}, and the switch \texttt{trigsign}. Two share packages extend the simplification rules built into Maxima, \texttt{ntrig} and \texttt{atrig1}. Do \texttt{describe(command)} for details.

10.5.2 Functions and Variables for Trigonometric

10.5.2.1 Trigonometric and Hyperbolic Functions

\begin{verbatim}
acos (x)  \hspace{1cm} \text{[Function]}  \\
    \hspace{2cm} \text{Arc Cosine.}  \\

acosh (x) \hspace{1cm} \text{[Function]}  \\
    \hspace{2cm} \text{Hyperbolic Arc Cosine.}  \\

acot (x) \hspace{1cm} \text{[Function]}  \\
    \hspace{2cm} \text{Arc Cotangent.}  \\

acoth (x) \hspace{1cm} \text{[Function]}  \\
    \hspace{2cm} \text{Hyperbolic Arc Cotangent.}  \\

acsc (x) \hspace{1cm} \text{[Function]}  \\
    \hspace{2cm} \text{Arc Cosecant.}  \\

acsch (x) \hspace{1cm} \text{[Function]}  \\
    \hspace{2cm} \text{Hyperbolic Arc Cosecant.}  \\

asec (x) \hspace{1cm} \text{[Function]}  \\
    \hspace{2cm} \text{Arc Secant.}  \\

asech (x) \hspace{1cm} \text{[Function]}  \\
    \hspace{2cm} \text{Hyperbolic Arc Secant.}  \\

asin (x) \hspace{1cm} \text{[Function]}  \\
    \hspace{2cm} \text{Arc Sine.}  \\

asinh (x) \hspace{1cm} \text{[Function]}  \\
    \hspace{2cm} \text{Hyperbolic Arc Sine.}  \\

atan (x) \hspace{1cm} \text{[Function]}  \\
    \hspace{2cm} \text{Arc Tangent.}  \\
See also \texttt{atan2}.  \\
\end{verbatim}
\textbf{atan2} \((y, x)\) \hspace{1cm} \text{[Function]}
\>
\text{yields the value of } \tan^{-1}(y/x) \text{ in the interval } -\pi \text{ to } \pi \text{ taking into consideration the quadrant of the point } (x, y).

See also \texttt{atan}.

\textbf{atanh} \((x)\) \hspace{1cm} \text{[Function]}
\>
\text{Hyperbolic Arc Tangent.}

\textbf{cos} \((x)\) \hspace{1cm} \text{[Function]}
\>
\text{Cosine.}

\textbf{cosh} \((x)\) \hspace{1cm} \text{[Function]}
\>
\text{Hyperbolic Cosine.}

\textbf{cot} \((x)\) \hspace{1cm} \text{[Function]}
\>
\text{Cotangent.}

\textbf{coth} \((x)\) \hspace{1cm} \text{[Function]}
\>
\text{Hyperbolic Cotangent.}

\textbf{csc} \((x)\) \hspace{1cm} \text{[Function]}
\>
\text{Cosecant.}

\textbf{csch} \((x)\) \hspace{1cm} \text{[Function]}
\>
\text{Hyperbolic Cosecant.}

\textbf{sec} \((x)\) \hspace{1cm} \text{[Function]}
\>
\text{Secant.}

\textbf{sech} \((x)\) \hspace{1cm} \text{[Function]}
\>
\text{Hyperbolic Secant.}

\textbf{sin} \((x)\) \hspace{1cm} \text{[Function]}
\>
\text{Sine.}

\textbf{sinh} \((x)\) \hspace{1cm} \text{[Function]}
\>
\text{Hyperbolic Sine.}

\textbf{tan} \((x)\) \hspace{1cm} \text{[Function]}
\>
\text{Tangent.}

\textbf{tanh} \((x)\) \hspace{1cm} \text{[Function]}
\>
\text{Hyperbolic Tangent.}

\subsection*{10.5.2.2 Options Controlling Simplification}

\texttt{%piargs} \hspace{1cm} \text{[Option variable]}
\>
\text{Default value: true}

\text{When } \texttt{%piargs} \text{ is true, trigonometric functions are simplified to algebraic constants when the argument is an integer multiple of } \pi, \ \pi/2, \ \pi/4, \text{ or } \pi/6.
Maxima knows some identities which can be applied when $\pi$, etc., are multiplied by an integer variable (that is, a symbol declared to be integer).

Examples:

```lisp
(%i1) %piargs : false$
(%i2) [sin (%pi), sin (%pi/2), sin (%pi/3)];
   %pi       %pi
(%o2) [sin(%pi), sin(---), sin(---)]
   2       3
(%i3) [sin (%pi/4), sin (%pi/5), sin (%pi/6)];
   %pi       %pi
(%o3) [sin(---), sin(---), sin(---)]
   4       5       6
(%i4) %piargs : true$
(%i5) [sin (%pi), sin (%pi/2), sin (%pi/3)];
   sqrt(3)
(%o5) [0, 1, -------]
      2
(%i6) [sin (%pi/4), sin (%pi/5), sin (%pi/6)];
   1 %pi 1
(%o6) [----------, sin(---), -]
   sqrt(2) 5 2
(%i7) [cos (%pi/3), cos (10*%pi/3), tan (10*%pi/3),
      cos (sqrt(2)*%pi/3)];
  1 1 sqrt(2) %pi
(%o7) [-, -, sqrt(2), cos(-----------)]
  2 2 3
```

Some identities are applied when $\pi$ and $\pi/2$ are multiplied by an integer variable.

```lisp
(%i1) declare (n, integer, m, even)$
(%i2) [sin (%pi * n), cos (%pi * m), sin (%pi/2 * m),
      cos (%pi/2 * m)];
        m/2
(%o2) [0, 1, 0, (- 1) ]
```

### %iargs

[Option variable]

Default value: true

When %iargs is true, trigonometric functions are simplified to hyperbolic functions when the argument is apparently a multiple of the imaginary unit $i$.

Even when the argument is demonstrably real, the simplification is applied; Maxima considers only whether the argument is a literal multiple of $i$.

Examples:

```lisp
(%i1) %iargs : false$
(%i2) [sin (%i * x), cos (%i * x), tan (%i * x)];
(%o2) [sin(%i x), cos(%i x), tan(%i x)]
(%i3) %iargs : true$
(%i4) [sin (%i * x), cos (%i * x), tan (%i * x)];
(%o4) [%i sinh(x), cosh(x), %i tanh(x)]
```
Even when the argument is demonstrably real, the simplification is applied.

(%i1) declare (x, imaginary)$
(%i2) [featurep (x, imaginary), featurep (x, real)];
(%o2) [true, false]
(%i3) sin (%i * x);
(%o3) %i sinh(x)

[Option variable]
halfangles
Default value: false
When halfangles is true, trigonometric functions of arguments expr/2 are simplified to functions of expr.
For a real argument \( x \) in the interval \( 0 \leq x < 2\pi \), \( \sin \frac{x}{2} \) simplifies to a simple formula:
\[
\frac{\sqrt{1 - \cos x}}{\sqrt{2}}
\]
A complicated factor is needed to make this formula correct for all complex arguments \( z = x + iy \):
\[
(-1)^{\lfloor x/(2\pi) \rfloor} \left[ 1 - \text{unit\_step}(y) \left( 1 + (-1)^{\lfloor x/(2\pi) \rfloor - \lceil x/(2\pi) \rceil} \right) \right]
\]
Maxima knows this factor and similar factors for the functions \( \sin, \cos, \sinh, \) and \( \cosh \). For special values of the argument \( z \) these factors simplify accordingly.
Examples:

(%i1) halfangles : false$
(%i2) sin (x / 2);
(%o2) sin(-)

(%i3) halfangles : true$
(%i4) sin (x / 2);
(%o4) \frac{\sqrt{1 - \cos x}}{\sqrt{2}}

(%i5) assume(x>0, x<2*%pi)$
(%i6) sin(x / 2);
(%o6) \frac{\sqrt{1 - \cos(x)}}{\sqrt{2}}

[Option variable]
trigsign
Default value: true
When trigsign is true, it permits simplification of negative arguments to trigonometric functions. E.g., \( \sin(-x) \) will become \( -\sin x \) only if trigsign is true.
10.5.2.3 Identities

**trigexpand (expr)**

Expands trigonometric and hyperbolic functions of sums of angles and of multiple angles occurring in `expr`. For best results, `expr` should be expanded. To enhance user control of simplification, this function expands only one level at a time, expanding sums of angles or multiple angles. To obtain full expansion into sines and cosines immediately, set the switch `trigexpand: true`.

`trigexpand` is governed by the following global flags:

- **trigexpand**
  - If `true` causes expansion of all expressions containing sin's and cos's occurring subsequently.

- **halfangles**
  - If `true` causes half-angles to be simplified away.

- **trigexpandplus**
  - Controls the "sum" rule for trigexpand, expansion of sums (e.g. `sin(x + y)`) will take place only if `trigexpandplus` is `true`.

- **trigexpandtimes**
  - Controls the "product" rule for trigexpand, expansion of products (e.g. `sin(2 x)`) will take place only if `trigexpandtimes` is `true`.

Examples:

```lisp
(%i1) x+sin(3*x)/sin(x),trigexpand=true,expand;
    2  2
(%o1) (- sin(x)) + 3 cos(x) + x
(%i2) trigexpand(sin(10*x+y));
(%o2) cos(10 x) sin(y) + sin(10 x) cos(y)
```

- **trigexpandplus**
  - Default value: `true`

- **trigexpandtimes**
  - Default value: `true`

- **triginverses**
  - Default value: `true`

`triginverses` controls the simplification of the composition of trigonometric and hyperbolic functions with their inverse functions.

If **all**, both e.g. `atan(tan(x))` and `tan(atan(x))` simplify to `x`. 
If `true`, the `arccos(f)(x)` simplification is turned off.

If `false`, both the `arccos(f)(x)` and `f(arccos(x))` simplifications are turned off.

`trigreduce`  
`trigreduce(expr, x)`  
`trigreduce(expr)`  
Combines products and powers of trigonometric and hyperbolic sin's and cos’s of `x` into those of multiples of `x`. It also tries to eliminate these functions when they occur in denominators. If `x` is omitted then all variables in `expr` are used.

See also `poissimp`.

```
(%i1) trigreduce(-sin(x)^2+3*cos(x)^2+x);
  cos(2 x) cos(2 x) 1 1
  -------- + 3 (- -------- + -) + x - -
  2 2 2 2
```

`trigsimp(expr)`  
Employs the identities \( \sin(x)^2 + \cos(x)^2 = 1 \) and \( \cosh(x)^2 - \sinh(x)^2 = 1 \) to simplify expressions containing \( \tan, \sec, \) etc., to \( \sin, \cos, \sinh, \cosh \).

`trigreduce`, `ratsimp`, and `radcan` may be able to further simplify the result.

demo ("trgsmp.dem") displays some examples of `trigsimp`.

`trigrat(expr)`  
Gives a canonical simplified quasilinear form of a trigonometrical expression; `expr` is a rational fraction of several `sin, cos` or `tan`, the arguments of them are linear forms in some variables (or kernels) and \( \%pi/n \) (\( n \) integer) with integer coefficients. The result is a simplified fraction with numerator and denominator linear in `sin` and `cos`. Thus `trigrat` linearize always when it is possible.

```
(%i1) trigrat(sin(3*a)/sin(a+%pi/3));
  %pi
  sin(a) sin(3 ((- b) - a + --))
  3
  --------------------------
  sin(b + a)
  (%o1) sqrt(3) sin(2 a) + cos(2 a) - 1
```

The following example is taken from Davenport, Siret, and Tournier, *Calcul Formel*, Masson (or in English, Addison-Wesley), section 1.5.5, Morley theorem.

```
(%i1) c: %pi/3 - a - b$
(%i2) bc: sin(a)*sin(3*c)/sin(a+b);
   %pi
   sin(a) sin(3 ((- b) - a + --))
   3
   --------------------------
   sin(b + a)

(%i3) ba : bc, c=a, a=c;
   %pi
   sin(3 a) sin(b + a - --)
   3
   --------------------------
   (%o3) %pi
   sin(a - --)
   3
```
\[
\begin{align*}
\text{(%i4) } \text{ac2} & : \text{ba}^2 + \text{bc}^2 - 2\text{bc}\text{ba}\cos(b) \quad \frac{\sin(3\ a) \sin(b + a - \frac{\pi}{3})}{2} \\
\text{(%o4) } & \frac{\sin(a - \frac{\pi}{3})}{2} \\
& - \frac{(2 \sin(a) \sin(3\ a) \sin(3((-b) - a + \frac{\pi}{3})) \cos(b)}{3} \\
& \left(\frac{\sin(b + a - \frac{\pi}{3})}{\sin(a - \frac{\pi}{3}) \sin(b + a)}\right) \\
& \frac{\sin(a) \sin(3((-b) - a + \frac{\pi}{3}))}{2} \\
\text{(%i5) } \text{trigrat (ac2);} \\
\text{(%o5) } & - (\sqrt{3} \sin(4\ b + 4\ a) - \cos(4\ b + 4\ a) \\
& - 2 \sqrt{3} \sin(4\ b + 2\ a) + 2 \cos(4\ b + 2\ a) \\
& - 2 \sqrt{3} \sin(2\ b + 4\ a) + 2 \cos(2\ b + 4\ a) \\
& + 4 \sqrt{3} \sin(2\ b + 2\ a) - 8 \cos(2\ b + 2\ a) - 4 \cos(2\ b - 2\ a) \\
& + \sqrt{3} \sin(4\ b) - \cos(4\ b) - 2 \sqrt{3} \sin(2\ b) + 10 \cos(2\ b) \\
& + \sqrt{3} \sin(4\ a) - \cos(4\ a) - 2 \sqrt{3} \sin(2\ a) + 10 \cos(2\ a) \\
& - 9)/4
\end{align*}
\]

10.5.2.4 Additional Functions

**atrig1**

The **atrig1** package contains several additional simplification rules for inverse trigonometric functions. Together with rules already known to Maxima, the following angles are fully implemented: 0, π/6, π/4, π/3, and π/2.

Corresponding angles in the other three quadrants are also available. Do `load("atrig1")` to use them.

**ntrig**

The **ntrig** package contains a set of simplification rules that are used to simplify trigonometric function whose arguments are of the form \(f(n \ %\pi/10)\) where \(f\) is any of the functions \(\sin, \cos, \tan, \csc, \sec\) and \(\cot\).
10.6 Random Numbers

**make_random_state**

- **make_random_state** \((n)\)
- **make_random_state** \((s)\)
- **make_random_state** \((true)\)
- **make_random_state** \((false)\)

A random state object represents the state of the random number generator. The state comprises 627 32-bit words.

**make_random_state** \((n)\) returns a new random state object created from an integer seed value equal to \(n\) modulo \(2^{32}\). \(n\) may be negative.

**make_random_state** \((s)\) returns a copy of the random state \(s\).

**make_random_state** \((true)\) returns a new random state object, using the current computer clock time as the seed.

**make_random_state** \((false)\) returns a copy of the current state of the random number generator.

**set_random_state** \((s)\)

Copies \(s\) to the random number generator state.

**set_random_state** always returns **done**.

**random** \((x)\)

Returns a pseudorandom number. If \(x\) is an integer, **random** \((x)\) returns an integer from 0 through \(x - 1\) inclusive. If \(x\) is a floating point number, **random** \((x)\) returns a nonnegative floating point number less than \(x\). **random** complains with an error if \(x\) is neither an integer nor a float, or if \(x\) is not positive.

The functions **make_random_state** and **set_random_state** maintain the state of the random number generator.

The Maxima random number generator is an implementation of the Mersenne twister MT 19937.

Examples:

```maxima
(%i1) s1: make_random_state (654321)$
(%i2) set_random_state (s1);  
   done
(%i3) random (1000);  
   768
(%i4) random (9573684);  
   7657880
(%i5) random (2^75);  
11804491615036831636390
(%i6) s2: make_random_state (false)$
(%i7) random (1.0);  
0.2310127244107132
(%i8) random (10.0);  
4.394553645870825
```
(%i9) random (100.0);  
(%o9) 32.28666704056853
(%i10) set_random_state (s2);  
(%o10) done
(%i11) random (1.0);  
(%o11) 0.2310127244107132
(%i12) random (10.0);  
(%o12) 4.394553645870825
(%i13) random (100.0);  
(%o13) 32.28666704056853
11 Maxima’s Database

11.1 Introduction to Maxima’s Database

11.2 Functions and Variables for Properties

**alphabetic**

alphabetic is a property type recognized by `declare`. The expression `declare(s, alphabetic)` tells Maxima to recognize as alphabetic all of the characters in `s`, which must be a string.

See also Section 6.3 [Identifiers], page 90.

Example:

```
(%i1) xx\-yy\'\@ : 1729;
(%o1) 1729
(%i2) declare ("-\@", alphabetic);
(%o2) done
(%i3) xx^-yy'@ + @yy'xx + 'xx@yy-;
(%o3) 'xx@yy- + @yy'xx + 1729
(%i4) listofvars (%);
(%o4) [@yy'xx, 'xx@yy-]
```

**bindtest**

The command `declare(x, bindtest)` tells Maxima to trigger an error when the symbol `x` is evaluated unbound.

```
(%i1) aa + bb;
(%o1) bb + aa
(%i2) declare (aa, bindtest);
(%o2) done
(%i3) aa + bb;
aa unbound variable
-- an error. Quitting. To debug this try debugmode(true);
(%i4) aa : 1234;
(%o4) 1234
(%i5) aa + bb;
(%o5) bb + 1234
```

**constant**

`declare(a, constant)` declares `a` to be a constant. The declaration of a symbol to be constant does not prevent the assignment of a nonconstant value to the symbol.

See `constantp` and `declare`.

Example:

```
(%i1) declare(c, constant);
(%o1) done
(%i2) constantp(c);
(%o2) true
```
constantp (expr)

Returns true if expr is a constant expression, otherwise returns false.

An expression is considered a constant expression if its arguments are numbers (including rational numbers, as displayed with /R/), symbolic constants such as %pi, %e, and %i, variables bound to a constant or declared constant by declare, or functions whose arguments are constant.

constantp evaluates its arguments.

See the property constant which declares a symbol to be constant.

Examples:

(%i1) constantp (7 * sin(2));
(%o1) true
(%i2) constantp (rat (17/29));
(%o2) true
(%i3) constantp (%pi * sin(%e));
(%o3) true
(%i4) constantp (exp (x));
(%o4) false
(%i5) declare (x, constant);
(%o5) done
(%i6) constantp (exp (x));
(%o6) true
(%i7) constantp (foo (x) + bar (%e) + baz (2));
(%o7) false
(%i8) declare (a_1, p_1, a_2, p_2, ...)

declare (a_1, p_1, a_2, p_2, . . .)

Assigns the atom or list of atoms a_i the property or list of properties p_i. When a_i and/or p_i are lists, each of the atoms gets all of the properties.

declare quotes its arguments. declare always returns done.

As noted in the description for each declaration flag, for some flags featurep(object, feature) returns true if object has been declared to have feature.

For more information about the features system, see features. To remove a property from an atom, use remove.

declare recognizes the following properties:

additive  Tells Maxima to simplify a_i expressions by the substitution a_i(x + y + z + ...) --> a_i(x) + a_i(y) + a_i(z) + .... The substitution is carried out on the first argument only.

alphabetic  Tells Maxima to recognize all characters in a_i (which must be a string) as alphabetic characters.
antisymmetric, commutative, symmetric  
Tells Maxima to recognize \( a_i \) as a symmetric or antisymmetric function.  
\texttt{commutative} is the same as \texttt{symmetric}.

bindtest  
Tells Maxima to trigger an error when \( a_i \) is evaluated unbound.

constant  
Tells Maxima to consider \( a_i \) a symbolic constant.

even, odd  
Tells Maxima to recognize \( a_i \) as an even or odd integer variable.

evenfun, oddfun  
Tells Maxima to recognize \( a_i \) as an odd or even function.

evflag  
Makes \( a_i \) known to the \texttt{ev} function so that \( a_i \) is bound to \texttt{true} during the execution of \texttt{ev} when \( a_i \) appears as a flag argument of \texttt{ev}. See \texttt{evflag}.

evfun  
Makes \( a_i \) known to \texttt{ev} so that the function named by \( a_i \) is applied when \( a_i \) appears as a flag argument of \texttt{ev}. See \texttt{evfun}.

feature  
Tells Maxima to recognize \( a_i \) as the name of a feature. Other atoms may then be declared to have the \( a_i \) property.

increasing, decreasing  
Tells Maxima to recognize \( a_i \) as an increasing or decreasing function.

integer, noninteger  
Tells Maxima to recognize \( a_i \) as an integer or noninteger variable.

integervalued  
Tells Maxima to recognize \( a_i \) as an integer-valued function.

lassociative, rassociative  
Tells Maxima to recognize \( a_i \) as a right-associative or left-associative function.

linear  
Equivalent to declaring \( a_i \) both \texttt{outative} and \texttt{additive}.

mainvar  
Tells Maxima to consider \( a_i \) a "main variable". A main variable succeeds all other constants and variables in the canonical ordering of Maxima expressions, as determined by \texttt{ordergreatp}.

multiplicative  
Tells Maxima to simplify \( a_i \) expressions by the substitution \( a_i(x \ast y \ast z \ast \ldots) \rightarrow a_i(x) \ast a_i(y) \ast a_i(z) \ast \ldots \). The substitution is carried out on the first argument only.

nary  
Tells Maxima to recognize \( a_i \) as an n-ary function.

The \texttt{nary} declaration is not the same as calling the \texttt{nary} function. The sole effect of \texttt{declare(foo, nary)} is to instruct the Maxima simplifier to flatten nested expressions, for example, to simplify \texttt{foo(x, foo(y, z))} to \texttt{foo(x, y, z)}.

nonarray  
Tells Maxima to consider \( a_i \) not an array. This declaration prevents multiple evaluation of a subscripted variable name.
nonscalar
Tells Maxima to consider \(a_i\) a nonscalar variable. The usual application is to declare a variable as a symbolic vector or matrix.

noun
Tells Maxima to parse \(a_i\) as a noun. The effect of this is to replace instances of \(a_i\) with \(\text{'a}_i\) or \(\text{nounify(a}_i)\), depending on the context.

outative
Tells Maxima to simplify \(a_i\) expressions by pulling constant factors out of the first argument.
When \(a_i\) has one argument, a factor is considered constant if it is a literal or declared constant.
When \(a_i\) has two or more arguments, a factor is considered constant if the second argument is a symbol and the factor is free of the second argument.

posfun
Tells Maxima to recognize \(a_i\) as a positive function.

rational, irrational
Tells Maxima to recognize \(a_i\) as a rational or irrational real variable.

real, imaginary, complex
Tells Maxima to recognize \(a_i\) as a real, pure imaginary, or complex variable.

scalar
Tells Maxima to consider \(a_i\) a scalar variable.

Examples of the usage of the properties are available in the documentation for each separate description of a property.

decreasing
[increasing]

The commands \(\text{declare(f, decreasing)}\) or \(\text{declare(f, increasing)}\) tell Maxima to recognize the function \(f\) as an decreasing or increasing function.

See also \text{declare} for more properties.

Example:

\begin{verbatim}
(%i1) assume(a > b);
(%o1) [a > b]
(%i2) is(f(a) > f(b));
(%o2) unknown
(%i3) declare(f, increasing);
(%o3) done
(%i4) is(f(a) > f(b));
(%o4) true
\end{verbatim}

even
[odd]

\(\text{declare(a, even)}\) or \(\text{declare(a, odd)}\) tells Maxima to recognize the symbol \(a\) as an even or odd integer variable. The properties \text{even} and \text{odd} are not recognized by the functions \text{evenp}, \text{oddp}, and \text{integerp}.

See also \text{declare} and \text{askinteger}.
Example:

```
(%i1) declare(n, even);
(%o1) done
(%i2) askinteger(n, even);
(%o2) yes
(%i3) askinteger(n);
(%o3) yes
(%i4) evenp(n);
(%o4) false
```

**feature**

Maxima understands two distinct types of features, system features and features which apply to mathematical expressions. See also **status** for information about system features. See also **features** and **featurep** for information about mathematical features.

**feature** itself is not the name of a function or variable.

**featurep** (a, f)

Attempts to determine whether the object a has the feature f on the basis of the facts in the current database. If so, it returns **true**, else **false**.

Note that **featurep** returns **false** when neither f nor the negation of f can be established.

**featurep** evaluates its argument.

See also **declare** and **features**.

```
(%i1) declare (j, even)$
(%i2) featurep (j, integer);
(%o2) true
```

**features**

Maxima recognizes certain mathematical properties of functions and variables. These are called "features".

**declare** (x, foo) gives the property foo to the function or variable x.

**declare** (foo, feature) declares a new feature foo. For example, **declare** ([red, green, blue], feature) declares three new features, red, green, and blue.

The predicate **featurep** (x, foo) returns **true** if x has the foo property, and **false** otherwise.

The infolist **features** is a list of known features. These are

- integer
- noninteger
- even
- odd
- rational
- irrational
- real
- imaginary
- complex
- analytic
- increasing
- decreasing
- oddfun
- evenfun
- posfun
- constant
- commutative
- lassociative
- rassociative
- symmetric
- antisymmetric
- integervalued
plus any user-defined features.

**features** is a list of mathematical features. There is also a list of non-mathematical, system-dependent features. See **status**.

Example:

```lisp
(%i1) declare (FOO, feature);
   done
(%i2) declare (x, FOO);
   done
(%i3) featurep (x, FOO);
   true
```

**get (a, i)**

Retrieves the user property indicated by i associated with atom a or returns **false** if a doesn’t have property i.

g get evaluates its arguments.

See also **put** and **qput**.

```lisp
(%i1) put (%e, 'transcendental, 'type);
   transcendental
(%i2) put (%pi, 'transcendental, 'type)
(%i3) put (%i, 'algebraic, 'type)
(%i4) typeof (expr) := block ([q],
   if numberp (expr)
   then return ('algebraic),
   if not atom (expr)
   then return (maplist ('typeof, expr)),
   q: get (expr, 'type),
   if q=false
   then errcatch (error(expr,"is not numeric.")) else q)
(%i5) typeof (2*%e + x*%pi);
   x is not numeric.
   [[transcendental, []], [algebraic, transcendental]]
(%i6) typeof (2*%e + %pi);
   [transcendental, [algebraic, transcendental]]
```

**integer**

 declared (a, integer) or declared (a, noninteger) tells Maxima to recognize a as an integer or noninteger variable.

See also **declare**.

Example:

```lisp
(%i1) declare(n, integer, x, noninteger);
   done
(%i2) askinteger(n);
   yes
(%i3) askinteger(x);
   no
```
**integervalued**  
Declare(f, integervalued) tells Maxima to recognize f as an integer-valued function.

See also declare.

Example:

```
(%i1) exp(%i)^f(x);
   %i f(x)
(%o1) (%e     )
(%%2) declare(f, integervalued);
   done
(%%3) exp(%i)^f(x);
   %i f(x)
(%o3) %e
```

**nonarray**  
The command declare(a, nonarray) tells Maxima to consider a not an array. This declaration prevents multiple evaluation, if a is a subscripted variable.

See also declare.

Example:

```
(%i4) a:'b$ b:'c$ c:'d$
(%%5) declare(a, nonarray);
   done
(%%6) a[x];
   a
```

**nonscalar**  
Makes atoms behave as does a list or matrix with respect to the dot operator.

See also declare.

**nonscalarp (expr)**  
Returns true if expr is a non-scalar, i.e., it contains atoms declared as non-scalars, lists, or matrices.

See also the predicate function scalarp and declare.

**posfun**  
Declare (f, posfun) declares f to be a positive function. is (f(x) > 0) yields true.

See also declare.
printprops  

printprops (a, i)  
printprops ([a_1, ..., a_n], i)  
printprops (all, i)

Displays the property with the indicator i associated with the atom a. a may also be a list of atoms or the atom all in which case all of the atoms with the given property will be used. For example, printprops ([f, g], atvalue). printprops is for properties that cannot otherwise be displayed, i.e., for atvalue, atomgrad, gradef, and matchdeclare.

properties (a)  

Returns a list of the names of all the properties associated with the atom a.

props  

Default value: []

props are atoms which have any property other than those explicitly mentioned in infolists, such as specified by atvalue, matchdeclare, etc., as well as properties specified in the declare function.

propvars (prop)  

Returns a list of those atoms on the props list which have the property indicated by prop. Thus propvars (atvalue) returns a list of atoms which have atvalues.

put (atom, value, indicator)  

Assigns value to the property (specified by indicator) of atom. indicator may be the name of any property, not just a system-defined property.

rem reverses the effect of put.

put evaluates its arguments. put returns value.

See also qput and get.

Examples:

(\%i1) put (foo, (a+b)^5, expr);
5
(\%o1) (b + a)
(\%i2) put (foo, "Hello", str);
(\%o2) Hello
(\%i3) properties (foo);
(\%o3) [[user properties, str, expr]]
(\%i4) get (foo, expr);
5
(\%o4) (b + a)
(\%i5) get (foo, str);
(\%o5) Hello

qput (atom, value, indicator)  

Assigns value to the property (specified by indicator) of atom. This is the same as put, except that the arguments are quoted.

See also get.
Example:

(%i1) foo: aa$
(%i2) bar: bb$
(%i3) baz: cc$
(%i4) put (foo, bar, baz);
   bb
(%o4)
(%i5) properties (aa);
   [[user properties, cc]]
(%o5)
(%i6) get (aa, cc);
   bb
(%o6)
(%i7) qput (foo, bar, baz);
   bar
(%o7)
(%i8) properties (foo);
   [value, [user properties, baz]]
(%o8)
(%i9) get ('foo, 'baz);
   bar
(%o9)

\textbf{rational} \quad \textbf{[Property]}
\textbf{irrational} \quad \textbf{[Property]}

\texttt{declare(a, rational)} or \texttt{declare(a, irrational)} tells Maxima to recognize \texttt{a} as a rational or irrational real variable.

See also \texttt{declare}.

\textbf{real} \quad \textbf{[Property]}
\textbf{imaginary} \quad \textbf{[Property]}
\textbf{complex} \quad \textbf{[Property]}

\texttt{declare(a, real)}, \texttt{declare(a, imaginary)}, or \texttt{declare(a, complex)} tells Maxima to recognize \texttt{a} as a real, pure imaginary, or complex variable.

See also \texttt{declare}.

\textbf{rem} \texttt{(atom, indicator)} \quad \textbf{[Function]}

Removes the property indicated by \texttt{indicator} from \texttt{atom}. \texttt{rem} reverses the effect of \texttt{put}.

\texttt{rem} returns \texttt{done} if \texttt{atom} had an \texttt{indicator} property when \texttt{rem} was called, or \texttt{false} if it had no such property.

\textbf{remove} \quad \textbf{[Function]}

\texttt{remove (a_{1}, p_{1}, \ldots, a_{n}, p_{n})}
\texttt{remove ([a_{1}, \ldots, a_{m}], [p_{1}, \ldots, p_{n}], \ldots)}
\texttt{remove ("a", operator)}
\texttt{remove (a, \texttt{transfun})}
\texttt{remove (all, p)}

Removes properties associated with atoms.
\texttt{remove (a_{1}, p_{1}, \ldots, a_{n}, p_{n})} removes property \texttt{p_{k}} from atom \texttt{a_{k}}.
\texttt{remove ([a_{1}, \ldots, a_{m}], [p_{1}, \ldots, p_{n}], \ldots)} removes properties \texttt{p_{1}, \ldots, p_{n}} from atoms \texttt{a_{1}, \ldots, a_{m}}. There may be more than one pair of lists.
\texttt{remove (all, p)} removes the property \texttt{p} from all atoms which have it.
The removed properties may be system-defined properties such as function, macro, or mode_declare. remove does not remove properties defined by put.

A property may be transfun to remove the translated Lisp version of a function. After executing this, the Maxima version of the function is executed rather than the translated version.

\texttt{remove ("a", operator)} or, equivalently, \texttt{remove ("a", op)} removes from a the operator properties declared by prefix, infix, \texttt{[function_nary]}, page 134, postfix, matchfix, or nolix. Note that the name of the operator must be written as a quoted string.

\texttt{remove} always returns done whether or not an atom has a specified property. This behavior is unlike the more specific remove functions \texttt{remvalue}, \texttt{remarray}, \texttt{remfunction}, and \texttt{remrule}.

\texttt{remove} quotes its arguments.

\textbf{scalar} \hfill \text{[Property]}

\texttt{declare(a, scalar)} tells Maxima to consider a a scalar variable.

See also \texttt{declare}.

\textbf{scalarp (expr)} \hfill \text{[Function]}

Returns \texttt{true} if \texttt{expr} is a number, constant, or variable declared \texttt{scalar} with \texttt{declare}, or composed entirely of numbers, constants, and such variables, but not containing matrices or lists.

See also the predicate function \texttt{nonscalarp}.

\section*{11.3 Functions and Variables for Facts}

\textbf{activate (context_1, \ldots, context_n)} \hfill \text{[Function]}

Activates the contexts \texttt{context_1}, \ldots, \texttt{context_n}. The facts in these contexts are then available to make deductions and retrieve information. The facts in these contexts are not listed by \texttt{facts ()}.

The variable \texttt{activecontexts} is the list of contexts which are active by way of the \texttt{activate} function.

\textbf{activecontexts} \hfill \text{[System variable]}

Default value: \texttt{[]} \hfill \text{[System variable]}

\texttt{activecontexts} is a list of the contexts which are active by way of the \texttt{activate} function, as opposed to being active because they are subcontexts of the current context.

\textbf{askequal} \hfill \text{[Function]}

\texttt{askequal (expr1, expr2)} \hfill \text{[Function]}

\texttt{askequal(expr1, expr2)} attempts to determine from the \texttt{assume} database whether \texttt{expr1} is equal to \texttt{expr2}, and prompts the user if it cannot tell.

If the user provides the answer, the answer is stored in the \texttt{assume} database for the duration of the evaluation of the expression currently in progress. When the evaluation is completed, the user-provided answer is removed from the database.
askequal returns yes or no, whether the answer was determined from the assume database or provided by the user.
See also equal.

askinteger

- askinteger(expr)
- askinteger(expr, integer)
- askinteger(expr, even)
- askinteger(expr, odd)

askinteger(expr) attempts to determine from the assume database whether expr is an integer. askinteger prompts the user if it cannot tell otherwise, and attempt to install the information in the database if possible. askinteger(expr) is equivalent to askinteger(expr, integer).

askinteger(expr, even) and askinteger(expr, odd) likewise attempt to determine if expr is an even integer or odd integer, respectively.

asksign(expr)

First attempts to determine whether the specified expression is positive, negative, or zero. If it cannot, it asks the user the necessary questions to complete its deduction. The user’s answer is recorded in the database for the duration of the current computation. The return value of asksign is one of pos, neg, or zero.

assume(pred_1, ..., pred_n)

Adds predicates pred_1, ..., pred_n to the current context. If a predicate is inconsistent or redundant with the predicates in the current context, it is not added to the context. The context accumulates predicates from each call to assume.

assume returns a list whose elements are the predicates added to the context or the atoms redundant or inconsistent where applicable.

The predicates pred_1, ..., pred_n can only be expressions with the relational operators < <= equal notequal >=. Predicates cannot be literal equality = or literal inequality # expressions, nor can they be predicate functions such as integerp.

Compound predicates of the form pred_1 and ... and pred_n are recognized, but not pred_1 or ... or pred_n. not pred_k is recognized if pred_k is a relational predicate. Expressions of the form not (pred_1 and pred_2) and not (pred_1 or pred_2) are not recognized.

Maxima’s deduction mechanism is not very strong; there are many obvious consequences which cannot be determined by is. This is a known weakness.

assume does not handle predicates with complex numbers. If a predicate contains a complex number assume returns inconsistent or redundant.

assume evaluates its arguments.
See also is, facts, forget, context, and declare.

Examples:

(%%i1) assume (xx > 0, yy < -1, zz >= 0);
(%%o1)                    [xx > 0, yy < - 1, zz >= 0]
(%%i2) assume (aa < bb and bb < cc);
(%o2) [bb > aa, cc > bb]
(%i3) facts ();
(%o3) [xx > 0, -1 > yy, zz >= 0, bb > aa, cc > bb]
(%i4) is (xx > yy);
(%o4) true
(%i5) is (yy < -yy);
(%o5) true
(%i6) is (sinh (bb - aa) > 0);
(%o6) true
(%i7) forget (bb > aa);
(%o7) [bb > aa]
(%i8) prederror : false;
(%o8) false
(%i9) is (sinh (bb - aa) > 0);
(%o9) unknown
(%i10) is (bb^2 < cc^2);
(%o10) unknown

assumescalar
[Option variable]
Default value: true
assumescalar helps govern whether expressions expr for which nonscalarp (expr)
is false are assumed to behave like scalars for certain transformations.
Let expr represent any expression other than a list or a matrix, and let [1, 2, 3]
represent any list or matrix. Then expr . [1, 2, 3] yields [expr, 2 expr, 3 expr]
if assumescalar is true, or scalarp (expr) is true, or constantp (expr) is true.
If assumescalar is true, such expressions will behave like scalars only for commuta-
tive operators, but not for noncommutative multiplication ..
When assumescalar is false, such expressions will behave like non-scalars.
When assumescalar is all, such expressions will behave like scalars for all the op-
erators listed above.

assume_pos
[Option variable]
Default value: false
When assume_pos is true and the sign of a parameter x cannot be determined from
the current context or other considerations, sign and asksign (x) return true. This
may forestall some automatically-generated asksign queries, such as may arise from
integrate or other computations.
By default, a parameter is x such that symbolp (x) or subvarp (x). The class of
expressions considered parameters can be modified to some extent via the variable
assume_pos_pred.

sign and asksign attempt to deduce the sign of expressions from the sign of operands
within the expression. For example, if a and b are both positive, then a + b is also
positive.
However, there is no way to bypass all asksign queries. In particular, when the
asksign argument is a difference x - y or a logarithm log(x), asksign always re-
quests an input from the user, even when \texttt{assume\_pos} is \texttt{true} and \texttt{assume\_pos\_pred} is a function which returns \texttt{true} for all arguments.

\texttt{assume\_pos\_pred} \\
\hspace*{1em} [Option variable] \\
\hspace*{1em} Default value: \texttt{false} \\
\hspace*{1em} When \texttt{assume\_pos\_pred} is assigned the name of a function or a lambda expression of one argument \texttt{x}, that function is called to determine whether \texttt{x} is considered a parameter for the purpose of \texttt{assume\_pos}. \texttt{assume\_pos\_pred} is ignored when \texttt{assume\_pos} is \texttt{false}.

The \texttt{assume\_pos\_pred} function is called by \texttt{sign} and \texttt{asksign} with an argument \texttt{x} which is either an atom, a subscripted variable, or a function call expression. If the \texttt{assume\_pos\_pred} function returns \texttt{true}, \texttt{x} is considered a parameter for the purpose of \texttt{assume\_pos}.

By default, a parameter is \texttt{x} such that \texttt{symbolp (x)} or \texttt{subvarp (x)}.

See also \texttt{assume} and \texttt{assume\_pos}.

Examples:

\begin{verbatim}
(%i1) assume_pos: true$
(%i2) assume_pos_pred: symbolp$
(%i3) sign (a); 
   pos
(%i4) sign (a[1]); 
   pnz
(%i5) assume_pos_pred: lambda ([x], display (x), true)$
(%i6) asksign (a); 
   x = a
   pos
(%i7) asksign (a[1]); 
   x = a
   1
   pos
(%i8) asksign (foo (a)); 
   x = foo(a)
   pos
(%i9) asksign (foo (a) + bar (b)); 
   x = foo(a)
   x = bar(b)
   pos
(%i10) asksign (log (a)); 
   x = a
   pos
Is a - 1 positive, negative, or zero?
\end{verbatim}
\begin{verbatim}
(%i10)
(a - b)

(%o10)
pos

Is b - a positive, negative, or zero?

(%o11)
neg
\end{verbatim}

\textbf{context} \hspace{1cm} [Option variable]

Default value: \textit{initial}

\textit{context} names the collection of facts maintained by \texttt{assume} and \texttt{forget}. \texttt{assume} adds facts to the collection named by \texttt{context}, while \texttt{forget} removes facts.

Binding \texttt{context} to a name \textit{foo} changes the current context to \textit{foo}. If the specified context \textit{foo} does not yet exist, it is created automatically by a call to \texttt{newcontext}.

The specified context is activated automatically.

See \texttt{contexts} for a general description of the context mechanism.

\textbf{contexts} \hspace{1cm} [Option variable]

Default value: \texttt{[initial, global]}

\texttt{contexts} is a list of the contexts which currently exist, including the currently active context.

The context mechanism makes it possible for a user to bind together and name a collection of facts, called a context. Once this is done, the user can have Maxima assume or forget large numbers of facts merely by activating or deactivating their context.

Any symbolic atom can be a context, and the facts contained in that context will be retained in storage until destroyed one by one by calling \texttt{forget} or destroyed as a whole by calling \texttt{kill} to destroy the context to which they belong.

Contexts exist in a hierarchy, with the root always being the context \texttt{global}, which contains information about Maxima that some functions need. When in a given context, all the facts in that context are "active" (meaning that they are used in deductions and retrievals) as are all the facts in any context which is a subcontext of the active context.

When a fresh Maxima is started up, the user is in a context called \texttt{initial}, which has \texttt{global} as a subcontext.

See also \texttt{facts}, \texttt{newcontext}, \texttt{supcontext}, \texttt{killcontext}, \texttt{activate}, \texttt{deactivate}, \texttt{assume}, and \texttt{forget}.
deactivate (context_1, ..., context_n)  [Function]
Deactivates the specified contexts context_1, ..., context_n.

facts  [Function]
  facts (item)
  facts ()
  If item is the name of a context, facts (item) returns a list of the facts in the
  specified context.
  If item is not the name of a context, facts (item) returns a list of the facts known
  about item in the current context. Facts that are active, but in a different context,
  are not listed.
  facts () (i.e., without an argument) lists the current context.

forget  [Function]
  forget (pred_1, ..., pred_n)
  forget (L)
  Removes predicates established by assume. The predicates may be expressions equivalent to (but not necessarily identical to) those previously assumed.
  forget (L), where L is a list of predicates, forgets each item on the list.

is (expr)  [Function]
Attempts to determine whether the predicate expr is provable from the facts in the
assume database.

If the predicate is provably true or false, is returns true or false, respectively.
Otherwise, the return value is governed by the global flag prederror. When
prederror is true, is complains with an error message. Otherwise, is returns
unknown.

ev(expr, pred) (which can be written expr, pred at the interactive prompt) is
equivalent to is(expr).

See also assume, facts, and maybe.

Examples:
is causes evaluation of predicates.
  (%i1) %pi > %e;
  (%o1) %pi > %e
  (%i2) is (%pi > %e);
  (%o2) true
is attempts to derive predicates from the assume database.
  (%i1) assume (a > b);
  (%o1) [a > b]
  (%i2) assume (b > c);
  (%o2) [b > c]
  (%i3) is (a < b);
  (%o3) false
  (%i4) is (a > c);
  (%o4) true
(%i5) is (equal (a, c));
(%o5) false

If is can neither prove nor disprove a predicate from the assume database, the global flag prederror governs the behavior of is.

(%i1) assume (a > b);
(%o1) [a > b]
(%i2) prederror: true$
(%i3) is (a > 0);
Maxima was unable to evaluate the predicate:
a > 0
-- an error. Quitting. To debug this try debugmode(true);
(%i4) prederror: false$
(%i5) is (a > 0);
(%o5) unknown

killcontext (context_1, ..., context_n) [Function]
Kills the contexts context_1, ..., context_n.

If one of the contexts is the current context, the new current context will become the first available subcontext of the current context which has not been killed. If the first available unkill context is global then initial is used instead. If the initial context is killed, a new, empty initial context is created.

killcontext refuses to kill a context which is currently active, either because it is a subcontext of the current context, or by use of the function activate.

killcontext evaluates its arguments. killcontext returns done.

maybe (expr) [Function]
Attempts to determine whether the predicate expr is provable from the facts in the assume database.

If the predicate is provably true or false, maybe returns true or false, respectively. Otherwise, maybe returns unknown.

maybe is functionally equivalent to is with prederror: false, but the result is computed without actually assigning a value to prederror.

See also assume, facts, and is.

Examples:

(%i1) maybe (x > 0);
(%o1) unknown
(%i2) assume (x > 1);
(%o2) [x > 1]
(%i3) maybe (x > 0);
(%o3) true

newcontext

newcontext (name)
newcontext ()

Creates a new, empty context, called name, which has global as its only subcontext. The newly-created context becomes the currently active context.
If `name` is not specified, a new name is created (via `gensym`) and returned.

`newcontext` evaluates its argument. `newcontext` returns `name` (if specified) or the new context name.

```
sign (expr)  \[Function\]
Attempts to determine the sign of `expr` on the basis of the facts in the current database. It returns one of the following answers: `pos` (positive), `neg` (negative), `zero`, `pz` (positive or zero), `nz` (negative or zero), `pn` (positive or negative), or `pnz` (positive, negative, or zero, i.e. nothing known).
See also `signum`.
```

```
supcontext  \[Function\]
```

```
supcontext  \[Function\]
```

11.4 Functions and Variables for Predicates

```
charfun (p)  \[Function\]
Return 0 when the predicate `p` evaluates to `false`; return 1 when the predicate evaluates to `true`. When the predicate evaluates to something other than `true` or `false` (unknown), return a noun form.
Examples:
```
(\%i1) charfun (x < 1);
(\%o1) charfun(x < 1)
(\%i2) subst (x = -1, \%);
(\%o2) 1
(\%i3) e : charfun ('"and" (-1 < x, x < 1))$
(\%i4) [subst (x = -1, e), subst (x = 0, e), subst (x = 1, e)];
(\%o4) [0, 1, 0]
```

```
compare (x, y)  \[Function\]
Return a comparison operator `op` (<, <=, >, >=, =, or #) such that `is (x op y)` evaluates to `true`; when either `x` or `y` depends on `\%i` and `x # y`, return `notcomparable`; when there is no such operator or Maxima isn’t able to determine the operator, return `unknown`.
Examples:
```
(\%i1) compare (1, 2);
(\%o1) <
(\%i2) compare (1, x);
```

The function `compare` doesn’t try to determine whether the real domains of its arguments are nonempty; thus

```lisp
(%i1) compare (acos (x^2 + 1), acos (x^2 + 1) + 1);
(%o1) <
```

The real domain of `acos (x^2 + 1)` is empty.

### equal (a, b)

[Function]

Represents equivalence, that is, equal value.

By itself, `equal` does not evaluate or simplify. The function `is` attempts to evaluate `equal` to a Boolean value. `is(equal(a, b))` returns `true` (or `false`) if and only if `a` and `b` are equal (or not equal) for all possible values of their variables, as determined by evaluating `ratsimp(a - b)`; if `ratsimp` returns 0, the two expressions are considered equivalent. Two expressions may be equivalent even if they are not syntactically equal (i.e., identical).

When `is` fails to reduce `equal` to `true` or `false`, the result is governed by the global flag `prederror`. When `prederror` is `true`, `is` complains with an error message. Otherwise, `is` returns `unknown`.

In addition to `is`, some other operators evaluate `equal` and `notequal` to `true` or `false`, namely `if`, `and`, `or`, and `not`.

The negation of `equal` is `notequal`.

Examples:

By itself, `equal` does not evaluate or simplify.

```lisp
(%i1) equal (x^2 - 1, (x + 1) * (x - 1));
(%o1) equal(x - 1, (x - 1) (x + 1))
(%i2) equal (x, x + 1);
(%o2) equal(x, x + 1)
(%i3) equal (x, y);
(%o3) equal(x, y)
```

The function `is` attempts to evaluate `equal` to a Boolean value. `is(equal(a, b))` returns `true` when `ratsimp(a - b)` returns 0. Two expressions may be equivalent even if they are not syntactically equal (i.e., identical).

```lisp
(%i1) ratsimp (x^2 - 1 - (x + 1) * (x - 1));
(%o1) 0
(%i2) is (equal (x^2 - 1, (x + 1) * (x - 1)));
```
true
is (x^2 - 1 = (x + 1) * (x - 1));
false
ratsimp (x - (x + 1));
- 1
is (equal (x, x + 1));
false
is (x = x + 1);
false
ratsimp (x - y);
x - y
is (equal (x, y));
unknown
is (x = y);
false

When is fails to reduce equal to true or false, the result is governed by the global flag prederror.

[aa : x^2 + 2*x + 1, bb : x^2 - 2*x - 1];
\[ \begin{align*}
  &x^2 + 2x + 1, \\
  &x^2 - 2x - 1
\end{align*} \]
ratsimp (aa - bb);
4 x + 2
prederror : true;
true
is (equal (aa, bb));
Maxima was unable to evaluate the predicate:
\[ \begin{align*}
  &x^2 + 2x + 1, \\
  &x^2 - 2x - 1
\end{align*} \]
-- an error. Quitting. To debug this try debugmode(true);
prederror : false;
false
is (equal (aa, bb));
unknown

Some operators evaluate equal and notequal to true or false.

if equal (y, y - 1) then FOO else BAR;
BAR
eq_1 : equal (x, x + 1);
equal(x, x + 1)
eq_2 : equal (y^2 + 2*y + 1, (y + 1)^2);
2 y + 1, (y + 1)^2)
\[ \begin{align*}
  &\text{false, true, true}
\end{align*} \]

Because not expr causes evaluation of expr, not equal(a, b) is equivalent to is(notequal(a, b)).

[notequal (2*z, 2*z - 1), not equal (2*z, 2*z - 1)];
\(\text{(\%o1) \ [notequal(2 \, z, \, 2 \, z \, - \, 1), \, \text{true}]\)}
\(\text{(\%i2) \ \text{is (notequal (2*z, 2*z - 1));}}\)
\(\text{(\%o2) \ \text{true}}\)

\text{\textbf{notequal (a, b)}}

Represents the negation of equal\((a, b)\).

Examples:

\(\text{(\%i1) \ \text{equal (a, b);}}\)
\(\text{(\%o1) \ \text{equal(a, b)}}\)
\(\text{(\%i2) \ \text{maybe (equal (a, b));}}\)
\(\text{(\%o2) \ \text{unknown}}\)
\(\text{(\%i3) \ \text{notequal (a, b);}}\)
\(\text{(\%o3) \ \text{notequal(a, b)}}\)
\(\text{(\%i4) \ \text{not equal (a, b);}}\)
\(\text{(\%o4) \ \text{notequal(a, b)}}\)
\(\text{(\%i5) \ \text{maybe (notequal (a, b));}}\)
\(\text{(\%o5) \ \text{unknown}}\)
\(\text{(\%i6) \ \text{assume (a \ > \ b);}}\)
\(\text{(\%o6) \ \text{[a \ > \ b]}}\)
\(\text{(\%i7) \ \text{equal (a, b);}}\)
\(\text{(\%o7) \ \text{equal(a, b)}}\)
\(\text{(\%i8) \ \text{maybe (equal (a, b));}}\)
\(\text{(\%o8) \ \text{false}}\)
\(\text{(\%i9) \ \text{notequal (a, b);}}\)
\(\text{(\%o9) \ \text{notequal(a, b)}}\)
\(\text{(\%i10) \ \text{maybe (notequal (a, b));}}\)
\(\text{(\%o10) \ \text{true}}\)

\text{\textbf{unknown (expr)}}

Returns true if and only if expr contains an operator or function not recognized by the Maxima simplifier.

\text{\textbf{zeroequiv (expr, v)}}

Tests whether the expression expr in the variable v is equivalent to zero, returning true, false, or dontknow.

zeroequiv has these restrictions:

1. Do not use functions that Maxima does not know how to differentiate and evaluate.
2. If the expression has poles on the real line, there may be errors in the result (but this is unlikely to occur).
3. If the expression contains functions which are not solutions to first order differential equations (e.g. Bessel functions) there may be incorrect results.
4. The algorithm uses evaluation at randomly chosen points for carefully selected subexpressions. This is always a somewhat hazardous business, although the algorithm tries to minimize the potential for error.
For example `zeroequiv (\sin(2 \cdot x) - 2 \cdot \sin(x) \cdot \cos(x), x)` returns `true` and `zeroequiv (\%e^x + x, x)` returns `false`. On the other hand `zeroequiv (\log(a \cdot b) - \log(a) - \log(b), a)` returns `dontknow` because of the presence of an extra parameter `b`. 
12 Plotting

12.1 Introduction to Plotting

To make the plots, Maxima can use an external plotting package or its own graphical interface Xmaxima (see the section on Section 12.2 Plotting Formats). The plotting functions calculate a set of points and pass them to the plotting package together with a set of commands specific to that graphic program. In some cases those commands and data are saved in a file and the graphic program is executed giving it the name of that file to be parsed.

When a file is created, it will begin with the name `maxout_xxx.format`, where `xxx` is a number that is unique to every concurrently-running instance of Maxima and `format` is the name of the plotting format being used (gnuplot, xmaxima, mgnuplot or geomview).

There are commands to save the plot in a graphic format file, rather than showing it in the screen. The default name for that graphic file is `maxplot.extension`, where `extension` is the extension normally used for the kind of graphic file selected, but that name can also be specified by the user.

The `maxout_xxx.format` and `maxplot.extension` files are created in the directory specified by the system variable `maxima_tempdir`. That location can be changed by assigning to that variable (or to the environment variable `MAXIMA_TEMPDIR`) a string that represents a valid directory where Maxima can create new files. The output of the Maxima plotting command will be a list with the names of the file(s) created, including their complete path, or empty if no files are created. Those files should be deleted after the maxima session ends.

If the format used is either gnuplot or xmaxima, and the `maxout_xxx.gnuplot` or `maxout_xxx.xmaxima` was saved, gnuplot or xmaxima can be run, giving it the name of that file as argument, in order to view again a plot previously created in Maxima. Thus, when a Maxima plotting command fails, the format can be set to gnuplot or xmaxima and the plain-text file `maxout_xxx.gnuplot` (or `maxout_xxx.xmaxima`) can be inspected to look for the source of the problem.

The additional package `draw`, page 817, provides functions similar to the ones described in this section with some extra features, but it only works with gnuplot. Note that some plotting options have the same name in both plotting packages, but their syntax and behavior is different. To view the documentation for a graphic option `opt`, type `?? opt` in order to choose the information for either of those two packages.

12.2 Plotting Formats

Maxima can use either Gnuplot, Xmaxima or Geomview as graphics program. Gnuplot and Geomview are external programs which must be installed separately, while Xmaxima is distributed with Maxima. To see which plotting format you are currently using, use the command `get_plot_option(plot_format);` and to change to another format, you can use `set_plot_option([plot_format, <format>])`, where `<format>` is the name of one of the formats described below. Those two commands show and change the global plot format, while each individual plotting command can use its own format, if it includes an option `[plot_format, <format>]` (see `get_plot_option` and `set_plot_option`).
The plotting formats are the following:

- **gnuplot**
  
  Used to launch the external program gnuplot, which must be installed in your system. All plotting commands and data are saved into the file `maxout_xxx.gnuplot`.

- **gnuplot_pipes** (default value)
  
  It is similar to the `gnuplot` format except that the commands and plot data are sent directly to `gnuplot` without creating any files. A single gnuplot process is kept open, with a single graphic window, and subsequent plot commands will be sent to the same process, replacing previous plots in that same window. Even if the graphic window is closed, the `gnuplot` process is still running until the end of the session or until it is killed with `gnuplot_close`. The function `gnuplot_replot` can be used to modify a plot that has already been displayed on the screen or to open again the graphic window after it was closed.

  This format does not work with some versions of Lisp under Windows and it is only used to plot to the screen; whenever graphic files are to be created, the format is silently switched to `gnuplot` and the commands needed to create the graphic file are saved with the data in file `maxout_xxx.gnuplot`.

- **mgnuplot**
  
  Mgnuplot is a Tk-based wrapper around gnuplot. It is an old interface still included in the Maxima distribution, but it is currently disabled because it does not have most of the features introduced by the newer versions of the plotting commands. Mgnuplot requires an external gnuplot installation and, in Unix systems, the Tcl/Tk system.

- **xmaxima**
  
  Xmaxima is a Tcl/Tk graphical interface for Maxima that can also be used to display plots created when Maxima is run from the console or from other graphical interfaces. To use this format, the xmaxima program, which is distributed together with Maxima, must be installed; in some Linux distributions Xmaxima is distributed in a package separate from other parts of Maxima. If Maxima is being run from the Xmaxima console, the data and commands are passed to xmaxima through the same socket used for the communication between Maxima and the Xmaxima console. When used from a terminal or from graphical interfaces different from Xmaxima, the commands and data are saved in the file `maxout_xxx.xmaxima` and xmaxima is run with the name of that file as argument.

- **geomview**
  
  Geomview, a Motif based interactive 3D viewing program for Unix. It can only be used to display plots created with `plot3d`. To use this format, the geomview program must be installed.

### 12.3 Functions and Variables for Plotting

- **geomview_command** [System variable]
  
  This variable stores the name of the command used to run the geomview program when the plot format is `geomview`. Its default value is "geomview". If the geomview program is not found unless you give its complete path or if you want to try a different version of it, you may change the value of this variable. For instance,
(%%i1) geomview_command: "/usr/local/bin/my_geomview"$

get_plot_option (keyword, index) [Function]
Returns the current default value of the option named keyword, which is a list. The optional argument index must be a positive integer which can be used to extract only one element from the list (element 1 is the name of the option).
See also set_plot_option, remove_plot_option and the section on Section 12.4 Plotting Options.

gnuplot_command [System variable]
This variable stores the name of the command used to run the gnuplot program when the plot format is gnuplot or gnuplot_pipes. Its default value is "gnuplot". If the gnuplot program is not found unless you give its complete path or if you want to try a different version of it, you may change the value of this variable. For instance,
(%%i1) gnuplot_command: "/usr/local/bin/my_gnuplot"$

gnuplot_file_args [System variable]
When a graphic file is going to be created using gnuplot, this variable is used to specify the format used to print the file name given to gnuplot. Its default value is "~a" in SBCL and Openmcl, and "~s" in other lisp versions, which means that the name of the file will be passed without quotes if SBCL or Openmcl are used and within quotes if other Lisp versions are used. The contents of this variable can be changed in order to add options for the gnuplot program, adding those options before the format directive "~s".

gnuplot_view_args [System variable]
This variable is the format used to parse the argument that will be passed to the gnuplot program when the plot format is gnuplot. Its default value is "-persist ~a" when SBCL or Openmcl are used, and "-persist ~s" with other Lisp variants, where "~a" or "~s" will be replaced with the name of the file where the gnuplot commands have been written (usually "maxout.xxx.gnuplot"). The option -persist tells gnuplot to exit after the commands in the file have been executed, without closing the window that displays the plot.
Those familiar with gnuplot, might want to change the value of this variable. For example, by changing it to:
(%%i1) gnuplot_view_args: "~s -"$
gnuplot will not be closed after the commands in the file have been executed; thus, the window with the plot will remain, as well as the gnuplot interactive shell where other commands can be issued in order to modify the plot.
In Windows versions of GnuPlot older than 4.6.3 the behavior of "~s -" and "-persist ~s" were the opposite; namely, "-persist ~s" made the plot window and the gnuplot interactive shell remain, while "~s -" closed the gnuplot shell keeping the plot window. Therefore, when older gnuplot versions are used in Windows, it might be necessary to adjust the value of gnuplot_view_args.

julia (x, y, ...options...)[Function]
Creates a graphic representation of the Julia set for the complex number (x + i y). The two mandatory parameters x and y must be real. This program is part of the
additional package `dynamics`, but that package does not have to be loaded; the first
time `julia` is used, it will be loaded automatically.

Each pixel in the grid is given a color corresponding to the number of iterations it
takes the sequence that starts at that point to move out of the convergence circle of
radius 2 centered at the origin. The number of pixels in the grid is controlled by the
`grid` plot option (default 30 by 30). The maximum number of iterations is set with
the option `iterations`. The program sets its own default palette: magenta, violet,
blue, cyan, green, yellow, orange, red, brown and black, but it can be changed by
adding an explicit `palette` option in the command.

The default domain used goes from -2 to 2 in both axes and can be changed with
the `x` and `y` options. By default, the two axes are shown with the same scale, unless
the option `yx_ratio` is used or the option `same_xy` is disabled. Other general plot
options are also accepted.

The following example shows a region of the Julia set for the number -0.55 + i0.6.
The option `color_bar_tics` is used to prevent Gnuplot from adjusting the color box
up to 40, in which case the points corresponding the maximum 36 iterations would
not be black.

```maxima
(%i1) julia (-0.55, 0.6, [iterations, 36], [x, -0.3, 0.2],
[y, 0.3, 0.9], [grid, 400, 400], [color_bar_tics, 0, 6, 36])$
```

```
```

```
```

The program sets its own default palette: magenta, violet, blue, cyan, green, yellow, orange, red, brown and black, but it can be changed by
adding an explicit `palette` option in the command.

The default domain used goes from -2 to 2 in both axes and can be changed with
the `x` and `y` options. By default, the two axes are shown with the same scale, unless
the option `yx_ratio` is used or the option `same_xy` is disabled. Other general plot
options are also accepted.

The following example shows a region of the Julia set for the number -0.55 + i0.6.
The option `color_bar_tics` is used to prevent Gnuplot from adjusting the color box
up to 40, in which case the points corresponding the maximum 36 iterations would
not be black.

```maxima
(%i1) julia (-0.55, 0.6, [iterations, 36], [x, -0.3, 0.2],
[y, 0.3, 0.9], [grid, 400, 400], [color_bar_tics, 0, 6, 36])$
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```
mandelbrot (options) [Function]

Creates a graphic representation of the Mandelbrot set. This program is part of the additional package dynamics, but that package does not have to be loaded; the first time mandelbrot is used, the package will be loaded automatically.

This program can be called without any arguments, in which case it will use a default value of 9 iterations per point, a grid with dimensions set by the grid plot option (default 30 by 30) and a region that extends from -2 to 2 in both axes. The options are all the same that plot2d accepts, plus an option iterations to change the number of iterations.

Each pixel in the grid is given a color corresponding to the number of iterations it takes the sequence starting at zero to move out of the convergence circle of radius 2, centered at the origin. The maximum number of iterations is set by the option iterations. The program uses its own default palette: magenta, violet, blue, cyan, green, yellow, orange, red, brown and black, but it can be changed by adding an explicit palette option in the command. By default, the two axes are shown with the same scale, unless the option yx_ratio is used or the option same_xy is disabled.

Example:

([grid,400,400])$

(%i1) mandelbrot ([iterations, 30], [x, -2, 1], [y, -1.2, 1.2],
[grid,400,400])$

polar_to_xy [System function]

It can be given as value for the transform_xy option of plot3d. Its effect will be to interpret the two independent variables in plot3d as the distance from the z axis and the azimuthal angle (polar coordinates), and transform them into x and y coordinates.
There are 5 types of plots that can be plotted by `plot2d`:

1. **Explicit functions.** `plot2d (expr, range_x, options)`, where `expr` is an expression that depends on only one variable, or the name of a function with one input parameter and numerical results. `range_x` is a list with three elements, the first one being the name of the variable that will be shown on the horizontal axis of the plot, and the other two elements should be two numbers, the first one smaller than the second, that define the minimum and maximum values to be shown on the horizontal axis. The name of the variable used in `range_x` must be the same variable on which `expr` depends. The result will show in the vertical axis the corresponding values of the expression or function for each value of the variable in the horizontal axis.

2. **Implicit functions.** `plot2d (expr_1=expr_2, range_x, range_y, options)`, where `expr_1` and `expr_2` are two expressions that can depend on one or two variables. `range_x` and `range_y` must be two lists of three elements that define the ranges for the variables in the two axes of the plot; the first element of each list is the name of the corresponding variable, and the other two elements are the minimum and maximum values for that variable. The two variables on which `expr_1` and `expr_2` can depend are those specified by `range_x` and `range_y`. The result will be a curve or a set of curves where the equation `expr_1=expr_2` is true.

3. **Parametric functions.** `plot2d ([parametric, expr_x, expr_y, range], options)`, where `expr_x` and `expr_y` are two expressions that depend on a single parameter. `range` must be a three-element list; the first element must be the name of the parameter on which `expr_x` and `expr_y` depend, and the other two elements must be the minimum and maximum values for that parameter. The result will be a curve in which the horizontal and vertical coordinates of each point are the values of `expr_x` and `expr_y` for a value of the parameter within the range given.

4. **Set of points.** `plot2d ([discrete, points], options)`, displays a list of points, joined by segments by default. The horizontal and vertical coordinates of each of those points can be specified in three different ways: With two lists of the same length, in which the elements of the first list are the horizontal coordinates of the points and the second list are the vertical coordinates, or with a list of two-element lists, each one corresponding to the two coordinates of one of the points, or with a single list that defines the vertical coordinates of the points; in this last case, the horizontal coordinates of the n points will be assumed to be the first n natural numbers.

5. **Contour lines.** `plot2d ([contour, expr], range_x, range_y, options)`, where `expr` is an expression that depends on two variables. `range_x` and `range_y` will be lists whose first elements are the names of those two variables, followed by two numbers that set the minimum and maximum values for them. The first variable
Chapter 12: Plotting

will be represented along the horizontal axis and the second along the vertical axis. The result will be a set of curves along which the given expression has certain values. If those values are not specified with the option levels, plotd2d will try to choose, at the most, 8 values of the form \(d \times 10^n\), where \(d\) is either 1, 2 or 5, all of them within the minimum and maximum values of \(expr\) within the given ranges.

At the end of a plot2d command several of the options described in Section 12.4 Plotting Options can be used. Many instances of the 5 types described above can be combined into a single plot, by putting them inside a list: plot2d ([type_1, ..., type_n], options). If one of the types included in the list require range_x or range_y, those ranges should come immediately after the list.

If there are several plots to be plotted, a legend will be written to identity each of the expressions. The labels that should be used in that legend can be given with the option legend. If that option is not used, Maxima will create labels from the expressions or function names.

Examples:

1. Explicit function.

   \[
   (%i1) \text{plot2d}\ (\sin(x), \ [x, -\pi, \pi])$
   \]

   \[
   \sin(x)
   \]

   \[
   x
   \]

   \[
   -1
   \]

   \[
   -0.5
   \]

   \[
   0
   \]

   \[
   0.5
   \]

   \[
   1
   \]

   \[
   -3 -2 -1 0 1 2 3
   \]

2. Implicit function.

   \[
   (%i1) \text{plot2d}\ (x^2-y^3+3*y=2, \ [x,-2.5,2.5], \ [y,-2.5,2.5])$
   \]

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3. Parametric function.

\[
\begin{align*}
\text{(\%i1) } & \text{ r: } (\exp(\cos(t)) - 2 \cos(4t) - \sin(t/12)^5) \\
\text{(\%i2) plot2d([parametric, r*\sin(t), r*\cos(t), [t, -8*\pi, 8*\pi]])} \\
\end{align*}
\]

4. Set of points.

\[
\begin{align*}
\text{(\%i1) plot2d ([discrete, makelist(i*\pi, i, 1, 5), [0.6, 0.9, 0.2, 1.3, 1]])} \\
\end{align*}
\]
5. Contour lines.

```
(%i1) plot2d ([contour, u^3 + v^2], [u, -4, 4], [v, -4, 4])$
```

Examples using options.

If an explicit function grows too fast, the y option can be used to limit the values in the vertical axis:

```
(%i1) plot2d (sec(x), [x, -2, 2], [y, -20, 20])$
```

When the plot box is disabled, no labels are created for the axes. In that case, instead of using xlabel and ylabel to set the names of the axes, it is better to use option label, which allows more flexibility. Option yx_ratio is used to change the default rectangular shape of the plot; in this example the plot will fill a square.

```
(%i1) plot2d ( x^2 - 1, [x, -3, 3], nobox, grid2d, [yx_ratio, 1], [axes, solid], [xtics, -2, 4, 2], [ytics, 2, 2, 6], [label, ["x", 2.9, -0.3], ["x^2-1", 0.1, 8]], [title, "A parabola"])$
```
A parabola

\[ x^2 - 1 \]

\[ x \]

A plot with a logarithmic scale in the vertical axis:

\[
\text{(\%i1) plot2d (exp(3*s), [s, -2, 2], logy)}$

\[
\text{(\%i1) plot2d (exp(3*s), [s, -2, 2], logy)}$

Plotting functions by name:

\[
\text{(\%i1) F(x) := x^2 }$
\]
\[
\text{(\%i2) :lisp (defun |$g| (x) (m* x x x))}$
\]
\[
\text{(\%i2) H(x) := if x < 0 then x^4 - 1 else 1 - x^5 }$
\]
\[
\text{(\%i3) plot2d ([F, G, H], [u, -1, 1], [y, -1.5, 1.5])}$
\]
Plot of a circle, using its parametric representation, together with the function $-|x|$. The circle will only look like a circle if the scale in the two axes is the same, which is done with the option `same_xy`.

(%i1) plot2d([[parametric, cos(t), sin(t), [t,0,2*%pi]], -abs(x)], [x, -sqrt(2), sqrt(2)], same_xy)$

A plot of 200 random numbers between 0 and 9:

(%i1) plot2d ([discrete, makelist ( random(10), 200)])$
In the next example a table with three columns is saved in a file “data.txt” which is then read and the second and third column are plotted on the two axes:

```maxima
(%i1) display2d:false$
(%i2) with_stdout ("data.txt", for x:0 thru 10 do
   print (x, x^2, x^3))$
(%i3) data: read_matrix ("data.txt")$
(%i4) plot2d ([discrete, transpose(data)[2], transpose(data)[3]],
   [style,points], [point_type,diamond], [color,red])$
```

A plot of discrete data points together with a continuous function:

```maxima
(%i1) xy: [[10, .6], [20, .9], [30, 1.1], [40, 1.3], [50, 1.4]]$
(%i2) plot2d ([[discrete, xy], 2*%pi*sqrt(l/980)], [l,0,50],
   [style, points, lines], [color, red, blue],
   [point_type, asterisk],
   [legend, "experiment", "theory"],
   [xlabel, "pendulum's length (cm)"],
   [ylabel, "period (s)"])$
```
See also the section about Plotting Options.

plot3d

\[ \text{plot3d} \left( \text{expr}, \text{x\_range}, \text{y\_range}, \ldots, \text{options}, \ldots \right) \]
\[ \text{plot3d} \left( \left[ \text{expr\_1}, \ldots, \text{expr\_n} \right], \text{x\_range}, \text{y\_range}, \ldots, \text{options}, \ldots \right) \]

Displays a plot of one or more surfaces defined as functions of two variables or in parametric form.

The functions to be plotted may be specified as expressions or function names. The mouse can be used to rotate the plot looking at the surface from different sides.

Examples.

Plot of a function of two variables:

\[
\text{plot3d} \left( u^2 - v^2, \left[ u, -2, 2 \right], \left[ v, -3, 3 \right], \left[ \text{grid}, 100, 100 \right], \text{nomesh\_lines} \right)
\]

Use of the \textit{z} option to limit a function that goes to infinity (in this case the function is minus infinity on the x and y axes); this also shows how to plot with only lines and no shading:

\[
\text{plot3d} \left( \log ( x^2*y^2 ), \left[ x, -2, 2 \right], \left[ y, -2, 2 \right], \left[ z, -8, 4 \right], \text{nopalette}, \left[ \text{color}, \text{magenta} \right] \right)
\]
The infinite values of \( z \) can also be avoided by choosing a grid that does not fall on any points where the function is undefined, as in the next example, which also shows how to change the palette and how to include a color bar that relates colors to values of the \( z \) variable:

\[
\text{(%i1) plot3d (log (x^2*y^2), [x, -2, 2], [y, -2, 2], [grid, 29, 29],}
\text{ [palette, [gradient, red, orange, yellow, green]],}
\text{ color_bar, [xtics, 1], [ytics, 1], [ztics, 4],}
\text{ [color_bar_tics, 4])};
\]

Two surfaces in the same plot. Ranges specific to one of the surfaces can be given by placing each expression and its ranges in a separate list; global ranges for the complete plot are also given after the function definitions.

\[
\text{(%i1) plot3d ([[-3*x - y, [x, -2, 2], [y, -2, 2]],}
\text{ 4*sin(3*(x^2 + y^2))/(x^2 + y^2), [x, -3, 3], [y, -3, 3]],}
\text{ [x, -4, 4], [y, -4, 4])};
\]
Plot of a Klein bottle, defined parametrically:

\[(\%i1) \text{ expr}_1: 5\cos(x)\cos(x/2)\cos(y) + \sin(x/2)\sin(2y) + 3 - 10\$
\[(\%i2) \text{ expr}_2: -5\sin(x)\cos(x/2)\cos(y) + \sin(x/2)\sin(2y) + 3\$
\[(\%i3) \text{ expr}_3: 5\sin(x/2)\cos(y) + \cos(x/2)\sin(2y)\$
\[(\%i4) \text{ plot3d} ([\text{expr}_1, \text{expr}_2, \text{expr}_3], [x, -\pi, \pi], [y, -\pi, \pi], [\text{grid}, 50, 50])\$

Plot of a “spherical harmonic” function, using the predefined transformation, \text{spherical\_to\_xyz} to transform from spherical coordinates to rectangular coordinates. See the documentation for \text{spherical\_to\_xyz}.

\[(\%i11) \text{ plot3d} (\sin(2\theta)\cos(\phi), [\theta, 0, \pi], [\phi, 0, 2\pi], [	ext{transform}\_\text{xy}, \text{spherical\_to\_xyz}], [\text{grid}, 30, 60], \text{nolegend})\$
Use of the pre-defined function `polar_to_xy` to transform from cylindrical to rectangular coordinates. See the documentation for `polar_to_xy`.

```
(%i1) plot3d (r^.33*cos(th/3), [r,0,1], [th,0,6*%pi], nobox, nolegend, [grid, 12, 80], [transform_xy, polar_to_xy])$
```

Plot of a sphere using the transformation from spherical to rectangular coordinates. Option `same_xyz` is used to get the three axes scaled in the same proportion. When transformations are used, it is not convenient to eliminate the mesh lines, because Gnuplot will not show the surface correctly.

```
(%i1) plot3d ( 5, [theta,0,%pi], [phi,0,2*%pi], same_xyz, nolegend, [transform_xy, spherical_to_xyz], [mesh_lines_color,blue], [palette,[gradient,"#1b1b4e", "#8c8cf8"]])$
```
Definition of a function of two-variables using a matrix. Notice the single quote in the definition of the function, to prevent \texttt{plot3d} from failing when it realizes that the matrix will require integer indices.

\[
\begin{array}{c}
\textbf{Maxima Code:}\\
\text{(%i1) } M: \text{matrix}([1,2,3,4], [1,2,3,2], [1,2,3,4], [1,2,3,3])\\
\text{(%i2) } f(x, y) := \text{float('M [round(x), round(y)])}\\
\text{(%i3) plot3d} \ (f(x,y), [x,1,4], [y,1,4], [grid,3,3], \text{nolegend})\\
\end{array}
\]

By setting the elevation equal to zero, a surface can be seen as a map in which each color represents a different level.

\[
\begin{array}{c}
\textbf{Maxima Code:}\\
\text{(%i1) plot3d} \ (\cos (-x^2 + y^3/4), [x,-4,4], [y,-4,4], [\text{zlabel,}""],\text{[mesh_lines_color,false]}, [\text{elevation,0}], [\text{azimuth,0}], \\
\text{color_bar}, [\text{grid,80,80}], \text{noztics, [color_bar_tics,1]}))\\
\end{array}
\]
See also Section 12.4 Plotting Options.

plot_options
[System variable]
This option is being kept for compatibility with older versions, but its use is deprecated. To set global plotting options, see their current values or remove options, use set_plot_option, get_plot_option and remove_plot_option.

remove_plot_option (name)
[Function]
Removes the default value of an option. The name of the option must be given.
See also set_plot_option, get_plot_option and Section 12.4 Plotting Options.

set_plot_option (option)
[Function]
Accepts any of the options listed in the section Plotting Options, and saves them for use in plotting commands. The values of the options set in each plotting command will have precedence, but if those options are not given, the default values set with this function will be used.
set_plot_option evaluates its argument and returns the complete list of options (after modifying the option given). If called without any arguments, it will simply show the list of current default options.
See also remove_plot_option, get_plot_option and the section on Plotting Options.
Example:
Modification of the grid values.

(%i1) set_plot_option ([grid, 30, 40]);
(%o1) [[plot_format, gnuplot_pipes], [grid, 30, 40],
        [run_viewer, true], [axes, true], [nticks, 29], [adapt_depth, 5],
        [color, blue, red, green, magenta, black, cyan],
        [point_type, bullet, box, triangle, plus, times, asterisk],
        [palette, [gradient, green, cyan, blue, violet],
        [gradient, magenta, violet, blue, cyan, green, yellow, orange,
        red, brown, black]], [gnuplot_preamble, ], [gnuplot_term, default]]

spherical_to_xyz
[System function]
It can be given as value for the transform_xy option of plot3d. Its effect will be to interpret the two independent variables and the function in plot3d as the spherical
coordinates of a point (first, the angle with the z axis, then the angle of the xy projection with the x axis and finally the distance from the origin) and transform them into x, y and z coordinates.

12.4 Plotting Options

All options consist of a list starting with one of the keywords in this section, followed by one or more values. If the option appears inside one of the plotting commands, its value will be local for that command. If the option is given as argument to \texttt{set\_plot\_option}, its value will be global and used in all plots, unless it is overridden by a local value.

Some of the options may have different effects in different plotting commands as it will be pointed out in the following list. The options that accept among their possible values \texttt{true} or \texttt{false}, can also be set to \texttt{true} by simply writing their names, and \texttt{false} by writing their names with the prefix \texttt{no}. For instance, typing \texttt{logx} as an option is equivalent to writing \texttt{[logx, true]} and \texttt{nobox} is equivalent to \texttt{[box, false]}. When just the name of the option is used for an option which cannot have a value \texttt{true}, it means that any value previously assigned to that option will be removed, leaving it to the graphic program to decide what to do.

\begin{description}
\item[\texttt{adapt\_depth \{adapt\_depth, integer\}}] Default value: 5
The maximum number of splittings used by the adaptive plotting routine of \texttt{plot2d}; \texttt{integer} must be a non-negative integer. A value of zero means that adaptive plotting will not be used and the resulting plot will have \(1+4^{*}\text{nticks}\) points (see option \texttt{nticks}). To have more control on the number of points and their positions, a list of points can be created and then plotted using the \texttt{discrete} method of \texttt{plot2d}.

\item[\texttt{axes \{axes, symbol\}, axes, noaxes}] Default value: \texttt{true}
Where \texttt{symbol} can be either \texttt{true}, \texttt{false}, \texttt{x}, \texttt{y} or \texttt{solid}. If \texttt{false}, no axes are shown; if equal to \texttt{x} or \texttt{y} only the \texttt{x} or \texttt{y} axis will be shown; if it is equal to \texttt{true}, both axes will be shown and \texttt{solid} will show the two axes with a solid line, rather than the default broken line. This option does not have any effect in the 3 dimensional plots. The single keywords \texttt{axes} and \texttt{noaxes} can be used as synonyms for \texttt{[axes, true]} and \texttt{[axes, false]}.

\item[\texttt{azimuth \{azimuth, number\}}] Default value: 30
A plot3d plot can be thought of as starting with the \texttt{x} and \texttt{y} axis in the horizontal and vertical axis, as in \texttt{plot2d}, and the \texttt{z} axis coming out of the screen. The \texttt{z} axis is then rotated around the \texttt{x} axis through an angle equal to \texttt{elevation} and then the new \texttt{x}\texttt{y} plane is rotated around the new \texttt{z} axis through an angle \texttt{azimuth}. This option sets the value for the azimuth, in degrees.
See also \texttt{elevation}.

\item[\texttt{box \{box, symbol\}, box, nobox}] Default value: \texttt{true}
If set to true, a bounding box will be drawn for the plot; if set to false, no box will be drawn. The single keywords box and nobox can be used as synonyms for [box, true] and [box, false].

color [color, color_1, . . ., color_n]  
In 2d plots it defines the color (or colors) for the various curves. In plot3d, it defines the colors used for the mesh lines of the surfaces, when no palette is being used.
If there are more curves or surfaces than colors, the colors will be repeated in sequence. The valid colors are red, green, blue, magenta, cyan, yellow, orange, violet, brown, gray, black, white, or a string starting with the character # and followed by six hexadecimal digits: two for the red component, two for green component and two for the blue component. If the name of a given color is unknown color, black will be used instead.

color_bar [color_bar, symbol], color_bar, nocolor_bar  
Default value: false in plot3d, true in mandelbrot and julia
Where symbol can be either true or false. If true, whenever plot3d, mandelbrot or julia use a palette to represent different values, a box will be shown on the right, showing the corresponding between colors and values. The single keywords color_bar and nocolor_bar can be used as synonyms for [color_bar, true] and [color_bar, false].

color_bar_tics [color_bar_tics, x1, x2, x3], color_bar_tics, nocolor_bar_tics  
Defines the values at which a mark and a number will be placed in the color bar. The first number is the initial value, the second the increments and the third is the last value where a mark is placed. The second and third numbers can be omitted. When only one number is given, it will be used as the increment from an initial value that will be chosen automatically. The single keyword color_bar_tics removes a value given previously, making the graphic program use its default for the values of the tics and nocolor_bar_tics will not show any tics on the color bar.

elevation [elevation, number]  
Default value: 60
A plot3d plot can be thought of as starting with the x and y axis in the horizontal and vertical axis, as in plot2d, and the z axis coming out of the screen. The z axis is then rotated around the x axis through an angle equal to elevation and then the new xy plane is rotated around the new z axis through an angle azimuth. This option sets the value for the elevation, in degrees.
See also azimuth.

grid [grid, integer, integer]  
Default value: 30, 30
Sets the number of grid points to use in the x- and y-directions for three-dimensional plotting or for the julia and mandelbrot programs.
For a way to actually draw a grid See grid2d.
grid2d [grid2d, value], grid2d, nogrid2d  
Default value: false  
Shows a grid of lines on the xy plane. The points where the grid lines are placed are 
the same points where tics are marked in the x and y axes, which can be controlled 
with the xtics and ytics options. The single keywords grid2d and nogrid2d can 
be used as synonyms for [grid2d, true] and [grid2d, false].  
See also grid.

iterations [iterations, value]  
Default value: 9  
Number of iterations made by the programs mandelbrot and julia.

label [label, [string, x, y], . . .]  
Writes one or several labels in the points with x, y coordinates indicated after each 
label.

legend [legend, string_1, . . ., string_n], legend, nolegend  
It specifies the labels for the plots when various plots are shown. If there are more 
plots than the number of labels given, they will be repeated. If given the value false, 
no legends will be shown. By default, the names of the expressions or functions will 
be used, or the words discretel, discrete2, . . ., for discrete sets of points. The single 
keyword legend removes any previously defined legends, leaving it to the plotting 
program to set up a legend. The keyword nolegend is a synonym for [legend, 
false].

levels [levels, number, . . .]  
This option is used by plot2d to do contour plots. If only one number is given after 
the keyword levels, it must be a natural number; plot2d will try to plot that number 
of contours with values between the minimum and maximum value of the expression 
given, with the form d*10^n, where d is either 1, 2 or 5. Since not always it will be 
possible to find that number of levels in that interval, the number of contour lines 
show will be smaller than the number specified by this option.  
If more than one number are given after the keyword levels, plot2d will show the 
contour lines corresponding to those values of the expression plotted, if they exist 
within the domain used.

logx [logx, value], logx, nologx  
Default value: false  
Makes the horizontal axes to be scaled logarithmically. It can be either true or false. 
The single keywords logx and nologx can be used as synonyms for [logx, true] 
and [logx, false].

logy [logy, value], logy, nology  
Default value: false  
Makes the vertical axes to be scaled logarithmically. It can be either true or false. 
The single keywords logy and nology can be used as synonyms for [logy, true] 
and [logy, false].
**mesh_lines_color** [mesh_lines_color, color], mesh_lines_color, [Plot option]

Default value: black

It sets the color used by plot3d to draw the mesh lines, when a palette is being used. It accepts the same colors as for the option color (see the list of allowed colors in color). It can also be given a value false to eliminate completely the mesh lines. The single keyword mesh_lines_color removes any previously defined colors, leaving it to the graphic program to decide what color to use. The keyword no_mesh_lines is a synonym for [mesh_lines_color, false]

**nticks** [nticks, integer] [Plot option]

Default value: 29

When plotting functions with plot2d, it is given the initial number of points used by the adaptive plotting routine for plotting functions. When plotting parametric functions with plot3d, it sets the number of points that will be shown for the plot.

**palette** [palette, [palette_1], ..., [palette_n]], palette, nopalette [Plot option]

It can consist of one palette or a list of several palettes. Each palette is a list with a keyword followed by values. If the keyword is gradient, it should be followed by a list of valid colors.

If the keyword is hue, saturation or value, it must be followed by 4 numbers. The first three numbers, which must be between 0 and 1, define the hue, saturation and value of a basic color to be assigned to the minimum value of z. The keyword specifies which of the three attributes (hue, saturation or value) will be increased according to the values of z. The last number indicates the increase corresponding to the maximum value of z. That last number can be bigger than 1 or negative; the corresponding values of the modified attribute will be rounded modulo 1.

Gnuplot only uses the first palette in the list; xmaxima will use the palettes in the list sequentially, when several surfaces are plotted together; if the number of palettes is exhausted, they will be repeated sequentially.

The color of the mesh lines will be given by the option mesh_lines_color. If palette is given the value false, the surfaces will not be shaded but represented with a mesh of curves only. In that case, the colors of the lines will be determined by the option color.

The single keyword palette removes any palette previously defined, leaving it to the graphic program to decide the palette to use and nopalette is a synonym for [palette, false].

**plotepsilon** [plotepsilon, value] [Plot option]

Default value: 1e-6

This value is used by plot2d when plotting implicit functions or contour lines. When plotting an explicit function expr_1=expr_2, it is converted into expr_1-expr_2 and the points where that equals zero are searched. When a contour line for expr equal to some value is going to be plotted, the points where expr minus that value are equal to zero are searched. The search is done by computing those expressions at a grid
of points (see sample). If at one of the points in that grid the absolute value of the expression is less than the value of plotepsilon, it will be assumed to be zero, and therefore, as being part of the curve to be plotted.

**plot_format [plot_format, format]**

[Plot option]

Default value: gnuplot, in Windows systems, or gnuplot_pipes in other systems.

Where format is one of the following: gnuplot, xmaxima, mgnuplot, gnuplot_pipes or geomview.

It sets the format to be used for plotting as explained in **Section 12.2 Plotting Formats**.

**plot_realpart [plot_realpart, symbol], plot_realpart, noplot_realpart**

[Plot option]

Default value: false

If set to true, the functions to be plotted will be considered as complex functions whose real value should be plotted; this is equivalent to plotting realpart(function). If set to false, nothing will be plotted when the function does not give a real value. For instance, when x is negative, log(x) gives a complex value, with real value equal to log(abs(x)); if plot_realpart were true, log(-5) would be plotted as log(5), while nothing would be plotted if plot_realpart were false. The single keyword plot_realpart can be used as a synonym for [plot_realpart, true] and noplot_realpart is a synonym for [plot_realpart, false].

**point_type [point_type, type_1, ..., type_n]**

[Plot option]

In gnuplot, each set of points to be plotted with the style “points” or “linespoints” will be represented with objects taken from this list, in sequential order. If there are more sets of points than objects in this list, they will be repeated sequentially. The possible objects that can be used are: bullet, circle, plus, times, asterisk, box, square, triangle, delta, wedge, nabla, diamond, lozenge.

**pdf_file [pdf_file, file_name]**

[Plot option]

Saves the plot into a PDF file with name equal to file_name, rather than showing it in the screen. By default, the file will be created in the directory defined by the variable maxima_tmpdir, unless file_name contains the character “/”, in which case it will be assumed to contain the complete path where the file should be created. The value of maxima_tmpdir can be changed to save the file in a different directory. When the option gnuplot_pdf_term_command is also given, it will be used to set up Gnuplot’s PDF terminal; otherwise, Gnuplot’s pdfcairo terminal will be used with solid colored lines of width 3, plot size of 17.2 cm by 12.9 cm and font of 18 points.

**png_file [png_file, file_name]**

[Plot option]

Saves the plot into a PNG graphics file with name equal to file_name, rather than showing it in the screen. By default, the file will be created in the directory defined by the variable maxima_tmpdir, unless file_name contains the character “/”, in which case it will be assumed to contain the complete path where the file should be created. The value of maxima_tmpdir can be changed to save the file in a different directory. When the option gnuplot_png_term_command is also given, it will be used to set up
Gnuplot’s PNG terminal; otherwise, Gnuplot’s pngcairo terminal will be used, with a font of size 12.

**ps_file** [ps_file, file_name]  
[Plot option]  
Saves the plot into a Postscript file with name equal to file_name, rather than showing it in the screen. By default, the file will be created in the directory defined by the variable maxima_tempdir, unless file_name contains the character “/”, in which case it will be assumed to contain the complete path where the file should be created. The value of maxima_tempdir can be changed to save the file in a different directory. When the option gnuplot_ps_term_command is also given, it will be used to set up Gnuplot’s Postscript terminal; otherwise, Gnuplot’s postscript terminal will be used with the EPS option, solid colored lines of width 2, plot size of 16.4 cm by 12.3 cm and font of 24 points.

**run_viewer** [run_viewer, symbol], run_viewer, norun_viewer  
[Plot option]  
Default value: true  
This option is only used when the plot format is gnuplot and the terminal is default or when the Gnuplot terminal is set to dumb (see gnuplot_term) and can have a true or false value.  
If the terminal is default, a file maxout_xxx.gnuplot (or other name specified with gnuplot_out_file) is created with the gnuplot commands necessary to generate the plot. Option run_viewer controls whether or not Gnuplot will be launched to execute those commands and show the plot.  
If the terminal is default, gnuplot is run to execute the commands in maxout_xxx.gnuplot, producing another file maxplot.txt (or other name specified with gnuplot_out_file). Option run_viewer controls whether or not that file, with an ASCII representation of the plot, will be shown in the Maxima or Xmaxima console. Its default value, true, makes the plots appear in either the console or a separate graphics window. run_viewer and norun_viewer are synonyms for [run_viewer, true] and [run_viewer, false].

**same_xy** [same_xy , value], same_xy, nosame_xy  
[Plot option]  
It can be either true or false. If true, the scales used in the x and y axes will be the same, in either 2d or 3d plots. See also yx_ratio. same_xy and nosame_xy are synonyms for [same_xy, true] and [same_xy, false].

**same_xyz** [same_xyz , value], same_xyz, nosame_xyz  
[Plot option]  
It can be either true or false. If true, the scales used in the 3 axes of a 3d plot will be the same. same_xyz and nosame_xyz are synonyms for [same_xyz, true] and [same_xyz, false].

**sample** [sample, nx, ny]  
[Plot option]  
Default value: [sample, 50, 50]  
x and ny must be two natural numbers that will be used by plot2d to look for the points that make part of the plot of an implicit function or a contour line. The domain is divided into nx intervals in the horizontal axis and ny intervals in the vertical axis and the numerical value of the expression is computed at the borders
of those intervals. Higher values of \( nx \) and \( ny \) will give smoother curves, but will increase the time needed to trace the plot. When there are critical points in the plot where the curve changes direction, to get better results it is more important to try to make those points to be at the border of the intervals, rather than increasing \( nx \) and \( ny \).

\[
\text{style} \ [\text{style}, \text{type}_1, \ldots, \text{type}_n], \ [\text{style}, [\text{style}_1], \ldots, [\text{style}_n]]
\]

[Plot option]

The styles that will be used for the various functions or sets of data in a 2d plot. The word \textit{style} must be followed by one or more styles. If there are more functions and data sets than the styles given, the styles will be repeated. Each style can be either \textit{lines} for line segments, \textit{points} for isolated points, \textit{linespoints} for segments and points, or \textit{dots} for small isolated dots. Gnuplot accepts also an \textit{impulses} style.

Each of the styles can be enclosed inside a list with some additional parameters. \textit{lines} accepts one or two numbers: the width of the line and an integer that identifies a color. The default color codes are: 1: blue, 2: red, 3: magenta, 4: orange, 5: brown, 6: lime and 7: aqua. If you use Gnuplot with a terminal different than X11, those colors might be different; for example, if you use the option \([\text{gnuplot-term, ps}]\), color index 4 will correspond to black, instead of orange.

\textit{points} accepts one two or three parameters; the first parameter is the radius of the points, the second parameter is an integer that selects the color, using the same code used for \textit{lines} and the third parameter is currently used only by Gnuplot and it corresponds to several objects instead of points. The default types of objects are: 1: filled circles, 2: open circles, 3: plus signs, 4: x, 5: *, 6: filled squares, 7: open squares, 8: filled triangles, 9: open triangles, 10: filled inverted triangles, 11: open inverted triangles, 12: filled lozenges and 13: open lozenges.

\textit{linespoints} accepts up to four parameters: line width, points radius, color and type of object to replace the points.

See also \textit{color} and \textit{point_type}.

\[
\text{svg_file} \ [\text{svg_file, file_name}]
\]

[Plot option]

Saves the plot into an SVG file with name equal to \textit{file_name}, rather than showing it in the screen. By default, the file will be created in the directory defined by the variable \texttt{maxima_tempdir}, unless \textit{file_name} contains the character “/”, in which case it will be assumed to contain the complete path where the file should be created. The value of \texttt{maxima_tempdir} can be changed to save the file in a different directory. When the option \texttt{gnuplot-svg-term-command} is also given, it will be used to set up Gnuplot’s SVG terminal; otherwise, Gnuplot’s svg terminal will be used with font of 14 points.

\[
t \ [t, \min, \max]
\]

[Plot option]

Default range for parametric plots.

\[
\text{title} \ [\text{title}, \text{text}]
\]

[Plot option]

Defines a title that will be written at the top of the plot.
transform_xy [transform_xy, symbol], notransform_xy  
[Plot option]
Where symbol is either false or the result obtained by using the function transform_xy. If different from false, it will be used to transform the 3 coordinates in plot3d.
notransform_xy removes any transformation function previously defined.
See make_transform, polar_to_xy and spherical_to_xyz.

window [window, n]  
[Plot option]
Opens the plot in window number n, instead of the default window 0. If window number n is already opened, the plot in that window will be replaced by the current plot.

x [x, min, max]  
[Plot option]
When used as the first option in a plot2d command (or any of the first two in plot3d), it indicates that the first independent variable is x and it sets its range. It can also be used again after the first option (or after the second option in plot3d) to define the effective horizontal domain that will be shown in the plot.

xlabel [xlabel, string]  
[Plot option]
Specifies the string that will label the first axis; if this option is not used, that label will be the name of the independent variable, when plotting functions with plot2d the name of the first variable, when plotting surfaces with plot3d, or the first expression in the case of a parametric plot.

xtics [xtics, x1, x2, x3], [xtics, false], xtics, noxtics  
[Plot option]
Defines the values at which a mark and a number will be placed in the x axis. The first number is the initial value and a number will be placed in the x axis. The first number is the initial value, the second the increments and the third is the last value where a mark is placed. The second and third numbers can be omitted, in which case the first number is the increment from an initial value that will be chosen by the graphic program. If [xtics, false] is used, no marks or numbers will be shown along the x axis.

The single keyword xtics removes any values previously defined, leaving it to the graphic program to decide the values to use and noxtics is a synonym for [xtics, false]

xy_scale [xy_scale, sx, sy]  
[Plot option]
In a 2d plot, it defines the ratio of the total size of the Window to the size that will be used for the plot. The two numbers given as arguments are the scale factors for the x and y axes.

This option does not change the size of the graphic window or the placement of the graph in the window. If you want to change the aspect ratio of the plot, it is better to use option yx_ratio. For instance, [yx_ratio, 10] instead of [xy_scale, 0.1, 1].

y [y, min, max]  
[Plot option]
When used as one of the first two options in plot3d, it indicates that one of the independent variables is y and it sets its range. Otherwise, it defines the effective domain of the second variable that will be shown in the plot.
**ylabel** [ylabel, string]  
Specifies the string that will label the second axis; if this option is not used, that label will be “y”, when plotting explicit functions with `plot2d`, or the name of the second variable, when plotting surfaces with `plot3d`, or the second expression in the case of a parametric plot.

**ytics** [ytics, y1, y2, y3], [ytics, false], ytics, noytics  
Defines the values at which a mark and a number will be placed in the y axis. The first number is the initial value, the second the increments and the third is the last value where a mark is placed. The second and third numbers can be omitted, in which case the first number is the increment from an initial value that will be chosen by the graphic program. If [ytics, false] is used, no marks or numbers will be shown along the y axis.

The single keyword ytics removes any values previously defined, leaving it to the graphic program to decide the values to use and noytics is a synonym for [ytics, false]

**yx_ratio** [yx_ratio, r]  
In a 2d plot, the ratio between the vertical and the horizontal sides of the rectangle used to make the plot. See also `same_xy`.

**z** [z, min, max]  
Used in `plot3d` to set the effective range of values of z that will be shown in the plot.

**zlabel** [zlabel, string]  
Specifies the string that will label the third axis, when using `plot3d`. If this option is not used, that label will be “z”, when plotting surfaces, or the third expression in the case of a parametric plot. It can not be used with `set_plot_option` and it will be ignored by `plot2d`.

**zmin** [zmin, z], zmin  
In 3d plots, the value of z that will be at the bottom of the plot box.

The single keyword zmin removes any value previously defined, leaving it to the graphic program to decide the value to use.

**ztics** [ztics, z1, z2, z3], [ztics, false], ztics, noztics  
Defines the values at which a mark and a number will be placed in the z axis. The first number is the initial value, the second the increments and the third is the last value where a mark is placed. The second and third numbers can be omitted, in which case the first number is the increment from an initial value that will be chosen by the graphic program. If [ztics, false] is used, no marks or numbers will be shown along the z axis.

The single keyword ztics removes any values previously defined, leaving it to the graphic program to decide the values to use and noztics is a synonym for [ztics, false]
12.5 Gnuplot Options

There are several plot options specific to gnuplot. All of them consist of a keyword (the name of the option), followed by a string that should be a valid gnuplot command, to be passed directly to gnuplot. In most cases, there exist a corresponding plotting option that will produce a similar result and whose use is more recommended than the gnuplot specific option.

\texttt{gnuplot\_term \tin \tin \tin \tin [\texttt{gnuplot\_term, terminal\_name}]\hfill \text{[Plot option]}}

Sets the output terminal type for gnuplot. The argument \textit{terminal\_name} can be a string or one of the following 3 special symbols:

- \texttt{default} (default value)
  Gnuplot output is displayed in a separate graphical window and the gnuplot terminal used will be specified by the value of the option \texttt{gnuplot\_default\_term\_command}.

- \texttt{dumb}
  Gnuplot output is saved to a file \texttt{maxout\_xxx.gnuplot} using "ASCII art" approximation to graphics. If the option \texttt{gnuplot\_out\_file} is set to \texttt{filename}, the plot will be saved there, instead of the default \texttt{maxout\_xxx.gnuplot}. The settings for the "dumb" terminal of Gnuplot are given by the value of option \texttt{gnuplot\_dumb\_term\_command}. If option \texttt{run\_viewer} is set to true and the plot format is \texttt{gnuplot} that ASCII representation will also be shown in the Maxima or Xmaxima console.

- \texttt{ps}
  Gnuplot generates commands in the PostScript page description language. If the option \texttt{gnuplot\_out\_file} is set to \texttt{filename}, gnuplot writes the PostScript commands to \texttt{filename}. Otherwise, it is saved as \texttt{maxplot.ps} file. The settings for this terminal are given by the value of the option \texttt{gnuplot\_dumb\_term\_command}.

- A string representing any valid gnuplot term specification
  Gnuplot can generate output in many other graphical formats such as png, jpeg, svg etc. To use those formats, option \texttt{gnuplot\_term} can be set to any supported gnuplot term name (which must be a symbol) or even a full gnuplot term specification with any valid options (which must be a string). For example \texttt{[gnuplot\_term, png]} creates output in PNG (Portable Network Graphics) format while \texttt{[gnuplot\_term, "png size 1000,1000"]} creates PNG of 1000 x 1000 pixels size. If the option \texttt{gnuplot\_out\_file} is set to \texttt{filename}, gnuplot writes the output to \texttt{filename}. Otherwise, it is saved as \texttt{maxplot.\term} file, where \texttt{term} is gnuplot terminal name.

\texttt{gnuplot\_out\_file \tin \tin \tin \tin [\texttt{gnuplot\_out\_file, file\_name}]\hfill \text{[Plot option]}}

It can be used to replace the default name for the file that contains the commands that will interpreted by gnuplot, when the terminal is set to \texttt{default}, or to replace the default name of the graphic file that gnuplot creates, when the terminal is different from \texttt{default}. If it contains one or more slashes, "/", the name of the file will be left as it is; otherwise, it will be appended to the path of the temporary directory. The complete name of the files created by the plotting commands is always sent as output of those commands so they can be seen if the command is ended by semi-colon.
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When used in conjunction with the `gnuplot_term` option, it can be used to save the plot in a file, in one of the graphic formats supported by Gnuplot. To create PNG, PDF, Postscript or SVG, it is easier to use options `png_file`, `pdf_file`, `ps_file`, or `svg_file`.

`gnuplot_script_file` [`gnuplot_script_file`,

`file_name_or_function`]  

[Plot option]

Creates a plot with `plot2d`, `plot3d`, `mandelbrot` or `julia` using the `gnuplot` `plot_format` and saving the script sent to Gnuplot in the file specified by `file_name_or_function`. The value `file_name_or_function` can be a string or a Maxima function of one variable that returns a string. If that string corresponds to a complete file path (directory and file name), the script will be created in that file and will not be deleted after Maxima is closed; otherwise, the string will give the name of a file to be created in the temporary directory and deleted when Maxima is closed.

In this example, the script file name is set to “sin.gnuplot”, in the current directory.

```
(%i1) plot2d( sin(x), [x,0,2*%pi], [gnuplot_script_file, "/sin.gnuplot"]);
```

```
(%o1) ["/sin.gnuplot"]
```

In this example, `gnuplot_maxout_prt(file)` is a function that takes the default file name, `file`. It constructs a full file name for the data file by interpolating a random 8-digit integer with a pad of zeros into the default file name (“maxout” followed by the id number of the Maxima process, followed by “.gnuplot”). The temporary directory is determined by `maxima_tempdir` (it is “/tmp” in this example).

```
(%i1) gnuplot_maxout_prt(file) := block(
    [beg,end]
    ,
    [beg,end] : split(file,"."),
    printf(false,"~a_~8,'0d.~a",beg,random(10^8-1),end)) $
```

```
(%i2) plot2d( sin(x), [x,0,2*%pi], [gnuplot_script_file, gnuplot_maxout_prt]);
```

```
(%o2) ["/tmp/maxout68715_09606909.gnuplot"]
```

By default, the script would have been saved in a file named `maxoutXXXXX.gnuplot` (XXXXX=68715 in this example) in the temporary directory. If the print statement in function `gnuplot_maxout_prt` above included a directory path, the file would have been saved in that directory rather than in the temporary directory.

`gnuplot_pm3d` [`gnuplot_pm3d`, value]  

[Plot option]

With a value of `false`, it can be used to disable the use of PM3D mode, which is enabled by default.

`gnuplot_preamble` [`gnuplot_preamble`, string]  

[Plot option]

This option inserts gnuplot commands before any other commands sent to Gnuplot. Any valid gnuplot commands may be used. Multiple commands should be separated with a semi-colon. See also `gnuplot_postamble`.

`gnuplot_postamble` [`gnuplot_postamble`, string]  

[Plot option]

This option inserts gnuplot commands after other commands sent to Gnuplot and right before the plot command is sent. Any valid gnuplot commands may
be used. Multiple commands should be separated with a semi-colon. See also \texttt{gnuplot_preamble}.

\texttt{gnuplot Default term command} \hspace{1cm} \texttt{[Plot option]}

\texttt{[gnuplot Default term command, command]}

The gnuplot command to set the terminal type for the default terminal. It this option is not set, the command used will be: \\
\texttt{"set term wxt size 640,480 font ",12";} \\
\texttt{set term pop"}.

\texttt{gnuplot Dumb term command} \hspace{1cm} \texttt{[Plot option]}

\texttt{[gnuplot Dumb term command, command]}

The gnuplot command to set the terminal type for the dumb terminal. It this option is not set, the command used will be: \\
\texttt{"set term dumb 79 22"}, which makes the text output 79 characters by 22 characters.

\texttt{gnuplot PDF term command} \hspace{1cm} \texttt{[Plot option]}

\texttt{[gnuplot PDF term command, command]}

The gnuplot command to set the terminal type for the PDF terminal. If this option is not set, the command used will be: \\
\texttt{"set term pdfcairo color solid lw 3 size 17.2 cm, 12.9 cm font ",18""}. See the gnuplot documentation for more information.

\texttt{gnuplot PNG term command} \hspace{1cm} \texttt{[Plot option]}

\texttt{[gnuplot PNG term command, command]}

The gnuplot command to set the terminal type for the PNG terminal. If this option is not set, the command used will be: \\
\texttt{"set term pngcairo font ",12""}. See the gnuplot documentation for more information.

\texttt{gnuplot Postscript term command} \hspace{1cm} \texttt{[Plot option]}

\texttt{[gnuplot Postscript term command, command]}

The gnuplot command to set the terminal type for the PostScript terminal. If this option is not set, the command used will be: \\
\texttt{"set term postscript eps color solid lw 2 size 16.4 cm, 12.3 cm font ",24""}. See the gnuplot documentation for set term postscript for more information.

\texttt{gnuplot Strings} \hspace{1cm} \texttt{[Plot option]}

\texttt{[gnuplot Strings, value]}

With a value of \texttt{true}, all strings used in labels and titles will be interpreted by gnuplot as “enhanced” text, if the terminal being used supports it. In enhanced mode, some characters such as \texttt{^} and \texttt{_} are not printed, but interpreted as formatting characters. For a list of the formatting characters and their meaning, see the documentation for enhanced in Gnuplot. The default value for this option is \texttt{false}, which will not treat any characters as formatting characters.

\texttt{gnuplot SVG term command} \hspace{1cm} \texttt{[Plot option]}

\texttt{[gnuplot SVG term command, command]}

The gnuplot command to set the terminal type for the SVG terminal. If this option is not set, the command used will be: \\
\texttt{"set term svg font ",14""}. See the gnuplot documentation for more information.

\texttt{gnuplot Curve titles} \hspace{1cm} \texttt{[Plot option]}

This is an obsolete option that has been replaced by \texttt{legend} described above.
gnuplot_curve_styles
   [Plot option]
   This is an obsolete option that has been replaced by style.

12.6 Gnuplot_pipes Format Functions

gnuplot_start()
   [Function]
   Opens the pipe to gnuplot used for plotting with the gnuplot_pipes format. Is not necessary to manually open the pipe before plotting.

gnuplot_close()
   [Function]
   Closes the pipe to gnuplot which is used with the gnuplot_pipes format.

gnuplot_send(s)
   [Function]
   Sends the command s to the gnuplot pipe. If that pipe is not currently opened, it will be opened before sending the command. s must be a string.

gnuplot_restart()
   [Function]
   Closes the pipe to gnuplot which is used with the gnuplot_pipes format and opens a new pipe.

gnuplot_replot
   gnuplot_replot()
   gnuplot_replot(s)
   Updates the gnuplot window. If gnuplot_replot is called with a gnuplot command in a string s, then s is sent to gnuplot before reploting the window.

gnuplot_reset()
   [Function]
   Resets the state of gnuplot used with the gnuplot_pipes format. To update the gnuplot window call gnuplot_replot after gnuplot_reset.
13 File Input and Output

13.1 Comments

A comment in Maxima input is any text between /* and */.

The Maxima parser treats a comment as whitespace for the purpose of finding tokens in the input stream; a token always ends at a comment. An input such as a/* foo */b contains two tokens, a and b, and not a single token ab. Comments are otherwise ignored by Maxima; neither the content nor the location of comments is stored in parsed input expressions.

Comments can be nested to arbitrary depth. The /* and */ delimiters form matching pairs. There must be the same number of /* as there are */.

Examples:

(%i1) /* aa is a variable of interest */  aa : 1234;
(%o1) 1234
(%i2) /* Value of bb depends on aa */  bb : aa^2;
(%o2) 1522756
(%i3) /* User-defined infix operator */  infix ("b");
(%o3) b
(%i4) /* Parses same as a b c, not abc */  a/* foo */b/* bar */c;
(%o4) a b c
(%i5) /* Comments /* can be nested /* to any depth */ */ */  1 + xyz;
(%o5) xyz + 1

13.2 Files

A file is simply an area on a particular storage device which contains data or text. Files on the disks are figuratively grouped into "directories". A directory is just a list of files. Commands which deal with files are:

appendfile  batch  batchload
closefile    file_output_append  filename_merge
file_search  file_search_maxima  file_search_lisp
file_search_demo  file_search_usage  file_search_tests
file_type  file_type_lisp  file_type_maxima
load  load_pathname  loadfile
loadprint  pathname_directory  pathname_name
pathname_type  printfile  save
stringout  with_stdout  writefile

When a file name is passed to functions like plot2d, save, or writefile and the file name does not include a path, Maxima stores the file in the current working directory. The current working directory depends on the system like Windows or Linux and on the installation.
### 13.3 Functions and Variables for File Input and Output

**appendfile (filename)**  
Appends a console transcript to `filename`. `appendfile` is the same as `writefile`, except that the transcript file, if it exists, is always appended.  
`closefile` closes the transcript file opened by `appendfile` or `writefile`.

**batch**  
- `batch (filename)`  
- `batch (filename, option)`  
- `batch (S)`  
- `batch (S, option)`

`batch(filename)` reads Maxima expressions from `filename` and evaluates them. `batch` searches for `filename` in the list `file_search_maxima`. See also `file_search`.  
`batch(S)` reads Maxima expressions from the input stream `S` as created by `openr`. The behavior of `batch` in this case is the same as if the input were a file name, and in the remainder of this description, what is said about input files applies to input streams as well, except that the comments about searching for files do not apply to streams.

`batch(filename, demo)` is like `demo(filename)`. In this case `batch` searches for `filename` in the list `file_search_demo`. See `demo`.

`batch(filename, test)` is like `run_testsuite` with the option `display_all=true`. For this case `batch` searches `filename` in the list `file_search_maxima` and not in the list `file_search_tests` like `run_testsuite`. Furthermore, `run_testsuite` runs tests which are in the list `testsuite_files`. With `batch` it is possible to run any file in a test mode, which can be found in the list `file_search_maxima`. This is useful, when writing a test file.

`filename` comprises a sequence of Maxima expressions, each terminated with ; or $. The special variable `%` and the function `%th` refer to previous results within the file. The file may include :lisp constructs. Spaces, tabs, and newlines in the file are ignored. A suitable input file may be created by a text editor or by the `stringout` function.

`batch` reads each input expression from `filename`, displays the input to the console, computes the corresponding output expression, and displays the output expression. Input labels are assigned to the input expressions and output labels are assigned to the output expressions. `batch` evaluates every input expression in the file unless there is an error. If user input is requested (by `asksign` or `askinteger`, for example) `batch` pauses to collect the requisite input and then continue.

It may be possible to halt `batch` by typing `control-C` at the console. The effect of `control-C` depends on the underlying Lisp implementation.

`batch` has several uses, such as to provide a reservoir for working command lines, to give error-free demonstrations, or to help organize one’s thinking in solving complex problems.

`batch` evaluates its arguments.
When called with no second argument or with the option demo, batch returns the path of filename, if the argument is a file name, or the path of the file for which the input stream was opened, if the argument is a file input stream. If the argument is a string input stream, a representation of the input stream is returned.

When called with the option test, the return value is a an empty list [] or a list with filename and the numbers of the tests which have failed.

See also load, batchload, and demo.

**batchload**

```
batchload (filename)
batchload (S)
```

Reads Maxima expressions from input file filename or input stream S and evaluates them, without displaying the input or output expressions and without assigning labels to output expressions. Printed output (such as produced by print or describe)) is displayed, however.

The special variable % and the function %th refer to previous results from the interactive interpreter, not results within the file. The file cannot include :lisp constructs.

batchload evaluates its argument.

batchload returns the path of filename, if the argument is a file name, or the path of the file for which the input stream was opened, if the argument is a file input stream. If the argument is a string input stream, a representation of the input stream is returned.

See also batch, and load.

**closefile ()**

Closes the transcript file opened by writefile or appendfile.

**file_output_append**

[Option variable]

Default value: false

file_output_append governs whether file output functions append or truncate their output file. When file_output_append is true, such functions append to their output file. Otherwise, the output file is truncated.

save, stringout, and with_stdout respect file_output_append. Other functions which write output files do not respect file_output_append. In particular, plotting and translation functions always truncate their output file, and tex and appendfile always append.

**filename_merge (path, filename)**

[Function]

Constructs a modified path from path and filename. If the final component of path is of the form ###.something, the component is replaced with filename.something. Otherwise, the final component is simply replaced by filename.

The result is a Lisp pathname object.

**file_search**

[Function]

```
file_search (filename)
file_search (filename, pathlist)
```

file_search searches for the file filename and returns the path to the file (as a string) if it can be found; otherwise file_search returns false. file_search (filename)
searches in the default search directories, which are specified by the `file_search_maxima`, `file_search_lisp`, and `file_search_demo` variables.

`file_search` first checks if the actual name passed exists, before attempting to match it to “wildcard” file search patterns. See `file_search_maxima` concerning file search patterns.

The argument `filename` can be a path and file name, or just a file name, or, if a file search directory includes a file search pattern, just the base of the file name (without an extension). For example,

```lisp
file_search ("/home/wfs/special/zeta.mac");
file_search ("zeta.mac");
file_search ("zeta");
```

all find the same file, assuming the file exists and `/home/wfs/special/###.mac` is in `file_search_maxima`.

`file_search (filename, pathlist)` searches only in the directories specified by `pathlist`, which is a list of strings. The argument `pathlist` supersedes the default search directories, so if the path list is given, `file_search` searches only the ones specified, and not any of the default search directories. Even if there is only one directory in `pathlist`, it must still be given as an one-element list.

The user may modify the default search directories. See `file_search_maxima`.

`file_search` is invoked by `load` with `file_search_maxima` and `file_search_lisp` as the search directories.

```lisp
file_search_maxima [Option variable]
file_search_lisp [Option variable]
file_search_demo [Option variable]
file_search_usage [Option variable]
file_search_tests [Option variable]
```

These variables specify lists of directories to be searched by `load`, `demo`, and some other Maxima functions. The default values of these variables name various directories in the Maxima installation.

The user can modify these variables, either to replace the default values or to append additional directories. For example,

```lisp
file_search_maxima: ["/usr/local/foo/###.mac",
                     "/usr/local/bar/###.mac"]$
```

replaces the default value of `file_search_maxima`, while

```lisp
file_search_maxima: append (file_search_maxima,
                         ["/usr/local/foo/###.mac", "/usr/local/bar/###.mac"])$
```

appends two additional directories. It may be convenient to put such an expression in the file `maxima-init.mac` so that the file search path is assigned automatically when Maxima starts. See also Section 32.1 [Introduction for Runtime Environment], page 577.

Multiple filename extensions and multiple paths can be specified by special “wildcard” constructions. The string `###` expands into the sought-after name, while a comma-separated list enclosed in curly braces `{foo,bar,baz}` expands into multiple strings. For example, supposing the sought-after name is `neumann`,

```lisp
file_search ("zeta, zeta.mac, zeta.mac", "/home/wfs/special")$
```
"/home/{wfs,gcj}###.{lisp,mac}"
expands into /home/wfs/neumann.lisp, /home/gcj/neumann.lisp,
/home/wfs/neumann.mac, and /home/gcj/neumann.mac.

file_type (filename) [Function]
Returns a guess about the content of filename, based on the filename extension.
filename need not refer to an actual file; no attempt is made to open the file and
inspect the content.
The return value is a symbol, either object, lisp, or maxima. If the extension is
matches one of the values in file_type_maxima, file_type returns maxima. If the
extension matches one of the values in file_type_lisp, file_type returns lisp. If
none of the above, file_type returns object.
See also pathname_type.
See file_type_maxima and file_type_lisp for the default values.
Examples:
(%i2) map('file_type,
   ["test.lisp", "test.mac", "test.dem", "test.txt"])
(%o2) [lisp, maxima, maxima, object]

file_type_lisp [Option variable]
Default value: [l, lsp, lisp]
file_type_lisp is a list of file extensions that maxima recognizes as denoting a Lisp
source file.
See also file_type.

file_type_maxima [Option variable]
Default value: [mac, mc, demo, dem, dm1, dm2, dm3, dmt, wxm]
file_type_maxima is a list of file extensions that maxima recognizes as denoting a
Maxima source file.
See also file_type.

load (filename) [Function]
Evaluates expressions in filename, thus bringing variables, functions, and other objects
into Maxima. The binding of any existing object is clobbered by the binding recovered
from filename.
filename must be a string, symbol, or Lisp pathname (as created by filename_merge). To find the file, load calls file_search with file_search_maxima and
file_search_lisp as the search directories. If load succeeds, it returns the name of the
file. Otherwise load prints an error message.
load works equally well for Lisp code and Maxima code. Files created by save,
translate_file, and compile_file, which create Lisp code, and stringout, which
creates Maxima code, can all be processed by load. load calls loadfile to load Lisp
files and batchload to load Maxima files.
load does not recognize :lisp constructs in Maxima files, and while processing
filename, the global variables _, __, %, and %th have whatever bindings they had
when load was called.
Note also that structures will only be read back as structures if they have been defined by \texttt{defstruct} before the \texttt{load} command is called.

See also \texttt{loadfile}, for Lisp files; and \texttt{batch}, \texttt{batchload}, and \texttt{demo}. for Maxima files. See \texttt{file_search} for more detail about the file search mechanism. The \texttt{numericalio} chapter describes many functions for loading csv and other data files.

During Maxima file loading, the variable \texttt{load_pathname} is bound to the pathname of the file being loaded.

\texttt{load} evaluates its argument.

\textbf{load_pathname} \hspace{1cm} [System variable]

Default value: \texttt{false}

When a file is loaded with the functions \texttt{load}, \texttt{loadfile} or \texttt{batchload} the system variable \texttt{load_pathname} is bound to the pathname of the file which is processed.

The variable \texttt{load_pathname} can be accessed from the file during the loading.

Example:

Suppose we have a batchfile \texttt{test.mac} in the directory 

"/home/dieter/workspace/mymaxima/temp/" with the following commands

```lisp
print("The value of load_pathname is: ", load_pathname)$
print("End of batchfile")$
```

then we get the following output

```lisp
(\%i1) load("/home/dieter/workspace/mymaxima/temp/test.mac")$
The value of load_pathname is:

/home/dieter/workspace/mymaxima/temp/test.mac

End of batchfile
```

\textbf{loadfile} \hspace{1cm} (filename) \hspace{1cm} [Function]

Evaluates Lisp expressions in \texttt{filename}. \texttt{loadfile} does not invoke \texttt{file_search}, so \texttt{filename} must include the file extension and as much of the path as needed to find the file.

\texttt{loadfile} can process files created by \texttt{save}, \texttt{translate_file}, and \texttt{compile_file}. The user may find it more convenient to use \texttt{load} instead of \texttt{loadfile}.

\textbf{loadprint} \hspace{1cm} [Option variable]

Default value: \texttt{true}

\texttt{loadprint} tells whether to print a message when a file is loaded.

- When \texttt{loadprint} is \texttt{true}, always print a message.
- When \texttt{loadprint} is \texttt{loadfile}, print a message only if a file is loaded by the function \texttt{loadfile}.
- When \texttt{loadprint} is \texttt{autoload}, print a message only if a file is automatically loaded. See \texttt{setup_autoload}.
- When \texttt{loadprint} is \texttt{false}, never print a message.
directory (path)  [Function]
Returns a list of the files and directories found in path in the file system.
path may contain wildcard characters (i.e., characters which represent unspecified parts of the path), which include at least the asterisk on most systems, and possibly other characters, depending on the system.
directory relies on the Lisp function DIRECTORY, which may have implementation-specific behavior.

pathname_directory (pathname)  [Function]
pathname_name (pathname)  [Function]
pathname_type (pathname)  [Function]
These functions return the components of pathname.
Examples:
(\%i1) pathname_directory("/home/dieter/maxima/changelog.txt");
(\%o1) /home/dieter/maxima/
(\%i2) pathname_name("/home/dieter/maxima/changelog.txt");
(\%o2) changelog
(\%i3) pathname_type("/home/dieter/maxima/changelog.txt");
(\%o3) txt

printfile (path)  [Function]
Prints the file named by path to the console. path may be a string or a symbol; if it is a symbol, it is converted to a string.
If path names a file which is accessible from the current working directory, that file is printed to the console. Otherwise, printfile attempts to locate the file by appending path to each of the elements of file_search_usage via filename_merge.
printfile returns path if it names an existing file, or otherwise the result of a successful filename merge.

save  [Function]

save (filename, name_1, name_2, name_3, ...)  save (filename, values, functions, labels, ...)  save (filename, [m, n])
save (filename, name_1=expr_1, ...)  save (filename, all)
save (filename, name_1=expr_1, name_2=expr_2, ...)
Stores the current values of name_1, name_2, name_3, ..., in filename. The arguments are the names of variables, functions, or other objects. If a name has no value or function associated with it, it is ignored. save returns filename.
save stores data in the form of Lisp expressions. If filename ends in .lisp the data stored by save may be recovered by load (filename). See load.
The global flag file_output_append governs whether save appends or truncates the output file. When file_output_append is true, save appends to the output file. Otherwise, save truncates the output file. In either case, save creates the file if it does not yet exist.
The special form `save (filename, values, functions, labels, ...)` stores the items named by `values`, `functions`, `labels`, etc. The names may be any specified by the variable `infolists`. `values` comprises all user-defined variables.

The special form `save (filename, [m, n])` stores the values of input and output labels m through n. Note that m and n must be literal integers. Input and output labels may also be stored one by one, e.g., `save ("foo.1", %i42, %o42)`. `save (filename, labels)` stores all input and output labels. When the stored labels are recovered, they clobber existing labels.

The special form `save (filename, name_1=expr_1, name_2=expr_2, ...)` stores the values of `expr_1`, `expr_2`, ..., with names `name_1`, `name_2`, ... It is useful to apply this form to input and output labels, e.g., `save ("foo.1", aa=%o88)`. The right-hand side of the equality in this form may be any expression, which is evaluated. This form does not introduce the new names into the current Maxima environment, but only stores them in `filename`.

These special forms and the general form of `save` may be mixed at will. For example, `save (filename, aa, bb, cc=42, functions, [11, 17])`.

The special form `save (filename, all)` stores the current state of Maxima. This includes all user-defined variables, functions, arrays, etc., as well as some automatically defined items. The saved items include system variables, such as `file_search_maxima` or `showtime`, if they have been assigned new values by the user; see `myoptions`.

`save` evaluates `filename` and quotes all other arguments.

**stringout**

*Function*

- `stringout (filename, expr_1, expr_2, expr_3, ...)`
- `stringout (filename, [m, n])`
- `stringout (filename, input)`
- `stringout (filename, functions)`
- `stringout (filename, values)`

`stringout` writes expressions to a file in the same form the expressions would be typed for input. The file can then be used as input for the `batch` or `demo` commands, and it may be edited for any purpose. `stringout` can be executed while `writefile` is in progress.

The global flag `file_output_append` governs whether `stringout` appends or truncates the output file. When `file_output_append` is `true`, `stringout` appends to the output file. Otherwise, `stringout` truncates the output file. In either case, `stringout` creates the file if it does not yet exist.

The general form of `stringout` writes the values of one or more expressions to the output file. Note that if an expression is a variable, only the value of the variable is written and not the name of the variable. As a useful special case, the expressions may be input labels (%i1, %i2, %i3, ...) or output labels (%o1, %o2, %o3, ...).

If `grind` is `true`, `stringout` formats the output using the `grind` format. Otherwise the `string` format is used. See `grind` and `string`.

The special form `stringout (filename, [m, n])` writes the values of input labels m through n, inclusive.
The special form `stringout (filename, input)` writes all input labels to the file.

The special form `stringout (filename, functions)` writes all user-defined functions (named by the global list `functions`) to the file.

The special form `stringout (filename, values)` writes all user-assigned variables (named by the global list `values`) to the file. Each variable is printed as an assignment statement, with the name of the variable, a colon, and its value. Note that the general form of `stringout` does not print variables as assignment statements.

```lisp
[Function]
with_stdout
  with_stdout (f, expr_1, expr_2, expr_3, ...)
  with_stdout (s, expr_1, expr_2, expr_3, ...)
```

Evaluates `expr_1, expr_2, expr_3, ...` and writes any output thus generated to a file `f` or output stream `s`. The evaluated expressions are not written to the output. Output may be generated by `print, display, grind`, among other functions.

The global flag `file_output_append` governs whether `with_stdout` appends or truncates the output file `f`. When `file_output_append` is `true`, `with_stdout` appends to the output file. Otherwise, `with_stdout` truncates the output file. In either case, `with_stdout` creates the file if it does not yet exist.

`with_stdout` returns the value of its final argument.

See also `writefile` and `display2d`.

```lisp
(%i1) with_stdout ("tmp.out", for i:5 thru 10 do
    print (i, "! yields", i!))$

5 ! yields 120
6 ! yields 720
7 ! yields 5040
8 ! yields 40320
9 ! yields 362880
10 ! yields 3628800
```

```lisp
[Function]
writefile (filename)
```

Begins writing a transcript of the Maxima session to `filename`. All interaction between the user and Maxima is then recorded in this file, just as it appears on the console. As the transcript is printed in the console output format, it cannot be reloaded into Maxima. To make a file containing expressions which can be reloaded, see `save` and `stringout`. `save` stores expressions in Lisp form, while `stringout` stores expressions in Maxima form.

The effect of executing `writefile` when `filename` already exists depends on the underlying Lisp implementation; the transcript file may be clobbered, or the file may be appended. `appendfile` always appends to the transcript file.

It may be convenient to execute `playback` after `writefile` to save the display of previous interactions. As `playback` displays only the input and output variables ( `%i1, %o1`, etc.), any output generated by a print statement in a function (as opposed to a return value) is not displayed by `playback`.

`closefile` closes the transcript file opened by `writefile` or `appendfile`. 
13.4 Functions and Variables for TeX Output

Note that the built-in TeX output functionality of wxMaxima makes no use of the functions described here but uses its own implementation instead.

\texttt{\textbf{tex}}

\texttt{\textbf{tex (expr)}}
\texttt{\textbf{tex (expr, destination)}}
\texttt{\textbf{tex (expr, false)}}
\texttt{\textbf{tex (label)}}
\texttt{\textbf{tex (label, destination)}}
\texttt{\textbf{tex (label, false)}}

Prints a representation of an expression suitable for the TeX document preparation system. The result is a fragment of a document, which can be copied into a larger document but not processed by itself.

\texttt{\textbf{tex (expr)}} prints a TeX representation of \texttt{expr} on the console.
\texttt{\textbf{tex (label)}} prints a TeX representation of the expression named by \texttt{label} and assigns it an equation label (to be displayed to the left of the expression). The TeX equation label is the same as the Maxima label.

\texttt{\textbf{destination}} may be an output stream or file name. When \texttt{destination} is a file name, \texttt{tex} appends its output to the file. The functions \texttt{openw} and \texttt{opena} create output streams.

\texttt{\textbf{tex (expr, false)}} and \texttt{\textbf{tex (label, false)}} return their TeX output as a string. \texttt{tex} evaluates its first argument after testing it to see if it is a label. Quote-quote \texttt{''} forces evaluation of the argument, thereby defeating the test and preventing the label.

See also \texttt{\textbf{tex1}} and \texttt{\textbf{texput}}.

Examples:

\begin{verbatim}
(%i1) integrate (1/(1+x^3), x);
2 x - 1
log(x - x + 1)  atan(------)
sqrt(3) 3
- --------------- + ------------- + ----------
6 sqrt(3) 
(%o1)
(%i2) tex (%o1);
\$-$-\left(\frac{1}{6}\right)\arctan\left(\frac{2 x - 1}{\sqrt{3}}\right)\log(x + 1)\log(x - x + 1)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)\\lt\tt (\%o1)\$\$
(%i3) tex (integrate (sin(x), x));
\$-$-\cos x\$\$
(%i4) tex (%o1, "foo.tex");
(%o4) \$-$-\left(\frac{1}{6}\right)\arctan\left(\frac{2 x - 1}{\sqrt{3}}\right)\log(x + 1)\log(x - x + 1)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)\\lt\tt (\%o1)\$\$
\end{verbatim}

\texttt{\textbf{tex (expr, false)}} returns its TeX output as a string.

\begin{verbatim}
(%i11) S : tex (x * y * z, false);
\end{verbatim}
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(%o1) $x\,y\,z$
(%i2) S;
(%o2) $x\,y\,z$

**tex1 (e)**
Returns a string which represents the TeX output for the expressions e. The TeX output is not enclosed in delimiters for an equation or any other environment.

See also **tex** and **texput**.

Examples:

(%i1) tex1 (sin(x) + cos(x));
(%o1) \sin x+\cos x

**texput**
Assign the TeX output for the atom a, which can be a symbol or the name of an operator.

**texput (a, s)** causes the **tex** function to interpolate the string s into the TeX output in place of a.

**texput (a, f)** causes the **tex** function to call the function f to generate TeX output. f must accept one argument, which is an expression which has operator a, and must return a string (the TeX output). f may call **tex1** to generate TeX output for the arguments of the input expression.

**texput (a, s, operator_type)**, where **operator_type** is **prefix**, **infix**, **postfix**, **nary**, or **nofix**, causes the **tex** function to interpolate s into the TeX output in place of a, and to place the interpolated text in the appropriate position.

**texput (a, [s_1, s_2], matchfix)** causes the **tex** function to interpolate s_1 and s_2 into the TeX output on either side of the arguments of a. The arguments (if more than one) are separated by commas.

**texput (a, [s_1, s_2, s_3], matchfix)** causes the **tex** function to interpolate s_1 and s_2 into the TeX output on either side of the arguments of a, with s_3 separating the arguments.

See also **tex** and **tex1**.

Examples:

Assign TeX output for a variable.

(%i1) texput (me, "\mu_e");
(%o1) \mu_e
(%i2) tex (me);
\$\mu_e\$
(%o2) false

Assign TeX output for an ordinary function (not an operator).

(%i1) texput (lcm, "\text{lcm}");
(%i1) \text{lcm}
(%i2) tex (\text{lcm} (a, b));
$$\text{lcm}(a, b)$$
(%o2) false

Call a function to generate TeX output.

(%i1) texfoo (e) := block ([a, b], [a, b] : args (e),
conc("\left[\stackrel{tex1(b)}{tex1(a)}\right]"))$
(%i2) texput (foo, texfoo);
(%o2) texfoo
(%i3) tex (foo (2^x, \pi));
$$\left[\stackrel{\pi}{2^x}\right]$$
(%o3) false

Assign TeX output for a prefix operator.

(%i1) prefix ("\text{grad}");
(%o1) \text{grad}
(%i2) texput ("\text{grad}", "\nabla", prefix);
(%o2) \nabla
(%i3) tex (\text{grad} f);
$$\nabla f$$
(%o3) false

Assign TeX output for an infix operator.

(%i1) infix ("\times");
(%o1) \times
(%i2) texput ("\times", "\times", infix);
(%o2) \times
(%i3) tex (a \times b);
$$a \times b$$
(%o3) false

Assign TeX output for a postfix operator.

(%i1) postfix ("!!");
(%o1) !!
(%i2) texput ("!!", "!!", postfix);
(%o2) !!
(%i3) tex (x !!);
$$x!!$$
(%o3) false

Assign TeX output for a nary operator.

(%i1) nary ("@@");
(%o1) @@
(%i2) texput ("@@", "\circ", nary);
(%o2) \circ
(%i3) tex (a @@ b @@ c @@ d);
$$a \circ b \circ c \circ d$$
(%o3) false
Assign TeX output for a nofix operator.

```
(%i1) nofix ("foo");
(%o1) foo
(%i2) texput ("foo", \textsc{foo}, nofix);
(%o2) \textsc{foo}
(%i3) tex (foo);
$$\textsc{foo}$$
(%o3) false
```

Assign TeX output for a matchfix operator.

```
(%i1) matchfix ("<<", ">>");
(%o1) <<
(%i2) texput ("<<", [" \langle ", " \rangle "], matchfix);
(%o2) [ \langle , \rangle ]
(%i3) tex (<<a>>);
$$ \langle a \rangle $$
(%o3) false
(%i4) tex (<<a, b>>);
$$ \langle a \rangle $$
(%o4) false
(%i5) texput ("<<", [" \langle ", " \rangle ", " \, | \, ", matchfix);
(%o5) [ \langle , \rangle , \, | \, ]
(%i6) tex (<<a>>);
$$ \langle a \rangle $$
(%o6) false
(%i7) tex (<<a, b>>);
$$ \langle a \rangle $$
(%o7) false
```

\begin{verbatim}
get_tex_environment (op) [Function]
set_tex_environment (op, before, after) [Function]
\end{verbatim}

Customize the TeX environment output by \texttt{tex}. As maintained by these functions, the TeX environment comprises two strings: one is printed before any other TeX output, and the other is printed after.

Only the TeX environment of the top-level operator in an expression is output; TeX environments associated with other operators are ignored.

\texttt{get_tex_environment} returns the TeX environment which is applied to the operator \texttt{op}; returns the default if no other environment has been assigned.

\texttt{set_tex_environment} assigns the TeX environment for the operator \texttt{op}.

Examples:

```
(%i1) get_tex_environment (":=");
(%o1) [
\begin{verbatim}

\end{verbatim}
```
(\%i2) tex (f (x) := 1 - x);

\begin{verbatim}
f(x):=1-x;
\end{verbatim}

(%o2) false

(%i3) set_tex_environment (":="; \\
\begin{equation}
\end{equation});

(%o3) \begin{equation}
\end{equation}

(%i4) tex (f (x) := 1 - x);

\begin{equation}
\end{equation}

(%o4) false

get_tex_environment_default ()

set_tex_environment_default (before, after)

Customize the TeX environment output by tex. As maintained by these functions, the TeX environment comprises two strings: one is printed before any other TeX output, and the other is printed after.

get_tex_environment_default returns the TeX environment which is applied to expressions for which the top-level operator has no specific TeX environment (as assigned by set_tex_environment).

set_tex_environment_default assigns the default TeX environment.

Examples:

(%i1) get_tex_environment_default ();
(%o1) \begin{equation}
\end{equation}

(%i2) tex (f(x) + g(x));

\begin{equation}
\end{equation}

(%o2) false

(%i3) set_tex_environment_default ("\begin{equation}
\end{equation}");

(%o3) \begin{equation}
\end{equation}

{%i4} tex (f(x) + g(x));

\begin{equation}
\end{equation}

(\%o4) false

13.5 Functions and Variables for Fortran Output

fortindent

Default value: 0
fortindent controls the left margin indentation of expressions printed out by the fortran command. 0 gives normal printout (i.e., 6 spaces), and positive values will causes the expressions to be printed farther to the right.

fortran (expr) [Function]
Prints expr as a Fortran statement. The output line is indented with spaces. If the line is too long, fortran prints continuation lines. fortran prints the exponentiation operator ^ as **, and prints a complex number a + b %i in the form (a,b).

expr may be an equation. If so, fortran prints an assignment statement, assigning the right-hand side of the equation to the left-hand side. In particular, if the right-hand side of expr is the name of a matrix, then fortran prints an assignment statement for each element of the matrix.

If expr is not something recognized by fortran, the expression is printed in grind format without complaint. fortran does not know about lists, arrays, or functions.

fortindent controls the left margin of the printed lines. 0 is the normal margin (i.e., indented 6 spaces). Increasing forindent causes expressions to be printed further to the right.

When fortspaces is true, fortran fills out each printed line with spaces to 80 columns.

fortran evaluates its arguments; quoting an argument defeats evaluation. fortran always returns done.

See also the function [function_f90], page 971 for printing one or more expressions as a Fortran 90 program.

Examples:

(%i1) expr: (a + b)^12$
(%i2) fortran (expr);
    (b+a)**12
    done
(%i3) fortran ('x=expr);
    x = (b+a)**12
    done
(%i4) fortran ('x=expand (expr));
    x = b**12+12*a*b**11+66*a**2*b**10+220*a**3*b**9+495*a**4*b**8+792
1   *a**5*b**7+924*a**6*b**6+792*a**7*b**5+495*a**8*b**4+220*a**9*b
2    **3+66*a**10*b**2+12*a**11*b+a**12
    done
(%i5) fortran ('x=7+5*%i);
    x = (7,5)
    done
(%i6) fortran ('x=[1,2,3,4]);
    x = [1,2,3,4]
    done
(%i7) f(x) := x^2$
(%i8) fortran (f);
    f
    done
fortspaces

Default value: false

When fortpaces is true, fortran fills out each printed line with spaces to 80 columns.
14 Polynomials

14.1 Introduction to Polynomials

Polynomials are stored in Maxima either in General Form or as Canonical Rational Expressions (CRE) form. The latter is a standard form, and is used internally by operations such as \texttt{factor}, \texttt{ratsimp}, and so on.

Canonical Rational Expressions constitute a kind of representation which is especially suitable for expanded polynomials and rational functions (as well as for partially factored polynomials and rational functions when \texttt{ratfac} is set to \texttt{true}). In this CRE form an ordering of variables (from most to least main) is assumed for each expression.

Polynomials are represented recursively by a list consisting of the main variable followed by a series of pairs of expressions, one for each term of the polynomial. The first member of each pair is the exponent of the main variable in that term and the second member is the coefficient of that term which could be a number or a polynomial in another variable again represented in this form. Thus the principal part of the CRE form of $3x^2 - 1$ is $(x 2 3 0 -1)$ and that of $2xy + x - 3$ is $(y 1 (x 1 2) 0 (x 1 1 0 -3))$ assuming $y$ is the main variable, and is $(x 1 (y 1 2 0 1) 0 -3)$ assuming $x$ is the main variable. "Main"-ness is usually determined by reverse alphabetical order.

The "variables" of a CRE expression needn’t be atomic. In fact any subexpression whose main operator is not +, -, *, / or \^ with integer power will be considered a "variable" of the expression (in CRE form) in which it occurs. For example the CRE variables of the expression $x + \sin(x+1) + 2\sqrt{x} + 1$ are $x$, $\sqrt{x}$, and $\sin(x+1)$. If the user does not specify an ordering of variables by using the \texttt{ratvars} function Maxima will choose an alphabetic one.

In general, CRE’s represent rational expressions, that is, ratios of polynomials, where the numerator and denominator have no common factors, and the denominator is positive. The internal form is essentially a pair of polynomials (the numerator and denominator) preceded by the variable ordering list. If an expression to be displayed is in CRE form or if it contains any subexpressions in CRE form, the symbol \texttt{/R/} will follow the line label.

See the \texttt{rat} function for converting an expression to CRE form.

An extended CRE form is used for the representation of Taylor series. The notion of a rational expression is extended so that the exponents of the variables can be positive or negative rational numbers rather than just positive integers and the coefficients can themselves be rational expressions as described above rather than just polynomials. These are represented internally by a recursive polynomial form which is similar to and is a generalization of CRE form, but carries additional information such as the degree of truncation. As with CRE form, the symbol \texttt{/T/} follows the line label of such expressions.

14.2 Functions and Variables for Polynomials

\textbf{algebraic} \hspace{2cm} \textbf{[Option variable]}

Default value: \texttt{false}

\texttt{algebraic} must be set to \texttt{true} in order for the simplification of algebraic integers to take effect.
**berlefact**

[Option variable]

Default value: true

When berlefact is false then the Kronecker factoring algorithm will be used otherwise the Berlekamp algorithm, which is the default, will be used.

**bezout** \((p1, p2, x)\)

[Function]

an alternative to the resultant command. It returns a matrix. determinant of this matrix is the desired resultant.

Examples:

\[
\begin{align*}
(\%i1) & \text{bezout}(a*x+b, c*x^2+d, x); \\
(\%o1) & \begin{bmatrix} b & c & -a & d \\ a & b \end{bmatrix} \\
(\%i2) & \text{determinant}(\%); \\
(\%o2) & 2^2 a d + b c \\
(\%i3) & \text{resultant}(a*x+b, c*x^2+d, x); \\
(\%o3) & 2^2 a d + b c
\end{align*}
\]

**bothcoef** \((expr, x)\)

[Function]

Returns a list whose first member is the coefficient of \(x\) in expr (as found by ratcoef if expr is in CRE form otherwise by coeff) and whose second member is the remaining part of expr. That is, \([A, B]\) where \(expr = A*x + B\).

Example:

\[
\begin{align*}
(\%i1) & \text{islinear} (expr, x) := \text{block} ([c], \\
& c: \text{bothcoef} (\text{rat} (expr, x), x), \\
& \text{is} (\text{freeof} (x, c) \text{ and } c[1] \neq 0))$
\end{align*}
\]

\[
\begin{align*}
(\%i2) & \text{islinear} ((r^2 - (x - r)^2)/x, x); \\
(\%o2) & \text{true}
\end{align*}
\]

**coeff**

[Function]

- coeff \((expr, x, n)\)
- coeff \((expr, x)\)

Returns the coefficient of \(x^n\) in expr, where expr is a polynomial or a monomial term in x. Other than ratcoef coeff is a strictly syntactical operation and will only find literal instances of \(x^n\) in the internal representation of expr.

\(\text{coeff}(expr, x^n)\) is equivalent to \(\text{coeff}(expr, x, n)\). coeff\((expr, x, 0)\) returns the remainder of expr which is free of x. If omitted, n is assumed to be 1.

\(x\) may be a simple variable or a subscripted variable, or a subexpression of expr which comprises an operator and all of its arguments.

It may be possible to compute coefficients of expressions which are equivalent to expr by applying expand or factor. coeff itself does not apply expand or factor or any other function.

coeff distributes over lists, matrices, and equations.

See also ratcoef.
Examples:

`coeff` returns the coefficient $x^n$ in `expr`.

```lisp
(%i1) coeff (b^3*a^3 + b^2*a^2 + b*a + 1, a^3); 3
(%o1) b
```

`coeff(expr, x^n)` is equivalent to `coeff(expr, x, n)`.

```lisp
(%i1) coeff (c[4]*z^4 - c[3]*z^3 - c[2]*z^2 + c[1]*z, z, 3);
(%o1) - c 3
(%i2) coeff (c[4]*z^4 - c[3]*z^3 - c[2]*z^2 + c[1]*z, z^3);
(%o2) - c 3
```

`coeff(expr, x, 0)` returns the remainder of `expr` which is free of `x`.

```lisp
(%i1) coeff (a*u + b^2*u^2 + c^3*u^3, b, 0); 3 3
(%o1) c u + a u
```

$x$ may be a simple variable or a subscripted variable, or a subexpression of `expr` which comprises an operator and all of its arguments.

```lisp
(%i1) coeff (h^4 - 2*%pi*h^2 + 1, h, 2);
(%o1) - 2 %pi
(%i2) coeff (v[1]^4 - 2*%pi*v[1]^2 + 1, v[1], 2);
(%o2) - 2 %pi
(%i3) coeff (sin(1+x)*sin(x) + sin(1+x)^3*sin(x)^3, sin(1+x)^3);
(%o3) sin(x)
(%i4) coeff ((d - a)^2*(b + c)^3 + (a + b)^4*(c - d), a + b, 4);
(%o4) c - d
```

`coeff` itself does not apply `expand` or `factor` or any other function.

```lisp
(%i1) coeff (c*(a + b)^3, a);
(%o1) 0
(%i2) expand (c*(a + b)^3);
(%o2) b c + 3 a b c + 3 a^2 b c + a^3 c
(%i3) coeff (%, a);
(%o3) 2
(%i4) coeff (b^3*c + 3*a*b^2*c + 3*a^2*b*c + a^3*c, (a + b)^3);
(%o4) 0
(%i5) factor (b^3*c + 3*a*b^2*c + 3*a^2*b*c + a^3*c);
(%o5) (b + a) c
(%i6) coeff (%, (a + b)^3);
(%o6) c
```

`coeff` distributes over lists, matrices, and equations.
content \((p_1, x_1, \ldots, x_n)\)

Returns a list whose first element is the greatest common divisor of the coefficients of the terms of the polynomial \(p_1\) in the variable \(x_n\) (this is the content) and whose second element is the polynomial \(p_1\) divided by the content.

Examples:

\[
(\%i1) \quad \text{content} \left(2x y + 4x^{2}y^{2}, y\right);
\]
\[
(\%o1) \quad \left[2, 2x y + y\right]
\]

denom \((expr)\)

Returns the denominator of the rational expression \(expr\).

See also num

\[
(\%i1) \quad g1: (x+2)*(x+1)/((x+3)^{2});
\]
\[
(\%o1) \quad \frac{(x + 1)(x + 2)}{(x + 3)^2}
\]
\[
(\%i2) \quad \text{denom}(g1);
\]
\[
(\%o2) \quad 2(x + 3)
\]
\[
(\%i3) \quad g2: \sin(x)/10*\cos(x)/y;
\]
\[
(\%o3) \quad \frac{\cos(x) \sin(x)}{10y}
\]
\[
(\%i4) \quad \text{denom}(g2);
\]
\[
(\%o4) \quad 10y
\]

divide \((p_1, p_2, x_1, \ldots, x_n)\)

computes the quotient and remainder of the polynomial \(p_1\) divided by the polynomial \(p_2\), in a main polynomial variable, \(x_n\). The other variables are as in the \texttt{ratvars} function. The result is a list whose first element is the quotient and whose second element is the remainder.

Examples:

\[
(\%i1) \quad \text{divide} \left(x + y, x - y, x\right);
\]
\[
(\%o1) \quad [1, 2y]
\]
\[
(\%i2) \quad \text{divide} \left(x + y, x - y\right);
\]
\[
(\%o2) \quad [-1, 2x]
\]

Note that \(y\) is the main variable in the second example.
eliminate ([eqn_1, ..., eqn_n], [x_1, ..., x_k])

Eliminates variables from equations (or expressions assumed equal to zero) by taking successive resultants. This returns a list of \( n-k \) expressions with the \( k \) variables \( x_1, \ldots, x_k \) eliminated. First \( x_1 \) is eliminated yielding \( n-1 \) expressions, then \( x_2 \) is eliminated, etc. If \( k = n \) then a single expression in a list is returned free of the variables \( x_1, \ldots, x_k \). In this case \texttt{solve} is called to solve the last resultant for the last variable.

Example:

\begin{verbatim}
(%i1) expr1: 2*x^2 + y*x + z;
       2
(%o1) z + x y + 2 x
(%i2) expr2: 3*x + 5*y - z - 1;
       2
(%o2) - z + 5 y + 3 x - 1
(%i3) expr3: z^2 + x - y^2 + 5;
       2 2
(%o3) z - y + x + 5
(%i4) eliminate ([expr3, expr2, expr1], [y, z]);
       8 7 6 5 4
(%o4) [7425 x - 1170 x + 1299 x + 12076 x + 22887 x
            3 2
- 5154 x - 1291 x + 7688 x + 15376]
\end{verbatim}

ezgcd (p_1, p_2, p_3, ...)

Returns a list whose first element is the greatest common divisor of the polynomials \( p_1, p_2, p_3, \ldots \) and whose remaining elements are the polynomials divided by the greatest common divisor. This always uses the \texttt{ezgcd} algorithm.

See also \texttt{gcd}, \texttt{gcdex}, \texttt{gcddivide}, and \texttt{poly_gcd}.

Examples:

The three polynomials have the greatest common divisor \( 2*x - 3 \). The \texttt{gcd} is first calculated with the function \texttt{gcd} and then with the function \texttt{ezgcd}.

\begin{verbatim}
(%i1) p1 : 6*x^3-17*x^2+14*x-3;
        3
(%o1) 6 x - 17 x + 14 x - 3
(%i2) p2 : 4*x^4-14*x^3+12*x^2+2*x-3;
        4
(%o2) 4 x - 14 x + 12 x + 2 x - 3
(%i3) p3 : -8*x^3+14*x^2-x-3;
        3
(%o3) - 8 x + 14 x - x - 3
(%i4) gcd(p1, gcd(p2, p3));
(%o4) 2 x - 3
(%i5) ezgcd(p1, p2, p3);
        2
2 3 2 2
\end{verbatim}
faceexpand [Option variable]

Default value: true

faceexpand controls whether the irreducible factors returned by factor are in expanded (the default) or recursive (normal CRE) form.

factor [Function]

Factors the expression expr, containing any number of variables or functions, into factors irreducible over the integers. factor(expr, p) factors expr over the field of rationals with an element adjoined whose minimum polynomial is p.

factor uses ifactors function for factoring integers.

factorflag if false suppresses the factoring of integer factors of rational expressions.

dontfactor may be set to a list of variables with respect to which factoring is not to occur. (It is initially empty). Factoring also will not take place with respect to any variables which are less important (using the variable ordering assumed for CRE form) than those on the dontfactor list.

savefactors if true causes the factors of an expression which is a product of factors to be saved by certain functions in order to speed up later factorizations of expressions containing some of the same factors.

berlefact if false then the Kronecker factoring algorithm will be used otherwise the Berlekamp algorithm, which is the default, will be used.

intfaclim if true maxima will give up factorization of integers if no factor is found after trial divisions and Pollard’s rho method. If set to false (this is the case when the user calls factor explicitly), complete factorization of the integer will be attempted. The user’s setting of intfaclim is used for internal calls to factor. Thus, intfaclim may be reset to prevent Maxima from taking an inordinately long time factoring large integers.

factor_max_degree if set to a positive integer n will prevent certain polynomials from being factored if their degree in any variable exceeds n.

Examples:

(%i1) factor (2^63 - 1);
   2
(%o1) 7 73 127 337 92737 649657

(%i2) factor (-8*y - 4*x + z^2*(2*y + x));
   2
(%o2) (2 y + x) (z - 2) (z + 2)

(%i3) -1 - 2*x - x^2 + y^2 + 2*x*y^2 + x^2*y^2;
    2 2 2 2
(%o3) x y + 2 x y + y - x - 2 x - 1

(%i4) block ([dontfactor: [x]], factor (%/36/(1 + 2*y + y^2)));
\(\frac{2}{36}\frac{(x + 2)(y - 1)}{(y + 1)}\)

\%(i5)\ factor (1 + \%e^-(3*x));
\%
\(\frac{2}{x^2 x x}\)

\%(i6)\ factor (1 + x^-4, a^-2 - 2);
\%
\(\frac{2}{x^2 x x}\)

\%(i7)\ factor (-y^-2*z^-2 - x*z^-2 + x^-2*y^-2 + x^-3);
\%
\(\frac{2}{x + 2}\)

\%(i8)\ factor (1 + x^4, a^2 - 2);
\%
\(\frac{2}{x^2 x x}\)

\%(i9)\ ratsimp (%);
\%
\(\frac{4}{x + 2}\)

\%(i10)\ partfrac (%); x
\%
\(\frac{2}{x^2 x x}\)
\[
\frac{2}{(c - 3)(c - b)(x + c)} - \frac{c - 2}{(c - 3)(c - b)(x + c)}
\]

\[
\frac{b - 2}{(b - 3)(c - b)(x + b)} - \frac{1}{(b - 3)(c - b)(x + 3)}
\]

\[
\text{(Maxima 5.46.893.g3fbfdcd58 Manual)}
\]

\[
\begin{align*}
\text{(i12)} & \quad \text{ratsimp } ((x^5 - 1)/(x - 1)); \\
& \quad x^4 + x^3 + x^2 + x + 1
\end{align*}
\]

\[
\text{(i13)} \quad \text{subst } (a, x, \%);
\]

\[
\begin{align*}
\text{(i14)} & \quad \text{factor } \text{(th}(2), \%); \\
& \quad (x - a)(x - a)(x - a)(x + a + a + a + 1)
\end{align*}
\]

\[
\text{(i15)} \quad \text{factor } (1 + x^{12});
\]

\[
\begin{align*}
\text{(i16)} & \quad (x + 1)(x - x + 1)(x - x + 1)
\end{align*}
\]

\[
\begin{align*}
\text{factor_max_degree} & \quad \text{[Option variable]}
\end{align*}
\]

Default value: 1000

When factor_max_degree is set to a positive integer n, it will prevent Maxima from attempting to factor certain polynomials whose degree in any variable exceeds n. If factor_max_degree_print_warning is true, a warning message will be printed. factor_max_degree can be used to prevent excessive memory usage and/or computation time and stack overflows. Note that "obvious" factoring of polynomials such as \(x^{2000}+x^{2001}\) to \(x^{2000}\times(x+1)\) will still take place. To disable this behavior, set factor_max_degree to 0.
Example:

```maxima
(%i1) factor_max_degree : 100$

(%i2) factor(x^100-1);

2  4  3  2
(\%o2) (x - 1) (x + 1) (x + 1) (x - x + x - x + 1)

4  3  2
(x + x + x + x + 1) (x - x + x - x + 1)

20  15  10  5
(x - x + x - x + 1) (x + x + x + x + 1)

40  30  20  10
(x - x + x - x + 1)

(%i3) factor(x^101-1);

Refusing to factor polynomial x - 1
because its degree exceeds factor_max_degree (100)

(%o3) x - 1
```

See also: `factor_max_degree_print_warning`

`factor_max_degree_print_warning` [Option variable]
Default value: true

When `factor_max_degree_print_warning` is true, then Maxima will print a warning message when the factoring of a polynomial is prevented because its degree exceeds the value of `factor_max_degree`.

See also: `factor_max_degree`

`factorflag` [Option variable]
Default value: false

When `factorflag` is false, suppresses the factoring of integer factors of rational expressions.

`factorout(expr, x_1, x_2, ...)` [Function]
Rearranges the sum `expr` into a sum of terms of the form `f(x_1, x_2, ...)*g` where `g` is a product of expressions not containing any `x_i` and `f` is factored.

Note that the option variable `keepfloat` is ignored by `factorout`.

Example:

```maxima
(%i1) expand (a*(x+1)*(x-1)*(u+1)^2);

2 2 2 2
(%o1) a u x + 2 a u x + a x - a u - 2 a u - a

(%i2) factorout(%),x);

2
(%o2) a u (x - 1) (x + 1) + 2 a u (x - 1) (x + 1)

+ a (x - 1) (x + 1)
```

`factorsum(expr)` [Function]
Tries to group terms in factors of `expr` which are sums into groups of terms such that their sum is factorable. `factorsum` can recover the result of `expand((x + y)^2 + (z`
\( + w \)^2) but it can’t recover \( \text{expand} \left( (x + 1)^2 + (x + y)^2 \right) \) because the terms have variables in common.

Example:

\[
\texttt{(%i1)} \quad \text{expand} \left( (x + 1)^2 + (u + v)^2 + a(w + z)^2 \right);
\]
\[
\texttt{(%o1)} \quad a \, x \, z + a \, z + 2 \, a \, w \, x \, z + 2 \, a \, w \, z + a \, w \, x + v \, x
\]
\[
+ 2 \, u \, v \, x + u \, x + a \, w + v + 2 \, u \, v + u
\]

\[
\texttt{(%i2)} \quad \text{factorsum} \left( \%ight);
\]
\[
\texttt{(%o2)} \quad (x + 1) \, (a \, (z + w) + (v + u))
\]

\textbf{fasttimes} \( (p_1, p_2) \) \hspace{1cm} \textbf{[Function]}

Returns the product of the polynomials \( p_1 \) and \( p_2 \) by using a special algorithm for multiplication of polynomials. \( p_1 \) and \( p_2 \) should be multivariate, dense, and nearly the same size. Classical multiplication is of order \( n_1 \, n_2 \) where \( n_1 \) is the degree of \( p_1 \) and \( n_2 \) is the degree of \( p_2 \). \text{fasttimes} is of order \( \max(n_1, n_2)^{1.585} \).

\textbf{fullratsimp} \( (expr) \) \hspace{1cm} \textbf{[Function]}

\text{fullratsimp} repeatedly applies \text{ratsimp} followed by non-rational simplification to an expression until no further change occurs, and returns the result.

When non-rational expressions are involved, one call to \text{ratsimp} followed as is usual by non-rational ("general") simplification may not be sufficient to return a simplified result. Sometimes, more than one such call may be necessary. \text{fullratsimp} makes this process convenient.

\text{fullratsimp} \( (expr, x_1, \ldots, x_n) \) takes one or more arguments similar to \text{ratsimp} and \text{rat}.

Example:

\[
\texttt{(%i1)} \quad \text{expr:} \quad (x^{(a/2)} + 1)^2*(x^{(a/2)} - 1)^2/(x^a - 1);
\]
\[
\texttt{(%i2)} \quad \text{ratsimp} \left( \text{expr} \right);
\]
\[
\texttt{(%i3)} \quad \text{fullratsimp} \left( \text{expr} \right);
\]
\[
\texttt{(%i4)} \quad \text{rat} \left( \text{expr} \right);
\]
fullratsubst \( \text{fullratsubst} \ (\text{new}, \ \text{old}, \ \text{expr}) \)

fullratsubst \( \text{fullratsubst} \ (\text{old} = \text{new}, \ \text{expr}) \)
fullratsubst \( \text{fullratsubst} \ ([\ \text{old}_1 = \text{new}_1, \ \ldots, \ \text{old}_n = \text{new}_n ], \ \text{expr}) \)

fullratsubst applies \text{lratsubst} \ repeatedly until \text{expr} \ stops changing (or \text{lrats_max_iter} \ is reached). This function is useful when the replacement expression and the replaced expression have one or more variables in common.

fullratsubst accepts its arguments in the format of \text{ratsubst} \ or \text{lratsubst}.

Examples:

- \text{subst} \ can carry out multiple substitutions. \text{lratsubst} \ is analogous to \text{subst}.
  \( \%i2 \) \text{subst} \ ([\ a = b, \ c = d], \ a + c) ;
  \( \%o2 \) \( d + b \)
  \( \%i3 \) \text{lratsubst} \ ([a^2 = b, \ c^2 = d], \ (a + e) * c * (a + c)) ;
  \( \%o3 \) \( (d + a c) e + a d + b c \)

- If only one substitution is desired, then a single equation may be given as first argument.
  \( \%i4 \) \text{lratsubst} \ (a^2 = b, \ a^3) ;
  \( \%o4 \) \( a b \)

- \text{fullratsubst} \ is equivalent to \text{ratsubst} \ except that it recurses until its result stops changing.
  \( \%i5 \) \text{ratsubst} \ (b * a, \ a^2, \ a^3) ;
  \( \%o5 \) \( a b \)
  \( \%i6 \) \text{fullratsubst} \ (b * a, \ a^2, \ a^3) ;
  \( \%o6 \) \( a b \)

- \text{fullratsubst} \ also accepts a list of equations or a single equation as first argument.
  \( \%i7 \) \text{fullratsubst} \ ([a^2 = b, \ b^2 = c, \ c^2 = a], \ a^3 * b * c) ;
  \( \%o7 \) \( b \)
  \( \%i8 \) \text{fullratsubst} \ (a^2 = b * a, \ a^3) ;
  \( \%o8 \) \( a b \)

- \text{fullratsubst} \ catches potential infinite recursions. [\text{lrats_max_iter}], \ page \ 279.
  \( \%i9 \) \text{fullratsubst} \ (b * a^2, \ a^2, \ a^3) ; \ \text{lrats_max_iter}=15 ;
  \text{Warning: fullratsubst1(substexpr,forexpr,expr): reached maximum iterations of 15}
  \( \%o7 \) \( a b \)

See also \text{lrats_max_iter} \ and \text{fullratsubstflag}.
fullratsubstflag

Default value: false

An option variable that is set to true in fullratsubst.

gcd (p_1, p_2, x_1, \ldots )

Returns the greatest common divisor of \( p_1 \) and \( p_2 \). The flag gcd determines which algorithm is employed. Setting gcd to ez, subres, red, or spmod selects the ezgcd, subresultant prs, reduced, or modular algorithm, respectively. If gcd=false then gcd \( (p_1, p_2, x) \) always returns 1 for all \( x \). Many functions (e.g. ratsimp, factor, etc.) cause gcd’s to be taken implicitly. For homogeneous polynomials it is recommended that gcd equal to subres be used. To take the gcd when an algebraic is present, e.g., gcd \( (x^2 - 2*sqrt(2) * x + 2, x - sqrt(2)) \), the option variable algebraic must be true and gcd must not be ez.

The gcd flag, default: spmod, if false will also prevent the greatest common divisor from being taken when expressions are converted to canonical rational expression (CRE) form. This will sometimes speed the calculation if gcds are not required.

See also ezgcd, gcdex, gcdivide, and poly_gcd.

Example:

\[(%i1) \quad p1:6*x^3+19*x^2+19*x+6;\]
\[\quad 3 \quad 2\]
\[(%o1) \quad 6 \ x^3 \ + \ 19 \ x^2 \ + \ 19 \ x \ + \ 6 \]
\[(%i2) \quad p2:6*x^5+13*x^4+12*x^3+13*x^2+6*x;\]
\[\quad 5 \quad 4 \quad 3 \quad 2\]
\[(%o2) \quad 6 \ x^5 \ + \ 13 \ x^4 \ + \ 12 \ x^3 \ + \ 13 \ x^2 \ + \ 6 \ x \]
\[(%i3) \quad \text{gcd}(p1, p2);\]
\[\quad 2\]
\[(%o3) \quad 6 \ x^5 \ + \ 13 \ x^4 \ + \ 6 \]
\[(%i4) \quad \text{p1/gcd(p1, p2)}, \text{ratsimp};\]
\[\quad x + 1\]
\[(%o4) \quad x + 1\]
\[(%i5) \quad \text{p2/gcd(p1, p2)}, \text{ratsimp};\]
\[\quad 3\]
\[(%o5) \quad x + x\]

\text{ezgcd} returns a list whose first element is the greatest common divisor of the polynomials \( p_1 \) and \( p_2 \), and whose remaining elements are the polynomials divided by the greatest common divisor.

\[(%i6) \quad \text{ezgcd}(p1, p2);\]
\[\quad 2 \quad 3\]
\[(%o6) \quad [6 \ x^5 + 13 \ x^4 + 6, x + 1, x + x]\]

gcdex

\text{gcdex} (f, g)
\text{gcdex} (f, g, x)

Returns a list \([a, b, u]\) where \( u \) is the greatest common divisor (gcd) of \( f \) and \( g \), and \( u \) is equal to \( a f + b g \). The arguments \( f \) and \( g \) should be univariate polynomials, or else polynomials in \( x \) a supplied main variable since we need to be in a principal
ideal domain for this to work. The gcd means the gcd regarding $f$ and $g$ as univariate polynomials with coefficients being rational functions in the other variables.

gcdex implements the Euclidean algorithm, where we have a sequence of $L[i]: [a[i], b[i], r[i]]$ which are all perpendicular to $[f, g, -1]$ and the next one is built as if $q = \text{quotient}(r[i]/r[i+1])$ then $L[i+2]: L[i] - q L[i+1]$, and it terminates at $L[i+1]$ when the remainder $r[i+2]$ is zero.

The arguments $f$ and $g$ can be integers. For this case the function igcdex is called by gcdex.

See also ezgcd, gcd, gcdivide, and poly_gcd.

Examples:

```
(%i1) gcdex (x^2 + 1, x^3 + 4);
    2
x + 4 x - 1 x + 4
(%o1)/R/ [- ------------, -----, 1]
        17        17

(%i2) % . [x^2 + 1, x^3 + 4, -1];
(%o2)/R/ 0
```

Note that the gcd in the following is $1$ since we work in $k(y)[x]$, not the $y+1$ we would expect in $k[y, x]$.

```
(%i1) gcdex (x*(y + 1), y^2 - 1, x);

(%o1)/R/ [0, -------, 1]
        2
        y - 1
```

gcfactor ($n$)

Factors the Gaussian integer $n$ over the Gaussian integers, i.e., numbers of the form $a + b \%i$ where $a$ and $b$ are rational integers (i.e., ordinary integers). Factors are normalized by making $a$ and $b$ non-negative.

gfactor ($expr$)

Factors the polynomial $expr$ over the Gaussian integers (that is, the integers with the imaginary unit $\%i$ adjoined). This is like factor ($expr$, $a^2+1$) where $a$ is $\%i$.

Example:

```
(%i1) gfactor (x^4 - 1);
    (%o1)
(x - 1) (x + 1) (x - \%i) (x + \%i)
```

gfactorsum ($expr$)

is similar to factorsum but applies gfactor instead of factor.

hipow ($expr$, $x$)

Returns the highest explicit exponent of $x$ in $expr$. $x$ may be a variable or a general expression. If $x$ does not appear in $expr$, hipow returns 0.

hipow does not consider expressions equivalent to $expr$. In particular, hipow does not expand $expr$, so hipow ($expr$, $x$) and hipow (expand ($expr$, $x$)) may yield different results.
Examples:

```
(%i1) hipow (y^3 * x^2 + x * y^4, x);
  (%o1) 2
(%i2) hipow ((x + y)^5, x);
  (%o2) 1
(%i3) hipow (expand ((x + y)^5), x);
  (%o3) 5
(%i4) hipow ((x + y)^5, x + y);
  (%o4) 5
(%i5) hipow (expand ((x + y)^5), x + y);
  (%o5) 0
```

**intfaclim**  
[Option variable]  
Default value: true  
If true, maxima will give up factorization of integers if no factor is found after trial divisions and Pollard’s rho method and factorization will not be complete.  
When intfaclim is false (this is the case when the user calls factor explicitly), complete factorization will be attempted. intfaclim is set to false when factors are computed in divisors, divsum and totient.  
Internal calls to factor respect the user-specified value of intfaclim. Setting intfaclim to true may reduce the time spent factoring large integers.

**keepfloat**  
[Option variable]  
Default value: false  
When keepfloat is true, prevents floating point numbers from being rationalized when expressions which contain them are converted to canonical rational expression (CRE) form.  
Note that the function solve and those functions calling it (eigenvalues, for example) currently ignore this flag, converting floating point numbers anyway.

Examples:

```
(%i1) rat(x/2.0);
rat: replaced 0.5 by 1/2 = 0.5
    x
(%o1)/R/
    -
    2
(%i2) rat(x/2.0), keepfloat;
    (%o2)/R/ 0.5 x

solve ignores keepfloat:

(%i1) solve(1.0-x,x), keepfloat;

rat: replaced 1.0 by 1/1 = 1.0
(%o1) [x = 1]
```
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lopow (expr, x) [Function]

Returns the lowest exponent of x which explicitly appears in expr. Thus

(%i1) lopow ((x+y)^2 + (x+y)^a, x+y);  
(%o1) min(a, 2)

lratsubst (new, old, expr) [Function]

lratsubst (old = new, expr)
lratsubst ([old_1 = new_1, ..., old_n = new_n], expr)

lratsubst is analogous to subst except that it uses ratsubst to perform substitutions.

The first argument of lratsubst is an equation, a list of equations or a list of unit length whose first element is a list of equations (that is, the first argument is identical in format to that accepted by subst). The substitutions are made in the order given by the list of equations, that is, from left to right.

Examples:

- subst can carry out multiple substitutions. lratsubst is analogous to subst.

(%i2) lratsubst ([a = b, c = d], a + c);  
(%o2) d + b
(%i3) lratsubst ([a^2 = b, c^2 = d], (a + e)*c*(a + c));  
(%o3) (d + a c) e + a d + b c

- If only one substitution is desired, then a single equation may be given as first argument.

(%i4) lratsubst (a^2 = b, a^3);  
(%o4) a b

- A nested list of substitutions can be used—but it must contain only one list.

(%i5) lratsubst ([[a^2=b*a, b=c]], a^3);  
(%o5) a c
(%i6) lratsubst ([[a^2=b*a, b=c],[a=b]], a^3);

2

lratsubst: improper argument: [[a = a b, b = c], [a = b]]
#0: lratsubst(listofeqns=[[a^2 = a*b,b = c],[a = b]],expr=a^3)
-- an error. To debug this try: debugmode(true);

See also fullratsubst.

lrats_max_iter [Option variable]

Default value: 100000

The upper limit on the number of iterations that fullratsubst and lratsubst may perform. It must be set to a positive integer. See the example for fullratsubst.

modulus [Option variable]

Default value: false
When \texttt{modulus} is a positive number \( p \), operations on canonical rational expressions (CREs, as returned by \texttt{rat} and related functions) are carried out modulo \( p \), using the so-called "balanced" modulus system in which \( n \) modulo \( p \) is defined as an integer \( k \) in \([-\frac{(p-1)}{2}, \ldots, 0, \ldots, \frac{(p-1)}{2}] \) when \( p \) is odd, or \([-\frac{(p/2 - 1)}{2}, \ldots, 0, \ldots, \frac{p}{2}] \) when \( p \) is even, such that \( a \cdot p + k \) equals \( n \) for some integer \( a \).

If \( \texttt{expr} \) is already in canonical rational expression (CRE) form when \texttt{modulus} is reset, then you may need to re-\texttt{rat} \( \texttt{expr} \), e.g., \( \texttt{expr: rat (ratdisrep (expr))} \), in order to get correct results.

Typically \texttt{modulus} is set to a prime number. If \texttt{modulus} is set to a positive non-prime integer, this setting is accepted, but a warning message is displayed. Maxima signals an error, when zero or a negative integer is assigned to \texttt{modulus}.

Examples:

```lisp
(%i1) modulus:7;
(%o1) 7
(%i2) polymod([0,1,2,3,4,5,6,7]);
(%o2) [0, 1, 2, 3, -3, -2, -1, 0]
(%i3) modulus:false;
(%o3) false
(%i4) poly:x^6+x^2+1;
6 2
(%o4) x + x + 1
(%i5) factor(poly);
6 2
(%o5) x + x + 1
(%i6) modulus:13;
(%o6) 13
(%i7) factor(poly);
2 4 2
(%o7) (x + 6) (x - 6 x - 2)
(%i8) polymod(%);
6 2
(%o8) x + x + 1
```

\texttt{num (expr)}

Returns the numerator of \( \texttt{expr} \) if it is a ratio. If \( \texttt{expr} \) is not a ratio, \( \texttt{expr} \) is returned. \texttt{num} evaluates its argument.

See also \texttt{denom}

```lisp
(%i1) g1:(x+2)*(x+1)/((x+3)^2);
3
2
(x + 1) (x + 2)
(%o1) -------------------
2
(x + 3)
(%i2) num(g1);
(x + 1) (x + 2)
(%o2) 2
```
(\%i3) \( g_2 : \frac{\sin(x)}{10\cos(x)} / y \); 
(\%o3) \( \frac{\cos(x) \sin(x)}{10\ y} \)
(\%i4) \( \text{num}(g_2) \); 
(\%o4) \( \cos(x) \sin(x) \)

\text{polydecomp} \ (p, x) \quad \text{[Function]}

Decomposes the polynomial \( p \) in the variable \( x \) into the functional composition of polynomials in \( x \). \text{polydecomp} returns a list \([p_1, \ldots, p_n]\) such that

\[
\lambda ([x], p_1) (\lambda ([x], p_2) (\ldots (\lambda ([x], p_n) (x)) \ldots))
\]

is equal to \( p \). The degree of \( p_i \) is greater than \( 1 \) for \( i \) less than \( n \).

Such a decomposition is not unique.

Examples:

(\%i1) \( \text{polydecomp} \ (x^{210}, x) \);
(\%o1) \[ x^7, x^5, x^3, x^2 \]
(\%i2) \( p : \text{expand} \ (\text{subst} \ (x^3 - x - 1, x, x^2 - a)) \);
(\%o2) \( x^6 - 2x^4 + 3x^2 + 2x - a + 1 \)
(\%i3) \( \text{polydecomp} \ (p, x) \);
(\%o3) \[ x - a, x - x - 1 \]

The following function composes \( L = [e_1, \ldots, e_n] \) as functions in \( x \); it is the inverse of \text{polydecomp}:

(\%i1) \( \text{compose} \ (L, x) := \) block ([r : x], for e in L do r : subst (e, x, r), r) $

Re-express above example using \text{compose}:

(\%i1) \( \text{polydecomp} \ (\text{compose} \ ([x^2 - a, x^3 - x - 1], x), x) \);
(\%o1) \[ \text{compose}([x - a, x - x - 1], x) \]

Note that though \( \text{compose} \ (\text{polydecomp} \ (p, x), x) \) always returns \( p \) (unexpanded), \( \text{polydecomp} \ (\text{compose} \ ([p_1, \ldots, p_n], x), x) \) does not necessarily return \([p_1, \ldots, p_n]\):

(\%i1) \( \text{polydecomp} \ (\text{compose} \ ([x^{*2} + 2*x + 3, x^{*2}], x), x) \);
(\%o1) \[ \text{compose}([x + 2 \ x + 3, x], x) \]
(\%i2) \( \text{polydecomp} \ (\text{compose} \ ([x^{*2} + x + 1, x^{*2} + x + 1], x), x) \);
(\%o2) \[ \text{compose}([x + x + 1, x + x + 1], x) \]
polymod [Function]
   polymod (p)
   polymod (p, m)
   Converts the polynomial p to a modular representation with respect to the current modulus which is the value of the variable modulus.

   polymod (p, m) specifies a modulus m to be used instead of the current value of modulus.

   See modulus.

polynomialp [Function]
   polynomialp (p, L, coeffp, exponp)
   polynomialp (p, L, coeffp)
   polynomialp (p, L)
   Return true if p is a polynomial in the variables in the list L. The predicate coeffp must evaluate to true for each coefficient, and the predicate exponp must evaluate to true for all exponents of the variables in L. If you want to use a non-default value for exponp, you must supply coeffp with a value even if you want to use the default for coeffp.

   The command polynomialp (p, L, coeffp) is equivalent to polynomialp (p, L, coeffp, 'nonnegintegerp) and the command polynomialp (p, L) is equivalent to polynomialp (p, L, 'constantp, 'nonnegintegerp).

   The polynomial needn’t be expanded:

   (%i1) polynomialp ((x + 1)*(x + 2), [x]);
   (%o1) true
   (%i2) polynomialp ((x + 1)*(x + 2)^a, [x]);
   (%o2) false

   An example using non-default values for coeffp and exponp:

   (%i1) polynomialp ((x + 1)*(x + 2)^3/2, [x], numberp, numberp);
   (%o1) true
   (%i2) polynomialp ((x^(1/2) + 1)*(x + 2)^3/2, [x], numberp, numberp);
   (%o2) true

   Polynomials with two variables:

   (%i1) polynomialp (x^2 + 5*x*y + y^2, [x]);
   (%o1) false
   (%i2) polynomialp (x^2 + 5*x*y + y^2, [x, y]);
   (%o2) true

   Polynomial in one variable and accepting any expression free of x as a coefficient.

   (%i1) polynomialp (a*x^2 + b*x + c, [x]);
   (%o1) false
   (%i2) polynomialp (a*x^2 + b*x + c, [x], lambda([ex], freeof(x, ex)));
   (%o2) true
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quote

quotient (p_1, p_2)
quotient (p_1, p_2, x_1, ..., x_n)

Returns the polynomial $p_1$ divided by the polynomial $p_2$. The arguments $x_1, ..., x_n$ are interpreted as in `ratvars`.

`quotient` returns the first element of the two-element list returned by `divide`.

rat

rat (expr)
rat (expr, x_1, ..., x_n)

Converts `expr` to canonical rational expression (CRE) form by expanding and combining all terms over a common denominator and cancelling out the greatest common divisor of the numerator and denominator, as well as converting floating point numbers to rational numbers within a tolerance of `ratepsilon`. The variables are ordered according to the $x_1, ..., x_n$, if specified, as in `ratvars`.

`rat` does not generally simplify functions other than addition $\oplus$, subtraction $\ominus$, multiplication $\otimes$, division $\oslash$, and exponentiation to an integer power, whereas `ratsimp` does handle those cases. Note that atoms (numbers and variables) in CRE form are not the same as they are in the general form. For example, `rat(x) - x` yields `rat(0)` which has a different internal representation than 0.

When `ratfac` is `true`, `rat` yields a partially factored form for CRE. During rational operations the expression is maintained as fully factored as possible without an actual call to the factor package. This should always save space and may save some time in some computations. The numerator and denominator are still made relatively prime (e.g., `rat((x^2 - 1)^4/(x + 1)^2)` yields $(x - 1)^4 (x + 1)^2$ when `ratfac` is `true`), but the factors within each part may not be relatively prime.

`ratprint` if `false` suppresses the printout of the message informing the user of the conversion of floating point numbers to rational numbers.

`keepfloat` if `true` prevents floating point numbers from being converted to rational numbers.

See also `ratexpand` and `ratsimp`.

Examples:

```lisp
(%i1) ((x - 2*y)^4/(x^2 - 4*y^2)^2 + 1)*(y + a)*(2*y + x) / (4*y^2 + x^2);
   4
   (x - 2 y)
   (y + a) (2 y + x) (------------- + 1)
       2 2
   (x - 4 y )

(%o1)---------------------------------------------------------------------
   2 2
   4 y + x

(%i2) rat (%i1, y, a, x);
   2 a + 2 y
---
   x + 2 y

(%o2)/R/
```
ratalgdenom

[Option variable]

Default value: true

When ratalgdenom is true, allows rationalization of denominators with respect to radicals to take effect. ratalgdenom has an effect only when canonical rational expressions (CRE) are used in algebraic mode.

ratcoef

[Function]

ratcoef (expr, x, n)
ratcoef (expr, x)

Returns the coefficient of the expression \(x^n\) in the expression \(expr\). If omitted, \(n\) is assumed to be 1.

The return value is free (except possibly in a non-rational sense) of the variables in \(x\). If no coefficient of this type exists, 0 is returned.

ratcoef expands and rationally simplifies its first argument and thus it may produce answers different from those of coeff which is purely syntactic. Thus ratcoef ((\(x + 1\))/\(y + x\), \(x\)) returns \((y + 1)/y\) whereas coeff returns 1.

ratcoef (expr, x, 0), viewing expr as a sum, returns a sum of those terms which do not contain \(x\). Therefore if \(x\) occurs to any negative powers, ratcoef should not be used.

Since expr is rationally simplified before it is examined, coefficients may not appear quite the way they were envisioned.

Example:

\[
%i1\) s: a*x + b*x + 5$
\%
%i2\) ratcoef (s, a + b);  
\%
o2\)

ratdenom (expr)

[Function]

Returns the denominator of \(expr\), after coercing \(expr\) to a canonical rational expression (CRE). The return value is a CRE.

expr is coerced to a CRE by rat if it is not already a CRE. This conversion may change the form of expr by putting all terms over a common denominator.

denom is similar, but returns an ordinary expression instead of a CRE. Also, denom does not attempt to place all terms over a common denominator, and thus some expressions which are considered ratios by ratdenom are not considered ratios by denom.

ratdenomdivide

[Option variable]

Default value: true

When ratdenomdivide is true, ratexpand expands a ratio in which the numerator is a sum into a sum of ratios, all having a common denominator. Otherwise, ratexpand collapses a sum of ratios into a single ratio, the numerator of which is the sum of the numerators of each ratio.

Examples:

\[
%i1\) expr: (x^2 + x + 1)/(y^2 + 7);
ratdiff (expr, x)

[Function]

Differentiates the rational expression expr with respect to x. expr must be a ratio of polynomials or a polynomial in x. The argument x may be a variable or a subexpression of expr.

The result is equivalent to diff, although perhaps in a different form. ratdiff may be faster than diff, for rational expressions.

ratdiff returns a canonical rational expression (CRE) if expr is a CRE. Otherwise, ratdiff returns a general expression.

ratdiff considers only the dependence of expr on x, and ignores any dependencies established by depends.

Example:

(%i11) expr: (4*x^3 + 10*x - 11)/(x^5 + 5);
\[ \frac{3}{4x^2 + 10x - 11} - \frac{5}{x + 5} \]

\(\texttt{(%o1)}\)

\(\texttt{ratdiff (expr, x);}\)

\[ \frac{7}{5} - \frac{4}{2} \]

\[ \frac{8x + 40}{x^2 - 55x - 60} - \frac{50}{x + 10} \]

\(\texttt{(%o2)}\)

\(\texttt{expr: f(x)^3 - f(x)^2 + 7;}\)

\[ f(x)^3 - f(x)^2 + 7 \]

\(\texttt{(%o3)}\)

\(\texttt{ratdiff (expr, f(x));}\)

\[ 3f(x) - 2f(x) \]

\(\texttt{(%o4)}\)

\(\texttt{expr: (a + b)^3 + (a + b)^2;}\)

\[ (a + b)^3 + (a + b)^2 \]

\(\texttt{(%o5)}\)

\(\texttt{ratdiff (expr, a + b);}\)

\[ 3b + (6a + 2)b + 3a + 2a \]

\(\texttt{(%o6)}\)

\[ \texttt{ratdisrep (expr)} \]

Returns its argument as a general expression. If expr is a general expression, it is returned unchanged.

Typically \texttt{ratdisrep} is called to convert a canonical rational expression (CRE) into a general expression. This is sometimes convenient if one wishes to stop the "contagion", or use rational functions in non-rational contexts.

See also \texttt{totaldisrep}.

\[ \texttt{ratexpand (expr)} \]

Expands expr by multiplying out products of sums and exponentiated sums, combining fractions over a common denominator, cancelling the greatest common divisor of the numerator and denominator, then splitting the numerator (if a sum) into its respective terms divided by the denominator.

The return value of \texttt{ratexpand} is a general expression, even if expr is a canonical rational expression (CRE).

The switch \texttt{ratexpand} if \texttt{true} will cause CRE expressions to be fully expanded when they are converted back to general form or displayed, while if it is \texttt{false} then they will be put into a recursive form. See also \texttt{ratsimp}.

When \texttt{ratdenomdivide} is \texttt{true}, \texttt{ratexpand} expands a ratio in which the numerator is a sum into a sum of ratios, all having a common denominator. Otherwise, \texttt{ratexpand} collapses a sum of ratios into a single ratio, the numerator of which is the sum of the numerators of each ratio.
When \texttt{keepfloat} is \texttt{true}, prevents floating point numbers from being rationalized when expressions which contain them are converted to canonical rational expression (CRE) form.

Examples:

\begin{verbatim}
(%i1) ratexpand ((2*x - 3*y)^3);
   3 2 2 3
   - 27 y + 54 x y - 36 x y + 8 x
(%o1)

(%i2) expr: (x - 1)/(x + 1)^2 + 1/(x - 1);
   x - 1   1
---------- + ------
   2  x - 1
(x + 1)
(%o2)

(%i3) expand (expr);
   x 1 1
---------- - ---------- + ------
   2 2 x - 1
x + 2 x + 1 x + 2 x + 1
(%o3)

(%i4) ratexpand (expr);
   2
   2 x
---------- + ----------
   3 2
x + x - x - 1 x + x - x - 1
(%o4)
\end{verbatim}

\textbf{ratfac} \hspace{1cm} [Option variable]

Default value: \texttt{false}

When \texttt{ratfac} is \texttt{true}, canonical rational expressions (CRE) are manipulated in a partially factored form.

During rational operations the expression is maintained as fully factored as possible without calling \texttt{factor}. This should always save space and may save time in some computations. The numerator and denominator are made relatively prime, for example \texttt{factor ((x^2 - 1)^4/(x + 1)^2)} yields \((x - 1)^4 (x + 1)^2\), but the factors within each part may not be relatively prime.

In the \texttt{ctensor} (Component Tensor Manipulation) package, Ricci, Einstein, Riemann, and Weyl tensors and the scalar curvature are factored automatically when \texttt{ratfac} is \texttt{true}. \texttt{ratfac should only be set for cases where the tensorial components are known to consist of few terms.}

The \texttt{ratfac} and \texttt{ratweight} schemes are incompatible and may not both be used at the same time.

\textbf{ratnumer (expr)} \hspace{1cm} [Function]

Returns the numerator of \texttt{expr}, after coercing \texttt{expr} to a canonical rational expression (CRE). The return value is a CRE.

\texttt{expr} is coerced to a CRE by \texttt{rat} if it is not already a CRE. This conversion may change the form of \texttt{expr} by putting all terms over a common denominator.
num is similar, but returns an ordinary expression instead of a CRE. Also, num does not attempt to place all terms over a common denominator, and thus some expressions which are considered ratios by ratnumer are not considered ratios by num.

ratp (expr)  [Function]
Returns true if expr is a canonical rational expression (CRE) or extended CRE, otherwise false.

CRE are created by rat and related functions. Extended CRE are created by taylor and related functions.

ratprint  [Option variable]
Default value: true
When ratprint is true, a message informing the user of the conversion of floating point numbers to rational numbers is displayed.

ratsimp (expr)  [Function]
ratsimp (expr, x_1, . . . , x_n)  [Function]
Simplifies the expression expr and all of its subexpressions, including the arguments to non-rational functions. The result is returned as the quotient of two polynomials in a recursive form, that is, the coefficients of the main variable are polynomials in the other variables. Variables may include non-rational functions (e.g., sin (x^2 + 1)) and the arguments to any such functions are also rationally simplified.

ratsimp (expr, x_1, . . . , x_n) enables rational simplification with the specification of variable ordering as in ratvars.
When ratsimpexpons is true, ratsimp is applied to the exponents of expressions during simplification.
See also ratexpand. Note that ratsimp is affected by some of the flags which affect ratexpand.

Examples:
(\%i1) sin (x/(x^2 + x)) = exp ((log(x) + 1)^2 - log(x)^2);
   x  2 2
   (log(x) + 1) - log (x)
(\%o1) sin(------) = %e
   x + x
(\%i2) ratsimp (%);
   1 2
   sin(------) = %e x
   x + 1
(\%o2) ratsimp %;
   1 2
   sin(------) = %e x
   x + 1
(\%i3) ((x - 1)^(3/2) - (x + 1)*sqrt(x - 1))/sqrt((x - 1)*(x + 1));
   3/2
   (x - 1) - sqrt(x - 1) (x + 1)
(\%o3) ------------------------
         sqrt((x - 1) (x + 1))
(\%i4) ratsimp %;
   2 sqrt(x - 1)
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(\%o4) \quad \frac{2}{\sqrt{x - 1}}

(\%i5) x^{(a + 1/a)}, \text{ratsimpexpons: true;}

\quad \frac{2}{a + 1}

\quad \frac{a}{a}

(\%o5) x

\text{ratsimpexpons} \quad \text{[Option variable]}

Default value: false

When \text{ratsimpexpons} is true, \text{ratsimp} is applied to the exponents of expressions during simplification.

\text{radsubstflag} \quad \text{[Option variable]}

Default value: false

\text{radsubstflag}, if true, permits \text{ratsubst} to make substitutions such as \( u \) for \( \sqrt{x} \) in \( x \).

\text{ratsubst} (a, b, c) \quad \text{[Function]}

Substitutes \( a \) for \( b \) in \( c \) and returns the resulting expression. \( b \) may be a sum, product, power, etc.

\text{ratsubst} knows something of the meaning of expressions whereas \text{subst} does a purely syntactic substitution. Thus \text{subst} \((a, x + y, x + y + z)\) returns \( x + y + z \) whereas \text{ratsubst} returns \( z + a \).

When \text{radsubstflag} is true, \text{ratsubst} makes substitutions for radicals in expressions which don't explicitly contain them.

\text{ratsubst} ignores the value true of the option variables \text{keepfloat}, \text{float}, and \text{numer}.

Examples:

(\%i1) ratsubst (a, x*y^2, x^4*y^3 + x^4*y^8);

\quad 3 \quad 4

\quad a \quad x \quad y \quad a

(\%o1) x^3 \quad y^4 \quad a

(\%i2) \cos(x)^4 + \cos(x)^3 + \cos(x)^2 + \cos(x) + 1;

\quad 4 \quad 3 \quad 2

\quad \cos (x) \quad + \quad \cos (x) \quad + \quad \cos (x) \quad + \quad \cos (x) \quad + \quad 1

(\%o2) \quad \cos (x) \quad + \quad \cos (x) \quad + \quad \cos (x) \quad + \quad \cos (x) \quad + \quad 1

(\%i3) ratsubst (1 - \sin(x)^2, \cos(x)^2, \%);

\quad 4 \quad 2

\quad \sin (x) - 3 \quad \sin (x) \quad + \quad \cos(x) \quad (2 \quad - \quad \sin (x)) \quad + \quad 3

(\%o3) \quad \sin (x) - 3 \quad \sin (x) \quad + \quad \cos(x) \quad (2 \quad - \quad \sin (x)) \quad + \quad 3

(\%i4) ratsubst (1 - \sin(x)^2, \sin(x)^2, \%);

\quad 4 \quad 2

\quad \cos (x) - 2 \quad \cos (x) \quad + \quad 1

(\%o4) \quad \cos (x) - 2 \quad \cos (x) \quad + \quad 1

(\%i5) \quad \text{radsubstflag: false}$

(\%i6) \quad \text{ratsubst (u, sqrt(x), x);}

(\%o6) \quad x

(\%i7) \quad \text{radsubstflag: true}$
ratvars (x_1, \ldots, x_n) \quad \text{[Function]}
ratvars () \quad \text{[Function]}
ratvars \quad \text{[System variable]}
Declares main variables x_1, \ldots, x_n for rational expressions. x_n, if present in a rational expression, is considered the main variable. Otherwise, x_{[n-1]} is considered the main variable if present, and so on through the preceding variables to x_1, which is considered the main variable only if none of the succeeding variables are present.

If a variable in a rational expression is not present in the ratvars list, it is given a lower priority than x_1.

The arguments to ratvars can be either variables or non-rational functions such as sin(x).

The variable ratvars is a list of the arguments of the function ratvars when it was called most recently. Each call to the function ratvars resets the list. ratvars () clears the list.

ratvarswitch \quad \text{[Option variable]}
Default value: true
Maxima keeps an internal list in the Lisp variable VARLIST of the main variables for rational expressions. If ratvarswitch is true, every evaluation starts with a fresh list VARLIST. This is the default behavior. Otherwise, the main variables from previous evaluations are not removed from the internal list VARLIST.

The main variables, which are declared with the function ratvars are not affected by the option variable ratvarswitch.

Examples:
If ratvarswitch is true, every evaluation starts with a fresh list VARLIST.

(\%i1) ratvarswitch: true$

(\%i2) rat(2*x+y^-2);
2
(\%o2)/R/ y + 2 x
(\%i3) :lisp varlist
($X $Y)
(\%i3) rat(2*a+b^-2);
2
(\%o3)/R/ b + 2 a
(\%i4) :lisp varlist
($A $B)

If ratvarswitch is false, the main variables from the last evaluation are still present.

(\%i4) ratvarswitch: false$
ratweight

ratweight (x_1, w_1, ..., x_n, w_n)
ratweight ()

Assigns a weight \( w_i \) to the variable \( x_i \). This causes a term to be replaced by 0 if its weight exceeds the value of the variable ratwtlvl (default yields no truncation). The weight of a term is the sum of the products of the weight of a variable in the term times its power. For example, the weight of \( 3 x_1^2 x_2 \) is \( 2 w_1 + w_2 \). Truncation according to ratwtlvl is carried out only when multiplying or exponentiating canonical rational expressions (CRE).

ratweight () returns the cumulative list of weight assignments.

Note: The ratfac and ratweight schemes are incompatible and may not both be used at the same time.

Examples:

(\%i1) ratweight (a, 1, b, 1);
(\%o1) [a, 1, b, 1]
(\%i2) expr1: rat(a + b + 1)$
(\%i3) expr1^2;
\( 2 \)
(\%o3)/R/ \( b + (2 a + 2) b + a + 2 a + 1 \)
(\%i4) ratwtlvl: 1$
(\%i5) expr1^2;
(\%o5)/R/ \( 2 b + 2 a + 1 \)

ratweights

Default value: []

ratweights is the list of weights assigned by ratweight. The list is cumulative: each call to ratweight places additional items in the list.

kill (ratweights) and save (ratweights) both work as expected.
**ratwtlvl**

[Option variable]

Default value: \texttt{false}

\texttt{ratwtlvl} is used in combination with the \texttt{ratweight} function to control the truncation of canonical rational expressions (CRE). For the default value of \texttt{false}, no truncation occurs.

**remainder**

[Function]

\begin{verbatim}
remainder (p_1, p_2)
remainder (p_1, p_2, x_1, \ldots, x_n)
\end{verbatim}

Returns the remainder of the polynomial \( p_1 \) divided by the polynomial \( p_2 \). The arguments \( x_1, \ldots, x_n \) are interpreted as in \texttt{ratvars}.

\texttt{remainder} returns the second element of the two-element list returned by \texttt{divide}.

**resultant \((p_1, p_2, x)\)**

[Function]

The function \texttt{resultant} computes the resultant of the two polynomials \( p_1 \) and \( p_2 \), eliminating the variable \( x \). The resultant is a determinant of the coefficients of \( x \) in \( p_1 \) and \( p_2 \), which equals zero if and only if \( p_1 \) and \( p_2 \) have a non-constant factor in common.

If \( p_1 \) or \( p_2 \) can be factored, it may be desirable to call \texttt{factor} before calling \texttt{resultant}.

The option variable \texttt{resultant} controls which algorithm will be used to compute the resultant. See the option variable [\texttt{option_resultant}], page 292.

The function \texttt{bezout} takes the same arguments as \texttt{resultant} and returns a matrix. The determinant of the return value is the desired resultant.

Examples:

\begin{verbatim}
(%i1) resultant(2*x^2+3*x+1, 2*x^2+x+1, x);
(%o1) 8
(%i2) resultant(x+1, x+1, x);
(%o2) 0
(%i3) resultant((x+1)*x, (x+1), x);
(%o3) 0
(%i4) resultant(a*x^2+b*x+1, c*x + 2, x);
\end{verbatim}

\begin{verbatim}
\hspace{1cm} 2
\hspace{1cm} c - 2 b c + 4 a
\end{verbatim}

\begin{verbatim}
(%i5) bezout(a*x^2+b*x+1, c*x+2, x);

\begin{verbatim}
[ 2 a 2 b - c ]
\end{verbatim}

\begin{verbatim}
[ c 2 ]
\end{verbatim}

(%i6) determinant(%);

\begin{verbatim}
(%o6) 4 a - (2 b - c) c
\end{verbatim}

**resultant**

[Option variable]

Default value: \texttt{subres}
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The option variable \texttt{resultant} controls which algorithm will be used to compute the resultant with the function \texttt{resultant}. The possible values are:

- \texttt{subres} for the subresultant polynomial remainder sequence (PRS) algorithm,
- \texttt{mod} (not enabled) for the modular resultant algorithm, and
- \texttt{red} for the reduced polynomial remainder sequence (PRS) algorithm.

On most problems the default value \texttt{subres} should be best.

\texttt{savefactors} \hspace{1cm} \text{[Option variable]} \hspace{1cm} Default value: \texttt{false}

When \texttt{savefactors} is \texttt{true}, causes the factors of an expression which is a product of factors to be saved by certain functions in order to speed up later factorizations of expressions containing some of the same factors.

\texttt{showratvars} \hspace{1cm} \text{[Function]} \hspace{1cm} Returns a list of the canonical rational expression (CRE) variables in expression \texttt{expr}.

See also \texttt{ratvars}.

\texttt{sqfr} \hspace{1cm} \text{[Function]} \hspace{1cm} is similar to \texttt{factor} except that the polynomial factors are "square-free." That is, they have factors only of degree one. This algorithm, which is also used by the first stage of \texttt{factor}, utilizes the fact that a polynomial has in common with its n'th derivative all its factors of degree greater than n. Thus by taking greatest common divisors with the polynomial of the derivatives with respect to each variable in the polynomial, all factors of degree greater than 1 can be found.

Example:

\begin{verbatim}
(%i1) sqfr (4*x^4 + 4*x^3 - 3*x^2 - 4*x - 1);
\end{verbatim}

\begin{verbatim}
2 2
(%o1) (2 x + 1) (x - 1)
\end{verbatim}

\texttt{tellrat} \hspace{1cm} \text{[Function]} \hspace{1cm} \texttt{tellrat} \hspace{1cm} \texttt{tellrat} (\texttt{p\_1}, \ldots, \texttt{p\_n})

\hspace{1cm} \texttt{tellrat} ()

Adds to the ring of algebraic integers known to Maxima the elements which are the solutions of the polynomials \texttt{p\_1}, \ldots, \texttt{p\_n}. Each argument \texttt{p\_i} is a polynomial with integer coefficients.

\texttt{tellrat} (\texttt{x}) effectively means substitute 0 for \texttt{x} in rational functions.

\texttt{tellrat} () returns a list of the current substitutions.

\texttt{algebraic} must be set to \texttt{true} in order for the simplification of algebraic integers to take effect.

Maxima initially knows about the imaginary unit \%i and all roots of integers.

There is a command \texttt{untellrat} which takes kernels and removes \texttt{tellrat} properties.

When \texttt{tellrat}'ing a multivariate polynomial, e.g., \texttt{tellrat (x^2 - y^2)}, there would be an ambiguity as to whether to substitute \texttt{y^2} for \texttt{x^2} or vice versa. Maxima picks
a particular ordering, but if the user wants to specify which, e.g. \texttt{tellrat (y^2 = x^2)} provides a syntax which says replace \(y^2\) by \(x^2\).

Examples:

\begin{verbatim}
(%i1) 10*('%i + 1)/('%i + 3^(1/3));
     10 (%i + 1)
(%o1) --------------
     1/3
     %i + 3

(%i2) ev (ratdisrep (rat(%)), algebraic);
    2/3    1/3    2/3    1/3
(%o2) (4 3 - 2 3 - 4) %i + 2 3 + 4 3 - 2

(%i3) tellrat (1 + a + a^2);
    2
(%o3) [a + a + 1]

(%i4) 1/(a*sqrt(2) - 1) + a/(sqrt(3) + sqrt(2));
     1      a
(%o4) -------------- + --------------
     sqrt(2) a - 1  sqrt(3) + sqrt(2)

(%i5) ev (ratdisrep (rat(%)), algebraic);
    7
(%o5) -----------------------------------

(%i6) tellrat (y^2 = x^2);
    2    2
(%o6) [y = x, a + a + 1]
\end{verbatim}

\texttt{totaldisrep (expr)} \hfill [Function]

Converts every subexpression of \texttt{expr} from canonical rational expressions (CRE) to general form and returns the result. If \texttt{expr} is itself in CRE form then \texttt{totaldisrep} is identical to \texttt{ratdisrep}.

\texttt{totaldisrep} may be useful for ratdisrepping expressions such as equations, lists, matrices, etc., which have some subexpressions in CRE form.

\texttt{untellrat (x_1, \ldots, x_n)} \hfill [Function]

Removes \texttt{tellrat} properties from \(x_1, \ldots, x_n\).
# 15 Special Functions

## 15.1 Introduction to Special Functions

Special function notation follows:

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
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<td>bessel_j (index, expr)</td>
<td>Bessel function, 1st kind</td>
</tr>
<tr>
<td>bessel_y (index, expr)</td>
<td>Bessel function, 2nd kind</td>
</tr>
<tr>
<td>bessel_i (index, expr)</td>
<td>Modified Bessel function, 1st kind</td>
</tr>
<tr>
<td>bessel_k (index, expr)</td>
<td>Modified Bessel function, 2nd kind</td>
</tr>
<tr>
<td>hankel_1 (v,z)</td>
<td>Hankel function of the 1st kind</td>
</tr>
<tr>
<td>hankel_2 (v,z)</td>
<td>Hankel function of the 2nd kind</td>
</tr>
<tr>
<td>struve_h (v,z)</td>
<td>Struve H function</td>
</tr>
<tr>
<td>struve_l (v,z)</td>
<td>Struve L function</td>
</tr>
<tr>
<td>assoc_legendre_p[v,u] (z)</td>
<td>Legendre function of degree v and order u</td>
</tr>
<tr>
<td>assoc_legendre_q[v,u] (z)</td>
<td>Legendre function, 2nd kind</td>
</tr>
<tr>
<td>%f[p,q] ([], [], expr)</td>
<td>Generalized Hypergeometric function</td>
</tr>
<tr>
<td>gamma (z)</td>
<td>Gamma function</td>
</tr>
<tr>
<td>gamma_incomplete_lower (a,z)</td>
<td>Lower incomplete gamma function</td>
</tr>
<tr>
<td>gamma_incomplete (a,z)</td>
<td>Tail of incomplete gamma function</td>
</tr>
<tr>
<td>hypergeometric (l1, l2, z)</td>
<td>Hypergeometric function</td>
</tr>
<tr>
<td>%s[u,v] (z)</td>
<td>Lommel &quot;small&quot; s function</td>
</tr>
<tr>
<td>slommel[u,v] (z)</td>
<td>Lommel &quot;big&quot; S function</td>
</tr>
<tr>
<td>%m[u,k] (z)</td>
<td>Whittaker function, 1st kind</td>
</tr>
<tr>
<td>%w[u,k] (z)</td>
<td>Whittaker function, 2nd kind</td>
</tr>
<tr>
<td>erf (z)</td>
<td>Complement of the erf function</td>
</tr>
<tr>
<td>expintegral_e (v,z)</td>
<td>Exponential integral E</td>
</tr>
<tr>
<td>expintegral_e1 (z)</td>
<td>Exponential integral E1</td>
</tr>
<tr>
<td>expintegral_ei (z)</td>
<td>Exponential integral Ei</td>
</tr>
<tr>
<td>expintegral_li (z)</td>
<td>Logarithmic integral Li</td>
</tr>
<tr>
<td>expintegral_si (z)</td>
<td>Exponential integral Si</td>
</tr>
<tr>
<td>expintegral_ci (z)</td>
<td>Exponential integral Ci</td>
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<tr>
<td>expintegral_shi (z)</td>
<td>Exponential integral Shi</td>
</tr>
<tr>
<td>expintegral_chi (z)</td>
<td>Exponential integral Chi</td>
</tr>
<tr>
<td>keliptic (z)</td>
<td>Complete elliptic integral of the first kind (K)</td>
</tr>
<tr>
<td>parabolic_cylinder_d (v,z)</td>
<td>Parabolic cylinder D function</td>
</tr>
</tbody>
</table>
15.2 Bessel Functions

\texttt{bessel\_j (v, z)} [Function]

The Bessel function of the first kind of order \( v \) and argument \( z \). See A&S eqn 9.1.10 and DLMF 10.2.E2.

\( \texttt{bessel\_j} \) is defined as

\[
J_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{v+2k}}{k! \Gamma(v+k+1)}
\]

although the infinite series is not used for computations.

When \texttt{besselexpand} is true, \texttt{bessel\_j} is expanded in terms of elementary functions when the order \( v \) is half of an odd integer. See \texttt{besselexpand}.

\texttt{bessel\_y (v, z)} [Function]

The Bessel function of the second kind of order \( v \) and argument \( z \). See A&S eqn 9.1.2 and DLMF 10.2.E3.

\( \texttt{bessel\_y} \) is defined as

\[
Y_v(z) = \frac{\cos(\pi v) J_v(z) - J_{-v}(z)}{\sin \pi v}
\]

when \( v \) is not an integer. When \( v \) is an integer \( n \), the limit as \( v \) approaches \( n \) is taken.

When \texttt{besselexpand} is true, \texttt{bessel\_y} is expanded in terms of elementary functions when the order \( v \) is half of an odd integer. See \texttt{besselexpand}.

\texttt{bessel\_i (v, z)} [Function]

The modified Bessel function of the first kind of order \( v \) and argument \( z \). See A&S eqn 9.6.10 and DLMF 10.25.E2.

\( \texttt{bessel\_i} \) is defined as

\[
I_v(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(v+k+1)} \left(\frac{z}{2}\right)^{v+2k}
\]

although the infinite series is not used for computations.

When \texttt{besselexpand} is true, \texttt{bessel\_i} is expanded in terms of elementary functions when the order \( v \) is half of an odd integer. See \texttt{besselexpand}.

\texttt{bessel\_k (v, z)} [Function]

The modified Bessel function of the second kind of order \( v \) and argument \( z \). See A&S eqn 9.6.2 and DLMF 10.27.E4.

\( \texttt{bessel\_k} \) is defined as

\[
K_v(z) = \pi \csc(\pi v) \frac{(I_{-v}(z) - I_v(z))}{2}
\]

when \( v \) is not an integer. If \( v \) is an integer \( n \), then the limit as \( v \) approaches \( n \) is taken.

When \texttt{besselexpand} is true, \texttt{bessel\_k} is expanded in terms of elementary functions when the order \( v \) is half of an odd integer. See \texttt{besselexpand}.
**hankel_1 (v, z)**  
The Hankel function of the first kind of order \( v \) and argument \( z \). See A&S eqn 9.1.3 and DLMF 10.4.E3.

\[ H_v^{(1)}(z) = J_v(z) + iy_v(z) \]

Maxima evaluates \( hankel_1 \) numerically for a complex order \( v \) and complex argument \( z \) in float precision. The numerical evaluation in bigfloat precision is not supported.

When `besselexpand` is `true`, \( hankel_1 \) is expanded in terms of elementary functions when the order \( v \) is half of an odd integer. See `besselexpand`.

Maxima knows the derivative of \( hankel_1 \) wrt the argument \( z \).

Examples:

Numerical evaluation:

\[
\begin{align*}
(\%i1) & \quad hankel_1(1,0.5); \\
(\%o1) & \quad 0.24226845767487 - 1.471472392670243 \frac{i}{z} \\
(\%i2) & \quad hankel_1(1,0.5+i); \\
(\%o2) & \quad -0.25582879948621 \frac{i}{z} - 0.23957560188301
\end{align*}
\]

Expansion of \( hankel_1 \) when `besselexpand` is `true`:

\[
\begin{align*}
(\%i1) & \quad hankel_1(1/2,z),\text{besselexpand}:\text{true}; \\
(\%o1) & \quad \frac{\sqrt{2} \sin(z)}{\sqrt{\pi} \sqrt{2} \cos(z)} - \frac{\sqrt{\pi} \cos(z)}{2}
\end{align*}
\]

Derivative of \( hankel_1 \) wrt the argument \( z \). The derivative wrt the order \( v \) is not supported. Maxima returns a noun form:

\[
\begin{align*}
(\%i1) & \quad \text{diff}(hankel_1(v,z),z); \\
(\%o1) & \quad \frac{hankel_1(v-1,z) - hankel_1(v+1,z)}{2}
\end{align*}
\]

\[
\begin{align*}
(\%i2) & \quad \text{diff}(hankel_1(v,z),v); \\
(\%o2) & \quad \frac{\text{d}}{\text{dv}} \left( hankel_1(v, z) \right)
\end{align*}
\]

**hankel_2 (v, z)**  
The Hankel function of the second kind of order \( v \) and argument \( z \). See A&S eqn 9.1.4 and DLMF 10.4.E3.

\[ H_v^{(2)}(z) = J_v(z) - iY_v(z) \]

Maxima evaluates \( hankel_2 \) numerically for a complex order \( v \) and complex argument \( z \) in float precision. The numerical evaluation in bigfloat precision is not supported.

When `besselexpand` is `true`, \( hankel_2 \) is expanded in terms of elementary functions when the order \( v \) is half of an odd integer. See `besselexpand`. 
Maxima knows the derivative of \texttt{hankel\_2} wrt the argument \( z \).

For examples see \texttt{hankel\_1}.

\textbf{besselexpand} \hspace{1cm} [Option variable]

Default value: \texttt{false}

Controls expansion of the Bessel, Hankel and Struve functions when the order is half of an odd integer. In this case, the functions can be expanded in terms of other elementary functions. When \texttt{besselexpand} is \texttt{true}, the Bessel function is expanded.

\begin{verbatim}
(%i1) besselexpand: false$
(%i2) bessel_j (3/2, z);
 3
(%o2) bessel_j(-, z)
   2

(%i3) besselexpand: true$
(%i4) bessel_j (3/2, z);
\sin(z) \cos(z)
\sqrt(2) \sqrt(z) \left(\frac{- \frac{\sqrt(z)}{2} - \frac{\sqrt(z)}{z}}{\sqrt(%pi)}\right)

(%i5) bessel_y(3/2,z);
\sin(z) \cos(z)
\sqrt(2) \sqrt(z) \left(\frac{- \frac{\sqrt(z)}{2} - \frac{\sqrt(z)}{z}}{\sqrt(%pi)}\right)

(%i6) bessel_i(3/2,z);
\cosh(z) \sinh(z)
\sqrt(2) \sqrt(z) \left(\frac{- \frac{\sqrt(z)}{2} - \frac{\sqrt(z)}{z}}{\sqrt(%pi)}\right)

(%i7) bessel_k(3/2,z);
\frac{1}{\sqrt(%pi)} \left(\frac{-1 + \frac{1}{z}}{\sqrt(2) \sqrt(z)}\right)
\end{verbatim}

\textbf{scaled\_bessel\_i}(v, z) \hspace{1cm} [Function]

The scaled modified Bessel function of the first kind of order \( v \) and argument \( z \). That is,
scaled_bessel_i(v, z) = e^{-|z|}I_v(z).

This function is particularly useful for calculating $I_v(z)$ for large $z$, which is large. However, maxima does not otherwise know much about this function. For symbolic work, it is probably preferable to work with the expression $\exp(-\text{abs}(z)) * \text{bessel_i}(v, z)$.

scaled_bessel_i0 (z)  
Identical to scaled_bessel_i(0,z).

scaled_bessel_i1 (z)  
Identical to scaled_bessel_i(1,z).

%s [u,v] (z)  
Lommel’s little $s_{\mu,\nu}(z)$ function. (DLMF 11.9.E3)(G&R 8.570.1).

This Lommel function is the particular solution of the inhomogeneous Bessel differential equation:

$$\frac{d^2 w}{dz^2} + \frac{1}{z} \frac{dw}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) w = z^{\mu - 1}$$

This can be defined by the series

$$s_{\mu,\nu}(z) = z^{\mu + 1} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{a_k(\mu, \nu)}$$

where

$$a_k(\mu, \nu) = \prod_{m=1}^{k} \left((\mu + 2m - 1)^2 - \nu^2\right) = 4^k \left(\frac{\mu - \nu + 1}{2}\right)_k \left(\frac{\mu + \nu + 1}{2}\right)_k$$

slommel [u,v] (z)  
Lommel’s big $S_{\mu,\nu}(z)$ function. (DLMF 11.9.E5)(G&R 8.570.2).

Lommel’s big S function is another particular solution of the inhomogeneous Bessel differential equation (see [%s], page 299) defined for all values of $\mu$ and $\nu$, where

$$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + 2^{\mu-1} \Gamma \left(\frac{\mu + \nu + 1}{2}\right) \Gamma \left(\frac{\mu - \nu + 1}{2}\right)$$

$$\times \left(\sin \left(\frac{(\mu - \nu)\pi}{2}\right) J_{\nu}(z) - \cos \left(\frac{(\mu - \nu)\pi}{2}\right) Y_{\nu}(z)\right)$$

When $\mu \pm \nu$ is an odd negative integer, the limit must be used.
15.3 Airy Functions

The Airy functions $\text{Ai}(x)$ and $\text{Bi}(x)$ are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, A&S Section 10.4 and DLMF 9.

The Airy differential equation is:

$$\frac{d^2 y}{dx^2} - xy = 0$$

The numerically satisfactory pair of solutions (DLMF 9.2#T1) on the real line are $y = \text{Ai}(x)$ and $y = \text{Bi}(x)$.

These two solutions are oscillatory for $x < 0$. $\text{Ai}(x)$ is the solution subject to the condition that $y \to 0$ as $x \to +\infty$, and $\text{Bi}(x)$ is the second solution with the same amplitude as $\text{Ai}(x)$ as $x \to -\infty$ which differs in phase by $\pi/2$. Also, $\text{Bi}(x)$ is unbounded as $x \to +\infty$.

If the argument $x$ is a real or complex floating point number, the numerical value of the function is returned.

**airy_ai(x)**

The Airy function $\text{Ai}(x)$. See A&S eqn 10.4.2 and DLMF 9.

See also **airy_bi**, **airy_dai**, and **airy_dbi**.

**airy_dai(x)**

The derivative of the Airy function $\text{Ai}(x)$:

$$\text{airy_dai}(x) = \frac{d}{dx} \text{Ai}(x)$$

See **airy_ai**.

**airy_bi(x)**

The Airy function $\text{Bi}(x)$. See A&S eqn 10.4.3 and DLMF 9.

See **airy_ai**, and **airy_dbi**.

**airy_dbi(x)**

The derivative of the Airy function $\text{Bi}(x)$:

$$\text{airy_dbi}(x) = \frac{d}{dx} \text{Bi}(x)$$

See **airy_ai**, and **airy_bi**.
15.4 Gamma and Factorial Functions

The gamma function and the related beta, psi and incomplete gamma functions are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, Chapter 6.

**bffac (expr, n)**  
Bigfloat version of the factorial (shifted gamma) function. The second argument is how many digits to retain and return, it’s a good idea to request a couple of extra.

```latex
(%i1) bffac(1/2,16);  
(%o1) 8.862269254527584b-1
(%i2) (1/2)!,numer;  
(%o2) 0.886226925452758
(%i3) bffac(1/2,32);  
(%o3) 8.862269254527580136490837416707b-1
```

**bfpsi (n, z, fpprec)**  
**bfpsi0 (z, fpprec)**

bfpsi is the polygamma function of real argument \( z \) and integer order \( n \). See [polygamma], page 313, for further information. bfpsi0 is the digamma function. \( \text{bfpsi0}(z, fpprec) \) is equivalent to \( \text{bfpsi}(0, z, fpprec) \).

These functions return bigfloat values. \( fpprec \) is the bigfloat precision of the return value.

```latex
(%i1) bfpsi0(1/3, 15);  
(%o1) - 3.13203378002081b0
(%i2) bfpsi0(1/3, 32);  
(%o2) - 3.1320337800208063229964190742873b0
(%i3) psi[0](1/3);  
(%o3) 3 \log(3) \frac{\%pi}{2 \sqrt{3}}
(%i4) float(%);  
(%o4) - 3.132033780020806
```

**cbffac (z, fpprec)**

Complex bigfloat factorial.

load ("bffac") loads this function.

```latex
(%i1) cbffac(1+%i,16);  
(%o1) 3.430658398165453b-1 %i + 6.529654964201666b-1
(%i2) (1+%i)!,numer;  
(%o2) 0.3430658398165453 %i + 0.6529654964201667
```

**gamma (z)**

The basic definition of the gamma function (DLMF 5.2.E1 and A&S eqn 6.1.1) is

\[
\Gamma (z) = \int_0^\infty t^{z-1} e^{-t} \, dt
\]
Maxima simplifies \texttt{gamma} for positive integer and positive and negative rational numbers. For half integral values the result is a rational number times $\sqrt{\pi}$. The simplification for integer values is controlled by \texttt{factlim}. For integers greater than \texttt{factlim} the numerical result of the factorial function, which is used to calculate \texttt{gamma}, will overflow. The simplification for rational numbers is controlled by \texttt{gammalim} to avoid internal overflow. See \texttt{factlim} and \texttt{gammalim}.

For negative integers \texttt{gamma} is not defined.

Maxima can evaluate \texttt{gamma} numerically for real and complex values in float and bigfloat precision.

\texttt{gamma} has mirror symmetry.

When \texttt{gamma\_expand} is \texttt{true}, Maxima expands \texttt{gamma} for arguments $z+n$ and $z-n$ where $n$ is an integer.

Maxima knows the derivative of \texttt{gamma}.

Examples:

Simplification for integer, half integral, and rational numbers:

\begin{verbatim}
(%i1) map('gamma,[1,2,3,4,5,6,7,8,9]);
(%o1) [1, 1, 2, 6, 24, 120, 720, 5040, 40320]
(%i2) map('gamma,[1/2,3/2,5/2,7/2]);
    sqrt(%pi) 3 sqrt(%pi) 15 sqrt(%pi)
(%o2) [sqrt(%pi), ---------, -----------, ----------]
        2  4  8
(%i3) map('gamma,[2/3,5/3,7/3]);
    2  1
2 gamma(-) 4 gamma(-)
(%o3) [gamma(-), --------, --------]  
   2 3 3
   3 3 9
\end{verbatim}

Numerical evaluation for real and complex values:

\begin{verbatim}
(%i4) map('gamma,[2.5,2.5b0]);
(%o4) [1.329340388179137, 1.3293403881791370205b0]
(%i5) map('gamma,[1.0+%i,1.0b0+%i]);
(%o5) [0.498015668118356 - .1549498283018107 %i,
        4.9801566811835604272b-1 - 1.5494982830181068513b-1 %i]
\end{verbatim}

\texttt{gamma} has mirror symmetry:

\begin{verbatim}
(%i6) declare(z,complex)$
(%i7) conjugate(gamma(z));
(%o7) gamma(conjugate(z))
\end{verbatim}

Maxima expands \texttt{gamma}\(z+n\) and \texttt{gamma}\(z-n\), when \texttt{gamma\_expand} is \texttt{true}:

\begin{verbatim}
(%i8) gamma\_expand:true$
(%i9) [gamma(z+1),gamma(z-1),gamma(z+2)/gamma(z+1)];
    gamma(z)
(%o9) [z gamma(z), -------, z + 1]
        z - 1
\end{verbatim}
The derivative of $\gamma$:

\begin{verbatim}
(%i10) diff(gamma(z),z);
(%o10) psi (z) gamma(z)
0
\end{verbatim}

See also $\texttt{makegamma}$.
The Euler-Mascheroni constant is $\%\gamma$.

$\log\_\gamma (z)$ \hfill [Function]
The natural logarithm of the gamma function.

\begin{verbatim}
(%i1) gamma(6);
(%o1) 120
(%i2) log\_\gamma (6);
(%o2) log(120)
(%i3) log\_\gamma (0.5);
(%o3) 0.5723649429247004
\end{verbatim}

$\texttt{gamma\_incomplete\_lower (a, z)}$ \hfill [Function]
The lower incomplete gamma function (DLMF 8.2.E1 and A&S eqn 6.5.2):

\[
\gamma (a, z) = \int_0^z t^{a-1} e^{-t} \, dt
\]

See also $\texttt{gamma\_incomplete}$ (upper incomplete gamma function).

$\texttt{gamma\_incomplete (a, z)}$ \hfill [Function]
The incomplete upper gamma function (DLMF 8.2.E2 and A&S eqn 6.5.3):

\[
\Gamma (a, z) = \int_z^\infty t^{a-1} e^{-t} \, dt
\]

See also $\texttt{gamma\_expand}$ for controlling how $\texttt{gamma\_incomplete}$ is expressed in terms of elementary functions and $\texttt{erfc}$.
Also see the related functions $\texttt{gamma\_incomplete\_regularized}$ and $\texttt{gamma\_incomplete\_generalized}$.

$\texttt{gamma\_incomplete\_regularized (a, z)}$ \hfill [Function]
The regularized incomplete upper gamma function (DLMF 8.2.E4):

\[
Q (a, z) = \frac{\Gamma (a, z)}{\Gamma (a)}
\]

See also $\texttt{gamma\_expand}$ for controlling how $\texttt{gamma\_incomplete}$ is expressed in terms of elementary functions and $\texttt{erfc}$.
Also see $\texttt{gamma\_incomplete}$.

$\texttt{gamma\_incomplete\_generalized (a, z1, z1)}$ \hfill [Function]
The generalized incomplete gamma function.

\[
\Gamma (a, z_1, z_2) = \int_{z_1}^{z_2} t^{a-1} e^{-t} \, dt
\]

Also see $\texttt{gamma\_incomplete}$ and $\texttt{gamma\_incomplete\_regularized}$.
gamma_expand

Default value: false

gamma_expand controls expansion of gamma_incomplete. When gamma_expand
is true, gamma_incomplete(v,z) is expanded in terms of z, exp(z), and
gamma_incomplete or erfc when possible.

(%i1) gamma_incomplete(2,z);
(%o1) gamma_incomplete(2, z)
(%i2) gamma_expand:true;
(%o2) true
(%i3) gamma_incomplete(2,z);
\(z\)
(%o3) (z + 1) %e
(%i4) gamma_incomplete(3/2,z);
\(- z\)
(%o4) sqrt(z) %e + -----------------------
2
(%i5) gamma_incomplete(4/3,z);
\(1\)
(%o5) gamma_incomplete(-, z)
\(1/3\)
\(- z\)
(%o6) z %e + -----------------------
3
(%i6) gamma_incomplete(a+2,z);
\(a\)
(%o6) z (z + a + 1) %e + a (a + 1) gamma_incomplete(a, z)
(%i7) gamma_incomplete(a-2, z);
\(a - 2\)
\(z\)
(%o7) gamma_incomplete(a, z) a - 2 (-------------- + -----) %e
(1 - a) (2 - a) (a - 2) (a - 1) a - 2

gammalim

Default value: 10000

gammalim controls simplification of the gamma function for integral and rational num-
ber arguments. If the absolute value of the argument is not greater than gammalim,
then simplification will occur. Note that the factlim switch controls simplification
of the result of gamma of an integer argument as well.

makegamma (expr)

Transforms instances of binomial, factorial, and beta functions in expr into gamma
functions.

See also makefact.

(%i1) makegamma(binomial(n,k));
\(gamma(n + 1)\)
(%o1) -----------------------------
\(gamma(k + 1) gamma(n - k + 1)\)
The beta function is defined as

\[ B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \]

(DLMF 5.12.E1 and A&S eqn 6.2.1).

Maxima simplifies the beta function for positive integers and rational numbers, which sum to an integer. When `beta_args_sum_to_integer` is `true`, Maxima simplifies also general expressions which sum to an integer.

For \(a\) or \(b\) equal to zero the beta function is not defined.

In general the beta function is not defined for negative integers as an argument. The exception is for \(a=-n\), \(n\) a positive integer and \(b\) a positive integer with \(b\leq n\), it is possible to define an analytic continuation. Maxima gives for this case a result.

When `beta_expand` is `true`, expressions like \(\text{beta}(a+n,b)\) and \(\text{beta}(a-n,b)\) or \(\text{beta}(a,b+n)\) and \(\text{beta}(a,b-n)\) with \(n\) an integer are simplified.

Maxima can evaluate the beta function for real and complex values in float and bigfloat precision. For numerical evaluation Maxima uses `log_gamma`:

\[ - \log\Gamma(b + a) + \log\Gamma(b) + \log\Gamma(a) \]

\%e

Maxima knows that the beta function is symmetric and has mirror symmetry.

Maxima knows the derivatives of the beta function with respect to \(a\) or \(b\).

To express the beta function as a ratio of gamma functions see `makegamma`.

Examples:

Simplification, when one of the arguments is an integer:

\(\%i1\) [\text{beta}(2,3), \text{beta}(2,1/3), \text{beta}(2,a)];

\(\%o1\)

\[
\begin{array}{ccc}
1 & 9 & 1 \\
12 & a & (a + 1)
\end{array}
\]

Simplification for two rational numbers as arguments which sum to an integer:

\(\%i2\) [\text{beta}(1/2,5/2), \text{beta}(1/3,2/3), \text{beta}(1/4,3/4)];

\(\%o2\)

\[
\begin{array}{ccc}
3 \%pi & 2 \%pi & 8 \\
--- & ---- & sqrt(2) \%pi & sqrt(3)
\end{array}
\]

When setting `beta_args_sum_to_integer` to `true` more general expression are simplified, when the sum of the arguments is an integer:

\(\%i3\) `beta_args_sum_to_integer:true$`
\( \text{beta}(a+1, -a+2); \)

\[\frac{\%\pi \ (a - 1) \ a}{2 \ \sin(\%\pi \ (2 - a))} \]

The possible results, when one of the arguments is a negative integer:

\( \text{beta}(-3, 1), \text{beta}(-3, 2), \text{beta}(-3, 3); \)

\[\frac{1}{3}, \frac{1}{6}, \frac{1}{3} \]

\( \text{beta}(a+n, b) \) or \( \text{beta}(a-n, b) \) with \( n \) an integer simplifies when \( \text{beta}_\text{expand} \) is true:

\( \text{beta}_\text{expand}: \text{true}; \)

\( \text{beta}(a+1, b), \text{beta}(a-1, b), \text{beta}(a+1, b)/\text{beta}(a, b+1); \)

\[\frac{a \ \text{beta}(a, b) \ \text{beta}(a, b) \ (b + a - 1) \ a}{b + a \ a - 1 \ b} \]

Beta is not defined, when one of the arguments is zero:

\( \text{beta}(0, b); \)

\text{beta: expected nonzero arguments; found 0, b} \)

-- an error. To debug this try debugmode(true); 

Numerical evaluation for real and complex arguments in float or bigfloat precision:

\( \text{beta}(2.5, 2.3); \)

\( 0.08694748611299981 \)

\( \text{beta}(2.5, 1.4+\%i); \)

\( 0.0640144950796695 - 0.1502078053286415 \ \%i \)

\( \text{beta}(2.5b0, 2.3b0); \)

\( 8.694748611299969b-2 \)

\( \text{beta}(2.5b0, 1.4b0+\%i); \)

\( 6.401449507966944b-2 - 1.502078053286415b-1 \ \%i \)

Beta is symmetric and has mirror symmetry:

\( \text{beta}(a, b)-\text{beta}(b, a); \)

\( 0 \)

\( \text{declare}(a, \text{complex}, b, \text{complex}); \)

\( \text{conjugate}(\text{beta}(a, b)); \)

\( \text{beta}(\text{conjugate}(a), \text{conjugate}(b)) \)

The derivative of the beta function wrt \( a \):

\( \text{diff}(\text{beta}(a, b), a); \)

\[-\text{beta}(a, b) \ (\psi (b + a) - \psi (a))\]

\( 0 \)
The basic definition of the incomplete beta function (DLMF 8.17.E1 and A&S eqn 6.6.1) is

$$B_z(a, b) = \int_0^z t^{a-1} (1-t)^{b-1} \, dt$$

This definition is possible for Re($a$) > 0 and Re($b$) > 0 and $|z| < 1$. For other values the incomplete beta function can be defined through a generalized hypergeometric function:

$$\text{gamma}(a) \ \text{hypergeometric\_generalized}([a, 1 - b], [a + 1], z) \ z$$

(See https://functions.wolfram.com/GammaBetaErf/Beta3/ for a complete definition of the incomplete beta function.)

For negative integers $a = -n$ and positive integers $b = m$ with $m \leq n$ the incomplete beta function is defined through

$$z^{n-1} \frac{\sum_{k=0}^{m-1} (1-m)_k z^k}{k!(n-k)}$$

Maxima uses this definition to simplify beta_incomplete for $a$ a negative integer.

For $a$ a positive integer, beta_incomplete simplifies for any argument $b$ and $z$ and for $b$ a positive integer for any argument $a$ and $z$, with the exception of $a$ a negative integer.

For $z = 0$ and Re($a$) > 0, beta_incomplete has the specific value zero. For $z = 1$ and Re($b$) > 0, beta_incomplete simplifies to the beta function $\text{beta}(a,b)$.

Maxima evaluates beta_incomplete numerically for real and complex values in float or bigfloat precision. For the numerical evaluation an expansion of the incomplete beta function in continued fractions is used.

When the option variable beta_expand is true, Maxima expands expressions like beta_incomplete(a+n,b,z) and beta_incomplete(a-n,b,z) where $n$ is a positive integer.

Maxima knows the derivatives of beta_incomplete with respect to the variables $a$, $b$ and $z$ and the integral with respect to the variable $z$.

Examples:

Simplification for $a$ a positive integer:

(\%i1) beta_incomplete(2,b,z);

\begin{align*}
\frac{b}{1 - (1 - z) (b z + 1)} & \quad \text{\%o1} \\
\frac{b}{b (b + 1)} &
\end{align*}

Simplification for $b$ a positive integer:

(\%i2) beta_incomplete(a,2,z);

\begin{align*}
\frac{a}{(a (1 - z) + 1) z} & \quad \text{\%o2} \\
\end{align*}
Simplification for $a$ and $b$ a positive integer:

$\text{(i3)}$ beta_incomplete(3, 2, z);

$\text{(o3)}$ \[
\frac{3 (1 - z) + 1}{12} \cdot z
\]

$a$ is a negative integer and $b \leq (-a)$, Maxima simplifies:

$\text{(i4)}$ beta_incomplete(-3, 1, z);

$\text{(o4)}$ \[
-\frac{1}{3} \cdot \frac{3}{z}
\]

For the specific values $z = 0$ and $z = 1$, Maxima simplifies:

$\text{(i5)}$ assume(a > 0, b > 0)$

$\text{(i6)}$ beta_incomplete(a, b, 0);

$\text{(o6)}$ 0

$\text{(i7)}$ beta_incomplete(a, b, 1);

$\text{(o7)}$ beta(a, b)

Numerical evaluation in float or bigfloat precision:

$\text{(i8)}$ beta_incomplete(0.25, 0.50, 0.9);

$\text{(o8)}$ 4.594959440269333

$\text{(i9)}$ fpprec: 25$

$\text{(i10)}$ beta_incomplete(0.25, 0.50, 0.9b0);

$\text{(o10)}$ 4.594959440269324086971203b0

For $\text{abs}(z) > 1$ beta_incomplete returns a complex result:

$\text{(i11)}$ beta_incomplete(0.25, 0.50, 1.7);

$\text{(o11)}$ 5.24115108584249 - 1.455187787844 %i

Results for more general complex arguments:

$\text{(i14)}$ beta_incomplete(0.25+%i, 1.0+%i, 1.7+%i);

$\text{(o14)}$ 2.726960675662536 - .3831175704269199 %i

$\text{(i15)}$ beta_incomplete(1/2, 5/4+%i, 2.8+%i);

$\text{(o15)}$ 13.04649635168716 %i - 5.802067956270001

Expansion, when beta_expand is true:

$\text{(i23)}$ beta_incomplete(a+1, b, z), beta_expand: true;

$\text{(o23)}$ \[
\frac{a \cdot \beta(a, b, z) - (1 - z) \cdot z}{b + a}
\]

$\text{(i24)}$ beta_incomplete(a-1, b, z), beta_expand: true;

$\text{(o24)}$ \[
\frac{b \cdot a - 1}{b + a}
\]
beta_incomplete(a, b, z) \frac{(-b-a+1)(1-z)z}{1-a} \quad 1-a

Derivative and integral for beta_incomplete:

(%i34) \text{diff(beta_incomplete(a, b, z), z);} 
\text{\quad b-1} \quad \text{a-1}

(%o34) \text{(1-z)z}

(%i35) \text{integrate(beta_incomplete(a, b, z), z);} 
\text{\quad b} \quad \text{a}

(%o35) \quad \text{a beta_incomplete(a, b, z)} \quad \text{\quad B(a+1, b+1)}

beta_incomplete_regularized (a, b, z) [Function]

The regularized incomplete beta function (DLMF 8.17.E2 and A&S eqn 6.6.2), defined as

\[ I_z(a, b) = \frac{B_z(a, b)}{B(a, b)} \]

As for beta_incomplete this definition is not complete. See https://functions.wolfram.com/GammaBetaErf/BetaRegularized/ for a complete definition of beta_incomplete_regularized.

beta_incomplete_regularized simplifies a or b a positive integer.

For z = 0 and Re(a) > 0, beta_incomplete_regularized has the specific value 0. For z = 1 and Re(b) > 0, beta_incomplete_regularized simplifies to 1.

Maxima can evaluate beta_incomplete_regularized for real and complex arguments in float and bigfloat precision.

When beta_expand is true, Maxima expands beta_incomplete_regularized for arguments a + n or a − n, where n is an integer.

Maxima knows the derivatives of beta_incomplete_regularized with respect to the variables a, b, and z and the integral with respect to the variable z.

Examples:

Simplification for a or b a positive integer:

(%i11) beta_incomplete_regularized(2,b,z);
\text{\quad b}

(%o11) 1 - (1-z)(b z + 1)

(%i12) beta_incomplete_regularized(a,2,z);
\text{\quad a}
(\%o2) \hskip 3.5em (a (1 - z) + 1) z

(\%i3) \text{beta_incomplete_regularized}(3,2,z); \hskip 3.5em \frac{3}{(1 - z) z}

(\%o3) \hskip 3.5em (3 (1 - z) + 1) z

For the specific values \(z = 0\) and \(z = 1\), Maxima simplifies:

(\%i4) \text{assume}(a>0,b>0)$
(\%i5) \text{beta_incomplete_regularized}(a,b,0); \hskip 3.5em 0
(\%o5) \hskip 3.5em 0

(\%i6) \text{beta_incomplete_regularized}(a,b,1); \hskip 3.5em 1
(\%o6) \hskip 3.5em 1

Numerical evaluation for real and complex arguments in float and bigfloat precision:

(\%i7) \text{beta_incomplete_regularized}(0.12,0.43,0.9); \hskip 3.5em .9114011367359802
(\%o7) \hskip 3.5em .9114011367359802

(\%i8) \text{fpprec}:32$
(\%i9) \text{beta_incomplete_regularized}(0.12,0.43,0.9b0); \hskip 3.5em 9.1140113673598075519946998779975b-1
(\%o9) \hskip 3.5em 9.1140113673598075519946998779975b-1

(\%i10) \text{beta_incomplete_regularized}(1+i,3/3,1.5*\%i); \hskip 3.5em .2865367499935403 %i - 0.122995963334684
(\%o10) \hskip 3.5em .2865367499935403 %i - 0.122995963334684

(\%i11) \text{fpprec}:20$
(\%i12) \text{beta_incomplete_regularized}(1+i,3/3,1.5b0*\%i); \hskip 3.5em 2.8653674999354036142b-1 %i - 1.2299596333468400163b-1
(\%o12) \hskip 3.5em 2.8653674999354036142b-1 %i - 1.2299596333468400163b-1

Expansion, when \text{beta_expand} is true:

(\%i13) \text{beta_incomplete_regularized}(a+1,b,z); \hskip 3.5em \frac{b a}{(1 - z) z}

(\%o13) \hskip 3.5em \frac{b a}{(1 - z) z} - \frac{a}{\text{beta}(a, b)} \text{beta}(a, b)

(\%i14) \text{beta_incomplete_regularized}(a-1,b,z); \hskip 3.5em \frac{b a - 1}{(1 - z) z}

(\%o14) \hskip 3.5em \frac{b a - 1}{(1 - z) z} - \frac{1}{\text{beta}(a, b)} \text{beta}(a, b) (b + a - 1)

The derivative and the integral wrt \(z\):

(\%i15) \text{diff(beta_incomplete_regularized(a,b,z),z)}; \hskip 3.5em \frac{b - 1}{\text{beta}(a, b)} \text{beta}(a, b)

(\%o15) \hskip 3.5em \frac{b - 1}{\text{beta}(a, b)} \text{beta}(a, b)

(\%i16) \text{integrate(beta_incomplete_regularized(a,b,z),z)}; \hskip 3.5em \frac{b a}{(1 - z) z}

(\%o16) \hskip 3.5em \frac{b a}{(1 - z) z} - \frac{1}{\text{beta}(a, b)} \text{beta}(a, b) (b + a - 1)
beta_incomplete_generalized \( a, b, z_1, z_2 \)

The basic definition of the generalized incomplete beta function is

\[
\int_{z_1}^{z_2} t^{a-1}(1-t)^{b-1} \, dt
\]

Maxima simplifies \( \text{beta_incomplete} \) for \( a \) and \( b \) a positive integer.

For \( \text{Re}(a) > 0 \) and \( z_1 = 0 \) or \( z_2 = 0 \), Maxima simplifies \( \text{beta_incomplete} \) to \( \text{beta_incomplete} \). For \( \text{Re}(b) > 0 \) and \( z_1 = 1 \) or \( z_2 = 1 \), Maxima simplifies to an expression with \( \text{beta} \) and \( \text{beta_incomplete} \).

Maxima evaluates \( \text{beta_incomplete} \) for real and complex values in float and bigfloat precision.

When \( \text{beta_expand} \) is true, Maxima expands \( \text{beta_incomplete_generalized} \) for \( a + n \) and \( a - n \), \( n \) a positive integer.

Maxima knows the derivative of \( \text{beta_incomplete} \) with respect to the variables \( a, b, z_1, \) and \( z_2 \) and the integrals with respect to the variables \( z_1 \) and \( z_2 \).

Examples:

Maxima simplifies \( \text{beta_incomplete} \) for \( a \) and \( b \) a positive integer:

\[
\begin{align*}
\text{(i1)} & \quad \text{beta_incomplete}(2,b,z_1,z_2); \\
& \quad \frac{\left(1 - z_1\right) \left(b z_1 + 1\right) - \left(1 - z_2\right) \left(b z_2 + 1\right)}{b \left(b + 1\right)} \\
\text{(i2)} & \quad \text{beta_incomplete}(a,2,z_1,z_2); \\
& \quad \frac{\left(a \left(1 - z_2\right) + 1\right) z_2 - \left(a \left(1 - z_1\right) + 1\right) z_1}{a \left(a + 1\right)} \\
\text{(i3)} & \quad \text{beta_incomplete}(3,2,z_1,z_2); \\
& \quad \frac{\left(1 - z_1\right) \left(3 z_1 + 2 z_2 + 1\right) - \left(1 - z_2\right) \left(3 z_2 + 2 z_2 + 1\right)}{12}
\end{align*}
\]

Simplification for specific values \( z_1 = 0, z_2 = 0, z_1 = 1, \) or \( z_2 = 1 \):

\[
\begin{align*}
\text{(i4)} & \quad \text{assume}(a > 0, b > 0)$ \\
\text{(i5)} & \quad \text{beta_incomplete}(a,b,z_1,0); \\
& \quad \text{beta_incomplete}(a, b, z_1) \\
\text{(i6)} & \quad \text{beta_incomplete}(a,b,0,z_2); \\
& \quad \text{beta_incomplete}(a, b, z_2) \\
\text{(i7)} & \quad \text{beta_incomplete}(a,b,z_1,1);
\end{align*}
\]
beta(a, b) - beta_incomplete(a, b, z1)

beta_incomplete_generalized(a,b,1,z2);

beta_incomplete(a, b, z2) - beta(a, b)

Numerical evaluation for real arguments in float or bigfloat precision:

beta_incomplete_generalized(1/2,3/2,0.25,0.31);

.09638178086368676

fpprec:32$

beta_incomplete_generalized(1/2,3/2,0.25,0.31b0);

9.6381780863686935309170054689964b-2

Numerical evaluation for complex arguments in float or bigfloat precision:

beta_incomplete_generalized(1/2+%i,3/2+%i,0.25,0.31);

- .096254630032053676 %i - .003323847735353769

fpprec:20$

beta_incomplete_generalized(1/2+%i,3/2+%i,0.25,0.31b0);

- 9.6254630032054178691b-2 %i - 3.3238477353543591914b-3

Expansion for \( a + n \) or \( a - n \), \( n \) a positive integer, when \( \text{beta}\_\text{expand} \) is \( \text{true} \):

beta_incomplete_generalized(a+1,b,z1,z2);

\[
\frac{(1 - z1) z1 - (1 - z2) z2}{b + a}
\]

beta_incomplete_generalized(a-1,b,z1,z2);

\[
\frac{\text{beta}\_\text{incomplete}\_\text{generalized}(a, b, z1, z2) (- b - a + 1)}{1 - a}
\]

Derivative wrt the variable \( z1 \) and integrals wrt \( z1 \) and \( z2 \):

\[
\frac{\text{diff(beta}\_\text{incomplete}\_\text{generalized}(a,b,z1,z2),z1)}{b - 1 \ a - 1}
\]

\[
\frac{\text{integrate(beta}\_\text{incomplete}\_\text{generalized}(a,b,z1,z2),z1)}{b - 1 \ a - 1}
\]

\[
\frac{\text{beta}\_\text{incomplete}\_\text{generalized}(a, b, z1, z2) z1}{1 - a}
\]

+ \text{beta}\_\text{incomplete}(a + 1, b, z1)
(\%i19) integrate(beta_incomplete_generalized(a, b, z1, z2), z2);
(\%o19) beta_incomplete_generalized(a, b, z1, z2) z2
- beta_incomplete(a + 1, b, z2)

beta_expand
[Option variable]
Default value: false
When beta_expand is true, beta(a, b) and related functions are expanded for arguments like a + n or a - n, where n is an integer.
See [beta], page 305, for examples.

beta_args_sum_to_integer
[Option variable]
Default value: false
When beta_args_sum_to_integer is true, Maxima simplifies beta(a, b), when the arguments a and b sum to an integer.
See [beta], page 305, for examples.

psi[n](x)
psi[n](x) is the polygamma function (DLMF 5.2E2, DLMF 5.15, A&S eqn 6.3.1 and A&S eqn 6.4.1) defined by

$$\psi^{(n)}(x) = \frac{d^{n+1}}{dx^{n+1}} \log \Gamma(x)$$

Thus, psi[0](x) is the first derivative, psi[1](x) is the second derivative, etc.
Maxima can compute some exact values for rational args as well for float and bfloat args. Several variables control what range of rational args $$\psi^{(n)}(x)$$ will return an exact value, if possible. See maxpsiposint, maxpsinegint, maxpsifracnum, and maxpsifracdenom. That is, x must lie between maxpsinegint and maxpsiposint. If the absolute value of the fractional part of x is rational and has a numerator less than maxpsifracnum and has a denominator less than maxpsifracdenom, $$\psi^{(0)}(x)$$ will return an exact value.

The function bfpsi in the bffac package can compute numerical values.

(\%i1) psi[0](.25);
(\%o1) - 4.227453533376265
(\%i2) psi[0](1/4);
(\%o2) (- 3 log(2)) - --- - %gamma
     2
(\%i3) float(\%);
(\%o3) - 4.227453533376265
(\%i4) psi[2](0.75);
(\%o4) 5.30263321633764
(\%i5) psi[2](3/4);
                1  3
(\%o5) psi (-) + 4 %pi
                2  4
(\%i6) float(\%);
(\%o6) - 5.30263321633764
maxpsiposint

Default value: 20

maxpsiposint is the largest positive integer value for which $\psi^{(n)}(m)$ gives an exact value for rational $x$.

(\%i1) psi[0](20);
(\%o1) 275295799
       ----- - %gamma
       77597520

(\%i2) psi[0](21);
(\%o2) psi (21)
       0

(\%i3) psi[2](20);
(\%o3) 1683118856778495358491487
       2 (------------------------- - zeta(3))
       1401731326612193601024000

(\%i4) psi[2](21);
(\%o4) psi (21)
       2

maxpsinegint

Default value: -10

maxpsinegint is the most negative value for which $\psi^{(n)}(x)$ will try to compute an exact value for rational $x$. That is if $x$ is less than maxpsinegint, $\psi^{(n)}(x)$ will not return simplified answer, even if it could.

(\%i1) psi[0](-100/9);
(\%o1) psi (- ---)
       0 9

(\%i2) psi[0](-100/11);
(\%o2) %pi cot(- ---) + psi (-) + ---------------
       11 0 11 381905105400

(\%i3) psi[2](-100/9);
(\%o3) psi (- ---)
       2 9

(\%i4) psi[2](-100/11);
(\%o4) 2 %pi cot(- ---) csc (- ---) + psi (-)
       11 11 2 11
       74191313259470963498957651385614962459
       +-----------------------------------
       27850718060013605318710152732000000
maxpsifracnum

Default value: 6

Let $x$ be a rational number of the form $p/q$. If $p$ is greater than \texttt{maxpsifracnum}, then $\psi^{(0)}(x)$ will not try to return a simplified value.

\begin{verbatim}
(%i1) psi[0](3/4);
\%pi
(- 3 log(2)) + --- - \%gamma
2
(%i2) psi[2](3/4);
1 3
psi (-) + 4 \%pi
2 4
(%i3) maxpsifracnum:2;
(%o3) 2
(%i4) psi[0](3/4);
3
psi (-)
0 4
(%i5) psi[2](3/4);
1 3
psi (-) + 4 \%pi
2 4
\end{verbatim}

maxpsifracdenom

Default value: 6

Let $x$ be a rational number of the form $p/q$. If $q$ is greater than \texttt{maxpsifracdenom}, then $\psi^{(0)}(x)$ will not try to return a simplified value.

\begin{verbatim}
(%i1) psi[0](3/4);
\%pi
(- 3 log(2)) + --- - \%gamma
2
(%i2) psi[2](3/4);
1 3
psi (-) + 4 \%pi
2 4
(%i3) maxpsifracdenom:2;
(%o3) 2
(%i4) psi[0](3/4);
3
psi (-)
0 4
(%i5) psi[2](3/4);
1 3
psi (-) + 4 \%pi
2 4
\end{verbatim}
makefact (expr)  
Transforms instances of binomial, gamma, and beta functions in expr into factorials.
See also makegamma.

(%i1) makefact(binomial(n,k));
(%o1) \frac{n!}{k! (n - k)!}

(%i2) makefact(gamma(x));
(%o2) (x - 1)!

(%i3) makefact(beta(a,b));
(%o3) \frac{(a - 1)! (b - 1)!}{(b + a - 1)!}

numfactor (expr)  
Returns the numerical factor multiplying the expression expr, which should be a single term.
content returns the greatest common divisor (gcd) of all terms in a sum.

(%i1) gamma (7/2);
(%o1) \frac{15 \sqrt{\pi}}{8}

(%i2) numfactor (%);
(%o2) \frac{15}{8}

15.5 Exponential Integrals
The Exponential Integral and related functions are defined in Abramowitz and Stegun, 
*Handbook of Mathematical Functions, A&S Chapter 5.*

expintegral_e1 (z)  
The Exponential Integral E1(z) defined as

\[ E_1(z) = \int_z^\infty \frac{e^{-t}}{t} \, dt \]

with |arg z| < \pi .  (A&S eqn 5.1.1) and (DLMF 6.2E2)
This can be written in terms of other functions. See [expintrep], page 318, for examples.

expintegral_ei (x)  
The Exponential Integral Ei(x) defined as

\[ Ei(x) = -\int_{-x}^\infty \frac{e^{-t}}{t} \, dt = \int_{-\infty}^x \frac{e^t}{t} \, dt \]
with \( x \) real and \( x > 0 \). (A&S eqn 5.1.2) and (DLMF 6.2E5)
This can be written in terms of other functions. See \citexpintrep\, page 318, for examples.

\textbf{expintegrral\_li \begin{math} (x) \end{math}}

The Exponential Integral \( li(x) \) defined as

\[
li(x) = \int_0^x \frac{dt}{\ln t}
\]

with \( x \) real and \( x > 1 \). (A&S eqn 5.1.3) and (DLMF 6.2E8)
This can be written in terms of other functions. See \citexpintrep\, page 318, for examples.

\textbf{expintegrral\_e \begin{math} (n,z) \end{math}}

The Exponential Integral \( En(z) \) (A&S eqn 5.1.4) defined as

\[
E_n(z) = \int_1^\infty e^{-zt} \frac{1}{t^n} dt
\]

with \( \Re(z) > 1 \) and \( n \) a non-negative integer.
For half-integral orders, this can be written in terms of \texttt{erfc} or \texttt{erf}. See \citexpintexpand\, page 322, for examples.

\textbf{expintegrral\_si \begin{math} (z) \end{math}}

The Exponential Integral \( Si(z) \) (A&S eqn 5.2.1 and DLMF 6.2#E9) defined as

\[
Si(z) = \int_0^z \frac{\sin t}{t} dt
\]

This can be written in terms of other functions. See \citexpintrep\, page 318, for examples.

\textbf{expintegrral\_ci \begin{math} (z) \end{math}}

The Exponential Integral \( Ci(z) \) (A&S eqn 5.2.2 and DLMF 6.2#E13) defined as

\[
Ci(z) = \gamma + \log z + \int_0^z \frac{\cos t - 1}{t} dt
\]

with \( |\arg z| < \pi \).
This can be written in terms of other functions. See \citexpintrep\, page 318, for examples.

\textbf{expintegrral\_shi \begin{math} (z) \end{math}}

The Exponential Integral \( Shi(z) \) (A&S eqn 5.2.3 and DLMF 6.2#E15) defined as

\[
Shi(z) = \int_0^z \frac{\sinh t}{t} dt
\]

This can be written in terms of other functions. See \citexpintrep\, page 318, for examples.
expintegral_chi (z)  

The Exponential Integral Chi(z) (A&S eqn 5.2.4 and DLMF 6.2#E16) defined as

\[
\mathrm{Chi}(z) = \gamma + \log z + \int_{0}^{z} \frac{\cosh t - 1}{t} dt
\]

with \(|\operatorname{arg} z| < \pi\).

This can be written in terms of other functions. See [expintrep], page 318, for examples.

expintrep  

[Option variable]

Default value: false

Change the representation of one of the exponential integrals, \(\text{expintegral\_e}(m, z)\), \(\text{expintegral\_e1}\), or \(\text{expintegral\_ei}\) to an equivalent form if possible.

Possible values for \text{expintrep} are \text{false}, \text{gamma\_incomplete}, \text{expintegral\_e1}, \text{expintegral\_ei}, \text{expintegral\_li}, \text{expintegral\_trig}, or \text{expintegral\_hyp}.

\text{false} means that the representation is not changed. Other values indicate the representation is to be changed to use the function specified where \text{expintegral\_trig} means \text{expintegral\_si}, \text{expintegral\_ci}; and \text{expintegral\_hyp} means \text{expintegral\_shi} or \text{expintegral\_chi}.

Here are some examples for \text{expintrep} set to \text{gamma\_incomplete}:

(%i1) expintrep:'gamma_incomplete;
(%o1) gamma_incomplete
(%i2) expintegral_e1(z);
(%o2) gamma_incomplete(0, z)
(%i3) expintegral_ei(z);
(%o3) \log(z) - \log(-z) - gamma_incomplete(0, -z)
(%i4) expintegral_li(z);
(%o4) \log(\log(z)) - \log(-\log(z)) - gamma_incomplete(0, -\log(z))
(%i5) expintegral_e(n,z);
(%o5) \frac{n - 1}{\gamma_incomplete(1 - n, z) z}
(%i6) expintegral_si(z);
(%o6) \frac{(- \log(\%i z)) + \log(-\%i z) - \gamma_incomplete(0, \%i z) + \gamma_incomplete(0, -\%i z))/2}{2}
(%i7) expintegral_ci(z);
(%o7) \frac{\log(z) - (\log(\%i z) + \log(-\%i z) + \gamma_incomplete(0, \%i z) + \gamma_incomplete(0, -\%i z))/2}{2}
(%i8) expintegral_shi(z);
(%o8) \frac{\log(z) - \log(-z) + \gamma_incomplete(0, z) - \gamma_incomplete(0, -z)}{2}
(%i9) expintegral_chi(z);
(%o9) \frac{(- \log(z)) + \log(-z) + \gamma_incomplete(0, z) + \gamma_incomplete(0, -z)}{2}
For `expintrep` set to `expintegral_e1`:

(\%i1) expintrep:'expintegral_e1;
(\%o1) \hspace{1cm} \text{expintegral}_e1

(\%i2) \text{expintegral}_e1(z);
(\%o2) \hspace{1cm} \log(z) - \log(-z) - \text{expintegral}_e1(-z)

(\%i3) \text{expintegral}_li(z);
(\%o3) \hspace{1cm} \log(\log(z)) - \log(-\log(z)) - \text{expintegral}_e1(-\log(z))

(\%i4) \text{expintegral}_e(n,z);
(\%o4) \hspace{1cm} \text{expintegral}_e(n,z)

(\%i5) \text{expintegral}_si(z);
(\%o5) \hspace{1cm} \left(\frac{(-\log(\%i z)) - \text{expintegral}_e1(\%i z) + \log(\%i z)}{2}\right)

(\%i6) \text{expintegral}_ci(z);
(\%o6) \hspace{1cm} \log(z)

(\%i7) \text{expintegral}_shi(z);
(\%o7) \hspace{1cm} \log(z) + \text{expintegral}_e1(z) - \log(-z) - \text{expintegral}_e1(-z)

(\%i8) \text{expintegral}_chi(z);
(\%o8) \hspace{1cm} \left(-\log(z) + \text{expintegral}_e1(z) + \log(-z) + \text{expintegral}_e1(-z)\right)

For `expintrep` set to `expintegral_ei`:

(\%i1) expintrep:'expintegral_ei;
(\%o1) \hspace{1cm} \text{expintegral}_ei

(\%i2) \text{expintegral}_ei(z);
(\%o2) \hspace{1cm} \frac{1}{2} \log(-z) - \log(-z)

(\%i3) \text{expintegral}_ei(z);
(\%o3) \hspace{1cm} \text{expintegral}_ei(z)

(\%i4) \text{expintegral}_li(z);
(\%o4) \hspace{1cm} \text{expintegral}_ei(\log(z))

(\%i5) \text{expintegral}_e(n,z);
(\%o5) \hspace{1cm} \text{expintegral}_e(n,z)

(\%i6) \text{expintegral}_si(z);
(\%o6) \hspace{1cm} \left(\frac{(-\log(\%i z) + 2(\text{expintegral}_ei(\%i z) - \text{expintegral}_ei(\%i z))}{\%i}

\log(-\%i z) + \log(\%i z) - \log(-\%i z))}{4}\right)
(%i7) expintegral_ci(z);
(%o7) ((- log(%i z)) + 2 (expintegral_ei(%i z) + expintegral_ei(- %i z)) \\
\hspace{1cm} %i %i \\
\hspace{1cm} - log(- %i z) + log(--) + log(- --))/4 + log(z) \\
\hspace{1cm} z z \\

(%i8) expintegral_shi(z);
(%o8) ((- 2 log(z)) + 2 (expintegral_ei(z) - expintegral_ei(- z)) + log(- z) \\
\hspace{1cm} 1 \\
\hspace{1cm} - log(- -))/4 \\
\hspace{1cm} z z \\

(%i9) expintegral_chi(z);
(%o9) \\
\hspace{1cm} 1 \\
\hspace{1cm} 2 log(z) + 2 (expintegral_ei(z) + expintegral_ei(- z)) - log(- z) + log(- -) \\
\hspace{1cm} ----------------------------------------------- \\
\hspace{1cm} z \\

For expintrep set to expintegral_li:

(%i11) expintrep:'expintegral_li;
(%o11) expintegral_li

(%i12) expintegral_e1(z);
(%o12) (- expintegral_li(%e )) - log(z) + ------------------- \\
\hspace{1cm} 2 \\
\hspace{1cm} z \\

(%i13) expintegral_ei(z);
(%o13) expintegral_li(%e ) \\

(%i14) expintegral_li(z);
(%o14) expintegral_li(z) \\

(%i15) expintegral_e(n,z);
(%o15) expintegral_e(n, z) \\

(%i16) expintegral_si(z);
(%o16) %i z - %e z %pi signum(z) \\
\hspace{1cm} %i (expintegral_li(%e ) - expintegral_li(%e )) - --------------- \\
\hspace{1cm} 2 \\
\hspace{1cm} 2 \\

(%i17) expintegral_ci(z);
(%o17) %i z - %i z \\
\hspace{1cm} expintegral_li(%e ) + expintegral_li(%e ) \\
\hspace{1cm} --------------------------------- - signum(z) + 1 \\
\hspace{1cm} 2 \\
\hspace{1cm} 2 \\

(%i18) expintegral_shi(z);
(%o18) z - z
expintegral_li(%e) - expintegral_li(%e)

(%o8) -------------------------------------------
    2

(%i9) expintegral_chi(z);
   z - z
expintegral_li(%e) + expintegral_li(%e)

(%o9) -------------------------------------------
    2

For expintrep set to expintegral_trig:

(%i1) expintrep:'expintegral_trig;
(%o1) expintegral_trig

(%i2) expintegral_e1(z);

(%o2) log(%i z) - %i expintegral_si(%i z) - expintegral_ci(%i z) - log(z)

(%i3) expintegral_ei(z);

(%i4) expintegral_li(z);

(%i5) expintegral_e(n,z);

(%i6) expintegral_si(z);

(%i7) expintegral_ci(z);

(%i8) expintegral_shi(z);

(%i9) expintegral_chi(z);

For expintrep set to expintegral_hyp:

(%i1) expintrep:'expintegral_hyp;
(%o1) expintegral_hyp

(%i2) expintegral_e1(z);

(%o2) expintegral_shi(z) - expintegral_ch(z)

(%i3) expintegral_ei(z);

(%i4) expintegral_li(z);

(%i5) expintegral_e(n,z);

(%i6) expintegral_si(z);

(%i7) expintegral_ci(z);

(%i8) expintegral_shi(z);

(%i9) expintegral_ch(z);
expintexpand

[Option variable]

Default value: false

Expand \texttt{expintegral\_e(n,z)} for half integral values in terms of \texttt{erfc} or \texttt{erf} and for positive integers in terms of \texttt{expintegral\_ei}.

\begin{verbatim}
(%i11) expintegral\_e(1/2,z);
(%o11) expintegral\_e(-, z)

(%i12) expintegral\_e(1,z);
(%o12) expintegral\_e(1, z)

(%i13) expintexpand:true;
(%o13) true

(%i14) expintegral\_e(1/2,z);
(%o14) sqrt(%pi) erfc(sqrt(z))

(%i15) expintegral\_e(1,z);
(%o15) (- log(z)) - \frac{\sqrt{%pi} \text{erfc}(sqrt(z))}{sqrt(z)}
\end{verbatim}

15.6 Error Function

The Error function and related functions are defined in Abramowitz and Stegun, \textit{Handbook of Mathematical Functions}, A\&S Chapter 7 and (DLMF 7)

\textbf{erf (z)}

The Error Function $\text{erf}(z)$:

$$
\text{erf } z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt
$$

(A\&S eqn 7.1.1) and (DLMF 7.2.E1).

See also flag \texttt{erfflag}. This can also be expressed in terms of a hypergeometric function. See [hypergeometric\_representation], page 325.

\textbf{erfc (z)}

The Complementary Error Function $\text{erfc}(z)$:

$$
\text{erfc } z = 1 - \text{erf } z
$$

(A\&S eqn 7.1.2) and (DLMF 7.2.E2).

This can also be expressed in terms of a hypergeometric function. See [hypergeometric\_representation], page 325.
erfi(z) [Function]
The Imaginary Error Function.

\[ \text{erfi}(z) = -i \text{erf}(iz) \]

erf\(_\text{generalized}(z_1,z_2)\) [Function]
Generalized Error function Erf(z_1,z_2):

\[ \text{erf}(z_1, z_2) = \frac{2}{\sqrt{\pi}} \int_{z_1}^{z_2} e^{-t^2} dt \]

This can also be expressed in terms of a hypergeometric function. See [hypergeometric_representation], page 325.

fresnel_c(z) [Function]
The Fresnel Integral \(C(z) = \int_0^z \cos\left(\frac{\pi}{2} t^2\right) dt\) (A&S eqn 7.3.1) and (DLMF 7.2.E7).
The simplification \(C(-x) = -C(x)\) is applied when flag \text{trigsign} is true.
The simplification \(C(ix) = iC(x)\) is applied when flag \text{%iargs} is true.
See flags \text{erf_representation} and \text{hypergeometric_representation}.

fresnel_s(z) [Function]
The Fresnel Integral \(S(z) = \int_0^z \sin\left(\frac{\pi}{2} t^2\right) dt\) (A&S eqn 7.3.2) and (DLMF 7.2.E8).
The simplification \(S(-x) = -S(x)\) is applied when flag \text{trigsign} is true.
The simplification \(S(ix) = iS(x)\) is applied when flag \text{%iargs} is true.
See flags \text{erf_representation} and \text{hypergeometric_representation}.

erf_representation [Option variable]
Default value: false
\text{erf_representation} controls how the error functions are represented. It must be set to one of \text{false}, \text{erf}, \text{erfc}, or \text{erfi}. When set to \text{false}, the error functions are not modified. When set to \text{erf}, all error functions (\text{erfc}, \text{erfi}, \text{erf}\_\text{generalized}, \text{fresnel_s} and \text{fresnel_c}) are converted to \text{erf} functions. Similary, \text{erfc} converts error functions to \text{erfc}. Finally \text{erfi} converts the functions to \text{erfi}.

Converting to \text{erf}:

\begin{verbatim}
(%i11) erf_representation: erf;
(%o11) true
(%i12) erfc(z);
(%o12) erfc(z)
\end{verbatim}
(%i3) erfi(z);
(%o3) erfi(z)
(%i4) erf_generalized(z1,z2);
(%o4) erf(z2) - erf(z1)
(%i5) fresnel_c(z);
   sqrt(%pi) (%i + 1) z sqrt(%pi) (1 - %i) z
   (1 - %i) (erf(------------------) + %i erf(------------------))
   2  2
(%o5) -------------------------------------------------------------------
       4
(%i6) fresnel_s(z);
   sqrt(%pi) (%i + 1) z sqrt(%pi) (1 - %i) z
   (%i + 1) (erf(------------------) - %i erf(------------------))
   2  2
(%o6) -------------------------------------------------------------------
       4

Converting to erfc:

(%i1) erf_representation:erfc;
(%o1) erfc
(%i2) erf(z);
(%o2) 1 - erfc(z)
(%i3) erfc(z);
(%o3) erfc(z)
(%i4) erf_generalized(z1,z2);
(%o4) erfc(z1) - erfc(z2)
(%i5) fresnel_c(c);
   sqrt(%pi) (%i + 1) c
   2
   sqrt(%pi) (1 - %i) c
   - %i (1 - erfc(------------------) + 1))/4
   2
(%o5) (2
(%i6) fresnel_s(c);
   sqrt(%pi) (%i + 1) c
   2
   sqrt(%pi) (1 - %i) c
   + %i (1 - erfc(------------------) + 1))/4
   2
(%o6) (2

Converting to erfc:

(%i1) erf_representation:erfi;
(%o1) erfi
(%i2) erf(z);
(%o2) - %i erfi(%i z)
(%i3) erfc(z);
Chapter 15: Special Functions

%o3) %i erfi(%i z) + 1
%o4) erfi(z);
%o4) erfi(z)
%o5) erf_generalized(z1,z2);
%o5) %i erfi(%i z1) - %i erfi(%i z2)
%o6) fresnel_s(z);
%o6) %i erfi(%i z) + 1
%o6) %i erfi(%i z) + 1
%o7) fresnel_c(z);
%o7) fresnel_c(z);
%o7) sqrt(\%pi) %i ((%i + 1) z
%o7) sqrt(\%pi) %i (%i + 1) z
%o8) erf(z);
%o8) erf(z)
%o9) erfi(z);
%o9) erfi(z)

hypergeometric_representation

Default value: false

Enables transformation to a Hypergeometric representation for fresnel_s and fresnel_c and other error functions.

(%i11) hypergeometric_representation:true;
(%o11) true
(%i12) fresnel_s(z);
%pi hypergeometric([-], [-, -], - -------) z
4 2 4 16
(%o12) 6
(%i13) fresnel_c(z);
hypergeometric([-], [-, -], - -------) z
4 2 4 16
(%o13) 6
(%i14) erf(z);
2 hypergeometric([-], [-], - z ) z
2
(%o14) 2
(%i15) erfi(z);
2 hypergeometric([-], [-], z ) z
2
(%o15) 2

[Option variable]
\( \text{erf}(z) = \frac{1}{\sqrt{\pi}} \) \( \text{erfc}(z) = \frac{2}{\sqrt{\pi}} \text{hypergeometric([-], [-], -z) z} \)

\( \text{erf}_{\text{generalized}}(z_1, z_2) = \frac{2}{\sqrt{\pi}} \text{hypergeometric([-], [-], -z_2) z_2} \)

\( \text{struve}_h(\nu, z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{\Gamma(k + \frac{3}{2}) \Gamma(k + \nu + \frac{3}{2})} \)

\( \text{struve}_l(\nu, z) = -ie^{-i\pi/2} \text{struve}_h(iz) \)

15.7 Struve Functions

The Struve functions are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions, A&S Chapter 12* and (DLMF 11). The Struve Function \( H_\nu(z) \) is a particular solution of the differential equation:

\[
\frac{z^2}{d^2w}{dz^2} + \frac{z}{dz}\frac{dw}{dz} + (z^2 - \nu^2)w = \frac{4}{\sqrt{\pi}} \Gamma(\nu + \frac{1}{2})
\]

which has the general solution

\[ w = aJ_\nu(z) + bY_\nu(z) + H_\nu(z) \]

\( \text{struve}_h(\nu, z) \)

The Struve Function \( H \) of order \( \nu \) and argument \( z \):

\[
H_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{\Gamma(k + \frac{3}{2}) \Gamma(k + \nu + \frac{3}{2})}
\]

(A&S eqn 12.1.3) and (DLMF 11.2.E1).

When \text{besselexpand} is true, \text{struve}_h is expanded in terms of elementary functions when the order \( \nu \) is half of an odd integer. See \text{besselexpand}.

\( \text{struve}_l(\nu, z) \)

The Modified Struve Function \( L \) of order \( \nu \) and argument \( z \):

\[
L_\nu(z) = -ie^{-i\pi/2} H_\nu(iz)
\]

(A&S eqn 12.2.1) and (DLMF 11.2.E2).

When \text{besselexpand} is true, \text{struve}_l is expanded in terms of elementary functions when the order \( \nu \) is half of an odd integer. See \text{besselexpand}.
15.8 Hypergeometric Functions

The Hypergeometric Functions are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, A&S Chapters 13 and A&S 15.

Maxima has very limited knowledge of these functions. They can be returned from function `hgfred`.

\[ %m \begin{bmatrix} k, u \end{bmatrix}(z) \]

Whittaker M function (A&S eqn 13.1.32):

\[
M_{\kappa, \mu}(z) = e^{-\frac{1}{2}z}z^{\frac{1}{2}+\mu}M\left(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, z\right)
\]

where \(M(a, b, z)\) is Kummer’s solution of the confluent hypergeometric equation.

This can also be expressed by the series (DLMF 13.14.E6):

\[
M_{\kappa, \mu}(z) = e^{-\frac{1}{2}z}z^{\frac{1}{2}+\mu} \sum_{s=0}^{\infty} \frac{\left(\frac{1}{2} + \mu - \kappa\right)}{(1 + 2\mu)s!} z^s
\]

\[ %w \begin{bmatrix} k, u \end{bmatrix}(z) \]

Whittaker W function (A&S eqn 13.1.33):

\[
W_{\kappa, \mu}(z) = e^{-\frac{1}{2}z}z^{\frac{1}{2}+\mu}U\left(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, z\right)
\]

where \(U(a, b, z)\) is Kummer’s second solution of the confluent hypergeometric equation.

\[ %f [p, q] ([a], [b], z) \]

The \( {}_pF_q(a_1, a_2, ..., a_p; b_1, b_2, ..., b_q; z) \) hypergeometric function, where \(a\) a list of length \(p\) and \(b\) a list of length \(q\).

\[ \text{hypergeometric} \left([a_1, ..., a_p],[b_1, ..., b_q], x\right) \]

The hypergeometric function. Unlike Maxima’s \%f hypergeometric function, the function `hypergeometric` is a simplifying function; also, `hypergeometric` supports complex double and big floating point evaluation. For the Gauss hypergeometric function, that is \( p = 2 \) and \( q = 1 \), floating point evaluation outside the unit circle is supported, but in general, it is not supported.

When the option variable `expand_hypergeometric` is true (default is false) and one of the arguments \(a_1\) through \(a_p\) is a negative integer (a polynomial case), `hypergeometric` returns an expanded polynomial.

Examples:

\[(\%i1) \quad \text{hypergeometric}([], [], x); \]
\[(\%o1) \quad %e^{-x} \]

Polynomial cases automatically expand when `expand_hypergeometric` is true:

\[(\%i2) \quad \text{hypergeometric}([-3], [7], x); \]
\[(\%o2) \quad \text{hypergeometric}([-3], [7], x) \]
(%i3) hypergeometric([-3],[7],x), expand_hypergeometric : true;
(%o3) -x^3/504+3*x^2/56-3*x/7+1

Both double float and big float evaluation is supported:

(%i4) hypergeometric([5.1],[7.1 + %i],0.42);
(%o4) 1.346250786375334 - 0.0559061414208204 %i

(%i5) hypergeometric([5,6],[8], 5.7 - %i);
(%o5) .007375824009774946 - .001049813688578674 %i

(%i6) hypergeometric([5,6],[8], 5.7b0 - %i), fpprec : 30;
(%o6) 7.37582400977494674506442010824b-3
- 1.04981368857867315858055393376b-3 %i

hypergeometric_simp (e)    [Function]

hypergeometric_simp simplifies hypergeometric functions by applying hgfred to the arguments of any hypergeometric functions in the expression e.

Only instances of hypergeometric are affected; any %f, %w, and %m in the expression e are not affected. Any unsimplified hypergeometric functions are returned unchanged (instead of changing to %f as hgfred would).

load("hypergeometric"); loads this function.

See also hgfred.

Examples:

(%i1) load ("hypergeometric")
(%i2) foo : [hypergeometric([1,1], [2], z), hypergeometric([1/2], [1], z)];
(%o2) [hypergeometric([1, 1], [2], z),
     hypergeometric([-], [1], z)]

(%i3) hypergeometric_simp (foo);
(%o3) [- -----, bessel_i(0, -) %e ]
     z 2

(%i4) bar : hypergeometric([n], [m], z + 1);
(%o4) hypergeometric([n], [m], z + 1)

(%i5) hypergeometric_simp (bar);
(%o5) hypergeometric([n], [m], z + 1)

hgfred (a, b, t)    [Function]

Simplify the generalized hypergeometric function in terms of other, simpler, forms. a is a list of numerator parameters and b is a list of the denominator parameters.

If hgfred cannot simplify the hypergeometric function, it returns an expression of the form %f[p,q]([a], [b], x) where p is the number of elements in a, and q is the number of elements in b. This is the usual \( {}_pF_q \) generalized hypergeometric function.

(%i1) assume(not(equal(z,0)));
(%o1) [notequal(z, 0)]

(%i2) hgfred([v+1/2],[2*v+1],2*%i+z);
\[
\frac{v/2 \ 4 \ \text{bessel}_j(v, z) \ \text{gamma}(v + 1) \ %e}{\text{gamma}(v + 1) \ %e}
\]

\[
\frac{v}{z}
\]

\[
\frac{\log(1 - z)}{z}
\]

\[
\frac{1 - 2 a}{2 (1 - 2 a) z}
\]

It can be beneficial to load orthopoly too as the following example shows. Note that \(L\) is the generalized Laguerre polynomial.

\[
\frac{(a - 1)}{2 L (z)}
\]

\[
\frac{2}{a (a + 1)}
\]

\[
\frac{2 z}{a (a + 1) + 1}
\]

15.9 Parabolic Cylinder Functions

The Parabolic Cylinder Functions are defined in Abramowitz and Stegun, Handbook of Mathematical Functions, A&S Chapter 19.

Maxima has very limited knowledge of these functions. They can be returned from function \text{hgfred}.

\texttt{parabolic\_cylinder\_d (v, z)}

The parabolic cylinder function \texttt{parabolic\_cylinder\_d(v,z)}. (A&S eqn 19.3.1).
The solution of the Weber differential equation

$$y''(z) + \left(\nu + \frac{1}{2} - \frac{1}{4}z^2\right)y(z) = 0$$

has two independent solutions, one of which is $D_\nu(z)$, the parabolic cylinder function.

### 15.10 Functions and Variables for Special Functions

**lambert_w (z)**

The principal branch of Lambert’s W function $W(z)$ (DLMF 4.13), the solution of

$$z = W(z)e^{W(z)}$$

**generalized_lambert_w (k, z)**

The $k$-th branch of Lambert’s W function $W(z)$ (DLMF 4.13), the solution of $z = W(z)e^{W(z)}$.

- The principal branch, denoted $W_p(z)$ in DLMF, is $\text{lambert_w(z)} = \text{generalized_lambert_w(0, z)}$.
- The other branch with real values, denoted $W_m(z)$ in DLMF, is $\text{generalized_lambert_w(-1, z)}$.

**kbateman [v] (x)**

The Bateman k function

$$k_v(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(x \tan \theta - v\theta) d\theta$$

It is a special case of the confluent hypergeometric function. Maxima can calculate the Laplace transform of $\text{kbateman}$ using $\text{laplace}$ or $\text{specint}$, but has no other knowledge of this function.

**nzeta (z)**

The Plasma Dispersion Function

$$\text{nzeta}(z) = i\sqrt{\pi}e^{-z^2}(1 - \text{erf}(-iz))$$

**nzetar (z)**

Returns $\text{realpart(nzeta(z))}$.

**nzetai (z)**

Returns $\text{imagpart(nzeta(z))}$.
16 Elliptic Functions

16.1 Introduction to Elliptic Functions and Integrals

Maxima includes support for Jacobian elliptic functions and for complete and incomplete elliptic integrals. This includes symbolic manipulation of these functions and numerical evaluation as well. Definitions of these functions and many of their properties can be found in Abramowitz and Stegun, A&S Chapter 16 and A&S Chapter 17. See also DLMF 22.2. As much as possible, we use the definitions and relationships given in Abramowitz and Stegun.

In particular, all elliptic functions and integrals use the parameter \( m \) instead of the modulus \( k \) or the modular angle \( \alpha \). The following relationships are true:

\[
  m = k^2 \\
  k = \sin \alpha
\]

Note that Abramowitz and Stegun uses the notation \( \text{sn}(u|m) \) where we use \( \text{sn}(u, m) \) instead. The DLMF uses modulus \( k \) instead of the parameter \( m \).

The elliptic functions and integrals are primarily intended to support symbolic computation. Therefore, most of derivatives of the functions and integrals are known. However, if floating-point values are given, a floating-point result is returned.

Support for most of the other properties of elliptic functions and integrals other than derivatives has not yet been written.

Some examples of elliptic functions:

\[
\begin{align*}
  &\text{(%i1)} \quad \text{jacobi_sn}(u, m); \\
  &\text{(%o1)} \quad \text{jacobi_sn}(u, m) \\
  &\text{(%i2)} \quad \text{jacobi_sn}(u, 1); \\
  &\text{(%o2)} \quad \tanh(u) \\
  &\text{(%i3)} \quad \text{jacobi_sn}(u, 0); \\
  &\text{(%o3)} \quad \sin(u) \\
  &\text{(%i4)} \quad \text{diff}(\text{jacobi_sn}(u, m), u); \\
  &\text{(%o4)} \quad \text{jacobi_cn}(u, m) \text{jacobi_dn}(u, m) \\
  &\text{(%i5)} \quad \text{diff}(\text{jacobi_sn}(u, m), m); \\
  &\text{(%o5)} \quad \text{jacobi_cn}(u, m) \text{jacobi_dn}(u, m) \\
  &\text{\quad elliptic_e(asin(jacobi_sn(u, m)), m)} \\
  &\quad (u - \frac{1}{2} \frac{1}{m(1 - m)})/2 \\
  &\quad \frac{1}{1 - m} \\
  &\quad 2 \frac{2}{1 - m} \\
  &\quad + \frac{1}{2} \frac{2}{1 - m} \\
\end{align*}
\]

Some examples of elliptic integrals:

\[
\begin{align*}
  &\text{(%i1)} \quad \text{elliptic_f}(\text{phi}, m); \\
  &\text{(%o1)} \quad \text{elliptic_f}(\text{phi}, m)
\end{align*}
\]
(%i2) elliptic_f (phi, 0);
(%o2) phi
(%i3) elliptic_f (phi, 1);
(%o3) phi %pi
      log(tan(--- + ---))
        2        4
(%i4) elliptic_e (phi, 1);
(%o4) sin(phi)
(%i5) elliptic_e (phi, 0);
(%o5) phi
(%i6) elliptic_kc (1/2);
(%o6) elliptic_kc(-)
        1
        2
(%i7) makegamma (%);
(%o7) --
     4
 1
2
4 sqrt(%pi)
(%i8) diff (elliptic_f (phi, m), phi);
(%o8) ---------------------
      2
sqrt(1 - m sin(phi))
(%i9) diff (elliptic_f (phi, m), m);
(%o9) (---------------------)
   2
- ----------------------/(2 (1 - m))
      2
 1
 2
 4 sqrt(1 - m sin(phi))
sqrt(1 - m sin(phi))

Support for elliptic functions and integrals was written by Raymond Toy. It is placed under the terms of the General Public License (GPL) that governs the distribution of Maxima.

16.2 Functions and Variables for Elliptic Functions

See A&S Section 6.12 and DLMF 22.2 for more information.

\textbf{jacobi_sn} (u, m)\hspace{1cm} [\text{Function}]

The Jacobian elliptic function sn(u, m).

\textbf{jacobi_cn} (u, m)\hspace{1cm} [\text{Function}]

The Jacobian elliptic function cn(u, m).
The Jacobian elliptic function $dn(u, m)$.

The Jacobian elliptic function $ns(u, m) = 1/sn(u, m)$.

The Jacobian elliptic function $sc(u, m) = sn(u, m)/cn(u, m)$.

The Jacobian elliptic function $sd(u, m) = sn(u, m)/dn(u, m)$.

The Jacobian elliptic function $nc(u, m) = 1/cn(u, m)$.

The Jacobian elliptic function $cs(u, m) = cn(u, m)/sn(u, m)$.

The Jacobian elliptic function $cd(u, m) = cn(u, m)/dn(u, m)$.

The Jacobian elliptic function $nd(u, m) = 1/dn(u, m)$.

The Jacobian elliptic function $ds(u, m) = dn(u, m)/sn(u, m)$.

The Jacobian elliptic function $dc(u, m) = dn(u, m)/cn(u, m)$.

The inverse of the Jacobian elliptic function $sn(u, m)$. For $-1 \leq u \leq 1$, it can also be written (DLMF 22.15.E12):

$$\text{inverse}_jacob\text{i}_sn(u, m) = \int_0^u \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}$$

The inverse of the Jacobian elliptic function $cn(u, m)$. For $-1 \leq u \leq 1$, it can also be written (DLMF 22.15.E13):

$$\text{inverse}_jacob\text{i}_cn(u, m) = \int_u^1 \frac{dt}{\sqrt{(1-t^2)(1-m+mt^2)}}$$

The inverse of the Jacobian elliptic function $dn(u, m)$. For $\sqrt{1-m} \leq u \leq 1$, it can also be written (DLMF 22.15.E14):

$$\text{inverse}_jacob\text{i}_dn(u, m) = \int_u^1 \frac{dt}{\sqrt{(1-t^2)(t^2 - (1-m))}}$$
inverse_jacobi_ns (u, m)  
The inverse of the Jacobian elliptic function ns(u, m). For 1 \leq u, it can also be written (DLMF 22.15.E121):

\[
\text{inverse}_\text{jacobi}_\text{ns}(u, m) = \int_u^\infty \frac{dt}{\sqrt{(1-t^2)(t^2-m)}}
\]

inverse_jacobi_sc (u, m)  
The inverse of the Jacobian elliptic function sc(u, m). For all u it can also be written (DLMF 22.15.E20):

\[
\text{inverse}_\text{jacobi}_\text{sc}(u, m) = \int_0^u \frac{dt}{\sqrt{(1+t^2)(1+(1-m)t^2)}}
\]

inverse_jacobi_sd (u, m)  
The inverse of the Jacobian elliptic function sd(u, m). For -1/\sqrt{1-m} \leq u \leq 1/\sqrt{1-m}, it can also be written (DLMF 22.15.E16):

\[
\text{inverse}_\text{jacobi}_\text{sd}(u, m) = \int_0^u \frac{dt}{\sqrt{(1-(1-m)t^2)(1+mt^2)}}
\]

inverse_jacobi_nc (u, m)  
The inverse of the Jacobian elliptic function nc(u, m). For 1 \leq u, it can also be written (DLMF 22.15.E19):

\[
\text{inverse}_\text{jacobi}_\text{nc}(u, m) = \int_1^u \frac{dt}{\sqrt{(t^2-1)(m+(1-m)t^2)}}
\]

inverse_jacobi_cs (u, m)  
The inverse of the Jacobian elliptic function cs(u, m). For all u it can also be written (DLMF 22.15.E23):

\[
\text{inverse}_\text{jacobi}_\text{cs}(u, m) = \int_u^\infty \frac{dt}{\sqrt{(1+t^2)(t^2+(1-m))}}
\]

inverse_jacobi_cd (u, m)  
The inverse of the Jacobian elliptic function cd(u, m). For -1 \leq u \leq 1, it can also be written (DLMF 22.15.E15):

\[
\text{inverse}_\text{jacobi}_\text{cd}(u, m) = \int_u^1 \frac{dt}{\sqrt{(t^2-1)(1-mt^2)}}
\]

inverse_jacobi_nd (u, m)  
The inverse of the Jacobian elliptic function nd(u, m). For 1 \leq u \leq 1/\sqrt{1-m}, it can also be written (DLMF 22.15.E17):

\[
\text{inverse}_\text{jacobi}_\text{nd}(u, m) = \int_1^u \frac{dt}{\sqrt{(t^2-1)(1-(1-m)t^2)}}
\]
inverse_jacobi_ds (u, m)  
The inverse of the Jacobian elliptic function ds(u, m). For $\sqrt{1 - m} \leq u$, it can also be written (DLMF 22.15.E22):

$$\text{inverse\_jacobi\_ds}(u, m) = \int_{u}^{\infty} \frac{dt}{\sqrt{(t^2 + m)(t^2 - (1 - m))}}$$

inverse_jacobi_dc (u, m)  
The inverse of the Jacobian elliptic function dc(u, m). For $1 \leq u$, it can also be written (DLMF 22.15.E18):

$$\text{inverse\_jacobi\_dc}(u, m) = \int_{1}^{u} \frac{dt}{\sqrt{(t^2 - 1)(t^2 - m)}}$$

16.3 Functions and Variables for Elliptic Integrals

elliptic_f (phi, m)  
The incomplete elliptic integral of the first kind, defined as

$$\int_{0}^{\phi} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}$$

See also [elliptic_e], page 335, and [elliptic_kc], page 336.

elliptic_e (phi, m)  
The incomplete elliptic integral of the second kind, defined as

$$\int_{0}^{\phi} \sqrt{1 - m \sin^2 \theta} \, d\theta$$

See also [elliptic_f], page 335, and [elliptic_ec], page 336.

elliptic_eu (u, m)  
The incomplete elliptic integral of the second kind, defined as

$$E(u, m) = \int_{0}^{u} \frac{dn(v, m) \, dv}{\sqrt{1 - m t^2}} = \int_{0}^{\tau} \sqrt{\frac{1 - m t^2}{1 - t^2}} \, dt$$

where $\tau = \text{sn}(u, m)$.

This is related to elliptic_e by

$$E(u, m) = E(\sin^{-1} \text{sn}(u, m), m)$$

See also [elliptic_e], page 335.
elliptic_pi (n, phi, m)

The incomplete elliptic integral of the third kind, defined as

\[
\int_0^\phi \frac{d\theta}{(1 - n \sin^2 \theta)\sqrt{1 - m \sin^2 \theta}}
\]

elliptic_kc (m)

The complete elliptic integral of the first kind, defined as

\[
\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}
\]

For certain values of \( m \), the value of the integral is known in terms of \( Gamma \) functions. Use \texttt{makegamma} to evaluate them.

elliptic_ec (m)

The complete elliptic integral of the second kind, defined as

\[
\int_0^{\pi/2} \sqrt{1 - m \sin^2 \theta} d\theta
\]

For certain values of \( m \), the value of the integral is known in terms of \( Gamma \) functions. Use \texttt{makegamma} to evaluate them.

carlson_rc (x, y)

Carlson’s RC integral is defined by

\[
R_C(x,y) = \frac{1}{2} \int_0^{\infty} \frac{1}{\sqrt{t + x(t + y)}} dt
\]

This integral is related to many elementary functions in the following way:

\[
\log x = (x - 1)R_C \left( \left( \frac{1 + x}{2} \right)^2, x \right), \quad x > 0
\]

\[
\sin^{-1} x = xR_C(1 - x^2, 1), \quad |x| \leq 1
\]

\[
\cos^{-1} x = \sqrt{1 - x^2}R_C(x^2, 1), \quad 0 \leq x \leq 1
\]

\[
\tan^{-1} x = xR_C(1, 1 + x^2)
\]

\[
\sinh^{-1} x = xR_C(1 + x^2, 1)
\]

\[
\cosh^{-1} x = \sqrt{x^2 - 1}R_C(x^2, 1), \quad x \geq 1
\]

\[
\tanh^{-1}(x) = xR_C(1, 1 - x^2), \quad |x| \leq 1
\]

Also, we have the relationship

\[
R_C(x, y) = R_F(x, y, y)
\]
Some special values:

\begin{align*}
R_C(0, 1) &= \frac{\pi}{2} \\
R_C(0, 1/4) &= \pi \\
R_C(2, 1) &= \log(\sqrt{2} + 1) \\
R_C(i, i + 1) &= \frac{\pi}{4} + \frac{i}{2} \log(\sqrt{2} - 1) \\
R_C(0, i) &= (1 - i) \frac{\pi}{2\sqrt{2}}
\end{align*}

\texttt{carlson\_rd}(x, y, z)

Carlson’s RD integral is defined by

\[ R_D(x, y, z) = \frac{3}{2} \int_0^\infty \frac{1}{\sqrt{t + x\sqrt{t + y\sqrt{t + z}}} (t + z)} \, dt \]

We also have the special values

\begin{align*}
R_D(x, x, x) &= x^{-\frac{3}{2}} \\
R_D(0, y, y) &= \frac{3}{4} \pi y^{-\frac{3}{2}} \\
R_D(0, 2, 1) &= 3\sqrt{\pi} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}
\end{align*}

It is also related to the complete elliptic E function as follows

\[ E(m) = R_F(0, 1 - m, 1) - \frac{m}{3} R_D(0, 1 - m, 1) \]

\texttt{carlson\_rf}(x, y, z)

Carlson’s RF integral is defined by

\[ R_F(x, y, z) = \frac{1}{2} \int_0^\infty \frac{1}{\sqrt{t + x\sqrt{t + y\sqrt{t + z}}} } \, dt \]

We also have the special values

\begin{align*}
R_F(0, 1, 2) &= \frac{\Gamma(\frac{1}{4})^2}{4\sqrt{2\pi}} \\
R_F(i, -i, 0) &= \frac{\Gamma(\frac{1}{4})^2}{4\sqrt{\pi}}
\end{align*}

It is also related to the complete elliptic E function as follows

\[ E(m) = R_F(0, 1 - m, 1) - \frac{m}{3} R_D(0, 1 - m, 1) \]

\texttt{carlson\_rj}(x, y, z, p)

Carlson’s RJ integral is defined by
\[ R_J(x, y, z) = \frac{1}{2} \int_0^\infty \frac{1}{\sqrt{t + x} \sqrt{t + y} \sqrt{t + z} (t + p)} \, dt \]
17 Limits

17.1 Functions and Variables for Limits

\texttt{lhospitallim} [Option variable]
\begin{itemize}
  \item Default value: 4
\end{itemize}
\texttt{lhospitallim} is the maximum number of times L'Hospital's rule is used in \texttt{limit}. This prevents infinite looping in cases like \texttt{limit (cot(x)/csc(x), x, 0)}.

\texttt{limit} [Function]
\begin{quote}
\begin{itemize}
  \item \texttt{limit (expr, x, val, dir)}
  \item \texttt{limit (expr, x, val)}
  \item \texttt{limit (expr)}
\end{itemize}
\end{quote}
Computes the limit of \texttt{expr} as the real variable \texttt{x} approaches the value \texttt{val} from the direction \texttt{dir}. \texttt{dir} may have the value \texttt{plus} for a limit from above, \texttt{minus} for a limit from below, or may be omitted (implying a two-sided limit is to be computed).

\texttt{limit} uses the following special symbols: \texttt{inf} (positive infinity) and \texttt{minf} (negative infinity). On output it may also use \texttt{und} (undefined), \texttt{ind} (indefinite but bounded) and \texttt{infinity} (complex infinity).

\texttt{infinity} (complex infinity) is returned when the limit of the absolute value of the expression is positive infinity, but the limit of the expression itself is not positive infinity or negative infinity. This includes cases where the limit of the complex argument is a constant, as in \texttt{limit(log(x), x, minf)}, cases where the complex argument oscillates, as in \texttt{limit((-2)^x, x, inf)}, and cases where the complex argument is different for either side of a two-sided limit, as in \texttt{limit(1/x, x, 0)} and \texttt{limit(log(x), x, 0)}.

\texttt{lhospitallim} is the maximum number of times L'Hospital's rule is used in \texttt{limit}. This prevents infinite looping in cases like \texttt{limit (cot(x)/csc(x), x, 0)}.

\texttt{tlimswitch} when true will allow the \texttt{limit} command to use Taylor series expansion when necessary.

\texttt{limsubst} prevents \texttt{limit} from attempting substitutions on unknown forms. This is to avoid bugs like \texttt{limit (f(n)/f(n+1), n, inf)} giving 1. Setting \texttt{limsubst} to true will allow such substitutions.

\texttt{limit} with one argument is often called upon to simplify constant expressions, for example, \texttt{limit (inf-1)}.

\texttt{example (limit)} displays some examples.


\texttt{limsubst} [Option variable]
\begin{itemize}
  \item Default value: false
\end{itemize}
\texttt{limsubst} prevents \texttt{limit} from attempting substitutions on unknown forms. This is to avoid bugs like \texttt{limit (f(n)/f(n+1), n, inf)} giving 1. Setting \texttt{limsubst} to true will allow such substitutions.
tlimit

\[ tlimit(\text{expr}, x, \text{val}, \text{dir}) \]
\[ tlimit(\text{expr}, x, \text{val}) \]
\[ tlimit(\text{expr}) \]

Take the limit of the Taylor series expansion of \text{expr} in \text{x} at \text{val} from direction \text{dir}.

tlimswitch

[Option variable]

Default value: true

When \text{tlimswitch} is true, the \text{limit} command will use a Taylor series expansion if the limit of the input expression cannot be computed directly. This allows evaluation of limits such as \text{limit}(\text{x}/(\text{x}-1)-1/\text{log(\text{x)}}, \text{x}, 1, \text{plus}). When \text{tlimswitch} is false and the limit of input expression cannot be computed directly, \text{limit} will return an unevaluated limit expression.

gruntz

[Function]

\[ gruntz(\text{expr}, \text{var}, \text{value}) \]
\[ gruntz(\text{expr}, \text{var}, \text{value}, \text{direction}) \]

Compute limit of expression \text{expr} with respect to variable \text{var} at \text{value}. When \text{value} is not infinite (i.e., not \text{inf} or \text{minf}), \text{direction} must be supplied, either \text{plus} for a limit from above, or \text{minus} for a limit from below.

If \text{gruntz} cannot find the limit, an unevaluated expression \text{gruntz}(...) is returned.

gruntz implements the method described in the dissertation of Dominik Gruntz, "On Computing Limits in a Symbolic Manipulation System" (ETH Zurich, 1996).

The algorithm identifies the most rapidly varying subexpression, replaces it with a new variable, rewrites the expression in terms of the new variable, and then repeats.

The algorithm doesn’t handle oscillating functions, so it can’t do things like \text{limit}(\sin(\text{x})/\text{x}, \text{x}, \text{inf}).

To handle limits involving functions such as \text{gamma(x)} and \text{erf(x)}, the Gruntz algorithm requires them to be written in terms of asymptotic expansions, which Maxima cannot currently do.

The Gruntz algorithm assumes that variables and expressions are real, so, for example, it can’t handle \text{limit}((\text{-2})^\text{x}, \text{x}, \text{inf}).

\text{gruntz} is one of the methods called from \text{limit}.
18 Differentiation

18.1 Functions and Variables for Differentiation

antid (expr, x, u(x)) [Function]

Returns a two-element list, such that an antiderivative of expr with respect to x can be constructed from the list. The expression expr may contain an unknown function u and its derivatives.

Let L, a list of two elements, be the return value of antid. Then L[1] + 'integrate (L[2], x) is an antiderivative of expr with respect to x.

When antid succeeds entirely, the second element of the return value is zero. Otherwise, the second element is nonzero, and the first element is nonzero or zero. If antid cannot make any progress, the first element is zero and the second nonzero.

load ("antid") loads this function. The antid package also defines the functions nonzero and freeof and linear.

antid is related to antidiff as follows. Let L, a list of two elements, be the return value of antid. Then the return value of antidiff is equal to L[1] + 'integrate (L[2], x) where x is the variable of integration.

Examples:

(%i1) load ("antid")$
(%i2) expr: exp (z(x)) * diff (z(x), x) * y(x);

(%o2) y(x) %e (z(x))

(%i3) a1: antid (expr, x, z(x));

(%o3) [y(x) %e (z(x)), - %e (y(x))]

(%i4) a2: antiddiff (expr, x, z(x));

(%i5) a2 - (first (a1) + 'integrate (second (a1), x));

(%i6) antid (expr, x, y(x));

(%i7) antiddiff (expr, x, y(x));


antidiff (expr, x, u(x))  

[Function]  
Returns an antiderivative of expr with respect to x. The expression expr may contain an unknown function u and its derivatives.

When antidiff succeeds entirely, the resulting expression is free of integral signs (that is, free of the integrate noun). Otherwise, antidiff returns an expression which is partly or entirely within an integral sign. If antidiff cannot make any progress, the return value is entirely within an integral sign.

load ("antid") loads this function. The antid package also defines the functions nonzeroandfreeof and linear.

antidiff is related to antid as follows. Let L, a list of two elements, be the return value of antid. Then the return value of antidiff is equal to L[1] + 'integrate (L[2], x) where x is the variable of integration.

Examples:

(%i1) load ("antid")$

(%i2) expr: exp (z(x)) * diff (z(x), x) * y(x);
   z(x) d
   y(x) %e (-- (z(x)))
   dx

(%i3) a1: antid (expr, x, z(x));
   z(x) z(x) d
   [y(x) %e , - %e (-- (y(x)))]
   dx

(%i4) a2: antidiff (expr, x, z(x));
   / z(x) d
  [ y(x) %e - I %e (-- (y(x))) dx
    dx ]
   /

(%i5) a2 - (first (a1) + 'integrate (second (a1), x));

(%i6) antid (expr, x, y(x));
   z(x) d
   [0, y(x) %e (-- (z(x)))]
   dx

(%i7) antidiff (expr, x, y(x));
   / z(x) d
  [ I y(x) %e (-- (z(x))) dx
    dx ]
at  

at (expr, [eqn_1, . . ., eqn_n])

at (expr, eqn)

Evaluates the expression expr with the variables assuming the values as specified for them in the list of equations [eqn_1, . . ., eqn_n] or the single equation eqn.

If a subexpression depends on any of the variables for which a value is specified but there is no atvalue specified and it can’t be otherwise evaluated, then a noun form of the at is returned which displays in a two-dimensional form.

at carries out multiple substitutions in parallel.

See also atvalue. For other functions which carry out substitutions, see also subst and ev.

Examples:

(%i1) atvalue (f(x,y), [x = 0, y = 1], a^2);

2

(%o1) a

(%i2) atvalue ('diff (f(x,y), x), x = 0, 1 + y);

(%o2) @2 + 1

(%i3) printprops (all, atvalue);

!d !
--- (f(@1, @2))! = @2 + 1

!d@1 !
!@1 = 0

2

f(0, 1) = a

(%o3) done

(%i4) diff (4*f(x, y)^2 - u(x, y)^2, x);

d d

(%o4) 8 f(x, y) (-- (f(x, y))) - 2 u(x, y) (-- (u(x, y))

dx dx

(%i5) at (%, [x = 0, y = 1]);

2

(%o5) 16 a - 2 u(0, 1) (-- (u(x, 1)))

dx

!x = 0

Note that in the last line y is treated differently to x as y isn’t used as a differentiation variable.

The difference between subst, at and ev can be seen in the following example:

(%i1) e1:I(t)=C*diff(U(t),t)$

(%i2) e2:U(t)=L*diff(I(t),t)$
%i3) at(e1,e2);

\( I(t) = C \left( \frac{d}{dt} \left( U(t) \right) \right) \)

\( U(t) = L \left( \frac{d}{dt} \left( I(t) \right) \right) \)

%o3

%i4) subst(e2,e1);

\( I(t) = C \left( \frac{d}{dt} \left( L \left( \frac{d}{dt} \left( I(t) \right) \right) \right) \right) \)

%o4

%i5) ev(e1,e2,diff);

\( I(t) = C L \left( \frac{d}{dt} \left( \frac{2}{2} \right) \right) \)

%o5

atomgrad [Property]

atomgrad is the atomic gradient property of an expression. This property is assigned by gradef.

atvalue [Function]

atvalue (expr, \([x_1 = a_1, \ldots, x_m = a_m]\), c)

atvalue (expr, \(x_1 = a_1, c\))

Assigns the value c to expr at the point \(x = a\). Typically boundary values are established by this mechanism.

expr is a function evaluation, \(f(x_1, \ldots, x_m)\), or a derivative, \(\text{diff}(f(x_1, \ldots, x_m), x_1, n_1, \ldots, x_n, n_m)\) in which the function arguments explicitly appear. \(n_i\) is the order of differentiation with respect to \(x_i\).

The point at which the atvalue is established is given by the list of equations \([x_1 = a_1, \ldots, x_m = a_m]\). If there is a single variable \(x_1\), the sole equation may be given without enclosing it in a list.

printprops ([f_1, f_2, \ldots], atvalue) displays the atvalues of the functions \(f_1, f_2, \ldots\) as specified by calls to atvalue. printprops (f, atvalue) displays the atvalues of one function \(f\). printprops (all, atvalue) displays the atvalues of all functions for which atvalues are defined.

The symbols \(\theta_1, \theta_2, \ldots\) represent the variables \(x_1, x_2, \ldots\) when atvalues are displayed.

atvalue evaluates its arguments. atvalue returns c, the atvalue.

See also at.

Examples:

\( \text{%i11) atvalue (f(x,y), [x = 0, y = 1], a^2);} \)

\( \text{%o11) a^2} \)
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(%i2) atvalue ('diff (f(x,y), x), x = 0, 1 + y);
(%o2)   @2 + 1
(%i3) printprops (all, atvalue);

\[ \frac{df}{d@1} (f(@1, @2)) = \frac{d@1}{@1} = @2 + 1 \]
\[ f(0, 1) = \text{a} \]

(%o3) done
(%i4) diff (4*f(x,y)^2 - u(x,y)^2, x);
\[ 8 f(x, y) \frac{df}{dx} - 2 u(x, y) \frac{du}{dx} \]
(%i5) at (%), [x = 0, y = 1];
\[ 16 \text{a} - 2 u(0, 1) \frac{du}{dx} \]
\[ x = 0 \]

\text{cartan} [\text{Function}]

The exterior calculus of differential forms is a basic tool of differential geometry developed by Elie Cartan and has important applications in the theory of partial differential equations. The \text{cartan} package implements the functions \text{ext_diff} and \text{lie_diff}, along with the operators \text{~} (wedge product) and \text{|} (contraction of a form with a vector.) Type \text{demo ("tensor")} to see a brief description of these commands along with examples.

\text{cartan} was implemented by F.B. Estabrook and H.D. Wahlquist.

\text{init_cartan ([x_1, ..., x_n])} [\text{Function}]

\text{init_cartan([x_1, ..., x_n])} initializes global variables for the \text{cartan} package. The sole argument is a list of symbols, from which the Cartan basis is constructed.

\text{init_cartan} returns the basis which is constructed.

\text{init_cartan} assigns values to the following global variables: \text{cartan_coords}, \text{cartan_dim}, \text{extdim}, and \text{cartan_basis}. In addition, the following arrays are assigned: \text{extsub} and \text{extsubb}.

Note: Because of the internal implementation of the \text{cartan} package, it is necessary for \text{init_cartan} to be called before any expression containing the Cartan coordinates \text{x_1}, \text{...}, \text{x_n} is parsed.

\text{del (x)} [\text{Function}]

\text{del (x)} represents the differential of the variable \text{x}.
diff returns an expression containing del if an independent variable is not specified. In this case, the return value is the so-called "total differential". See also diff, del and derivdegree.

Examples:

(\%i1) diff (log (x));
\frac{del(x)}{x}
(\%o1)

(\%i2) diff (exp (x*y));
\frac{x \cdot y \cdot del(y) + y \cdot e^x \cdot del(x)}{x \cdot y}
(\%o2)

(\%i3) diff (x*y*z);
\frac{x \cdot y \cdot \frac{del(z)}{z} + x \cdot z \cdot \frac{del(y)}{y} + y \cdot z \cdot \frac{del(x)}{x}}{x \cdot y}
(\%o3)

delta (t) [Function]
The Dirac Delta function.
Currently only laplace knows about the delta function.
Example:

(\%i1) laplace (delta (t - a) * sin(b*t), t, s);
Is a positive, negative, or zero?
p;
\frac{-a \cdot s \cdot sin(a \cdot b \cdot e^t)}{s}
(\%o1)

dependencies [System variable]
dependencies (f_1, \ldots, f_n) [Function]
The variable dependencies is the list of atoms which have functional dependencies, assigned by depends, the function dependencies, or gradef. The dependencies list is cumulative: each call to depends, dependencies, or gradef appends additional items. The default value of dependencies is [].

The function dependencies(f_1, \ldots, f_n) appends f_1, \ldots, f_n, to the dependencies list, where f_1, \ldots, f_n are expressions of the form f(x_1, \ldots, x_m), and x_1, \ldots, x_m are any number of arguments.

dependencies(f(x_1, \ldots, x_m)) is equivalent to depends(f, [x_1, \ldots, x_m]). See also depends and gradef.

(\%i1) dependencies;
[]
(\%o1)

(\%i2) depends (foo, [bar, baz]);
[foo(bar, baz)]
(\%o2)

(\%i3) depends ([g, h], [a, b, c]);
[g(a, b, c), h(a, b, c)]
(\%o3)

(\%i4) dependencies;
[]
(\%o4)

(\%i5) dependencies (quux (x, y), mumble (u));
[quux(x, y), mumble(u)]
(\%o5)
depends \( (f_1, x_1, \ldots, f_n, x_n) \)  
Declares functional dependencies among variables for the purpose of computing derivatives. In the absence of declared dependence, \texttt{diff (f, x)} yields zero. If \texttt{depends (f, x)} is declared, \texttt{diff (f, x)} yields a symbolic derivative (that is, a \texttt{diff} noun).

Each argument \( f_i, x_i \), etc., can be the name of a variable or array, or a list of names. Every element of \( f_i \) (perhaps just a single element) is declared to depend on every element of \( x_i \) (perhaps just a single element). If some \( f_i \) is the name of an array or contains the name of an array, all elements of the array depend on \( x_i \).

\texttt{diff} recognizes indirect dependencies established by \texttt{depends} and applies the chain rule in these cases.

\texttt{remove (f, dependency)} removes all dependencies declared for \( f \).

\texttt{depends} returns a list of the dependencies established. The dependencies are appended to the global variable \texttt{dependencies}. \texttt{depends} evaluates its arguments.

\texttt{diff} is the only Maxima command which recognizes dependencies established by \texttt{depends}. Other functions (\texttt{integrate}, \texttt{laplace}, etc.) only recognize dependencies explicitly represented by their arguments. For example, \texttt{integrate} does not recognize the dependence of \( f \) on \( x \) unless explicitly represented as \texttt{integrate (f(x), x)}.

\texttt{depends(f, \{x_1, \ldots, x_n\})} is equivalent to \texttt{dependencies(f(x_1, \ldots, x_n))}.

See also \texttt{diff}, \texttt{del}, \texttt{derivdegree} and \texttt{derivabbrev}.
(o7) done
(o8) r . -- --
du du
ds dt

derivabbrev

[Option variable]
Default value: false
When derivabbrev is true, symbolic derivatives (that is, diff nouns) are displayed as subscripts. Otherwise, derivatives are displayed in the Leibniz notation dy/dx.

derivdegree (expr, y, x)

[Function]
Returns the highest degree of the derivative of the dependent variable y with respect to the independent variable x occurring in expr.
Example:

(i1) 'diff (y, x, 2) + 'diff (y, z, 3) + 'diff (y, x) * x^2;

(o1) d y d y 2 dy
    --- + --- + x --
    3 2 dx
dz dx
dz dx

(i2) derivdegree (%o1, y, x);

(o2) 2

derivlist (var_1, ..., var_k)

[Function]
Causes only differentiations with respect to the indicated variables, within the ev command.

derivsubst

[Option variable]
Default value: false
When derivsubst is true, a non-syntactic substitution such as subst (x, 'diff (y, t), 'diff (y, t, 2)) yields 'diff (x, t).

diff

[Function]

diff (expr, x_1, n_1, ..., x_m, n_m)
diff (expr, x, n)
diff (expr, x)
diff (expr)

Returns the derivative or differential of expr with respect to some or all variables in expr.
diff (expr, x, n) returns the n'th derivative of expr with respect to x.
diff (expr, x_1, n_1, ..., x_m, n_m) returns the mixed partial derivative of expr with respect to x_1, ..., x_m. It is equivalent to diff (... (diff (expr, x_m, n_m) ...) ..., x_1, n_1).
diff (expr, x) returns the first derivative of expr with respect to the variable x.
diff (expr) returns the total differential of expr, that is, the sum of the derivatives of expr with respect to each its variables times the differential del of each variable. No further simplification of del is offered.
The noun form of \texttt{diff} is required in some contexts, such as stating a differential equation. In these cases, \texttt{diff} may be quoted (as \texttt{'diff}) to yield the noun form instead of carrying out the differentiation.

When \texttt{derivabbrev} is \texttt{true}, derivatives are displayed as subscripts. Otherwise, derivatives are displayed in the Leibniz notation, \( \frac{dy}{dx} \).

See also \texttt{depends}, \texttt{del}, \texttt{derivdegree} and \texttt{derivabbrev}.

Examples:

\begin{verbatim}
(%i1) diff (exp (f(x)), x, 2);
               2
  f(x) d     f(x) d
(%o1) %e (--- (f(x))) + %e (-- (f(x)))
         2
     dx     dx

(%i2) derivabbrev: true$

(%i3) 'integrate (f(x, y), y, g(x), h(x));
               h(x)
  I f(x, y) dy
(%o3) / [ ]
            g(x)

(%i4) diff (%x, x);
               h(x)
(%o4) I f(x, y) dy + f(x, h(x)) h(x) - f(x, g(x)) g(x)
  x         x
            x
          / 
        g(x)
\end{verbatim}

For the tensor package, the following modifications have been incorporated:

1. The derivatives of any indexed objects in \texttt{expr} will have the variables \texttt{x_i} appended as additional arguments. Then all the derivative indices will be sorted.

2. The \texttt{x_i} may be integers from 1 up to the value of the variable \texttt{dimension} [default value: 4]. This will cause the differentiation to be carried out with respect to the \texttt{x_i}'th member of the list \texttt{coordinates} which should be set to a list of the names of the coordinates, e.g., \([x, y, z, t]\). If \texttt{coordinates} is bound to an atomic variable, then that variable subscripted by \texttt{x_i} will be used for the variable of differentiation. This permits an array of coordinate names or subscripted names like \(X[1], X[2], \ldots\) to be used. If \texttt{coordinates} has not been assigned a value, then the variables will be treated as in (1) above.

\texttt{diff} \hfill \textbf{[Special symbol]}

When \texttt{diff} is present as an \texttt{evflag} in call to \texttt{ev}, all differentiations indicated in \texttt{expr} are carried out.
**express** (expr)

[Function]

Expands differential operator nouns into expressions in terms of partial derivatives. **express** recognizes the operators **grad**, **div**, **curl**, **laplacian**. **express** also expands the cross product "\".

Symbolic derivatives (that is, **diff** nouns) in the return value of **express** may be evaluated by including **diff** in the **ev** function call or command line. In this context, **diff** acts as an **evfun**.

**load** ("vect") loads this function.

Examples:

```
(%i11) load ("vect")$
(%i12) grad (x^2 + y^2 + z^2);
    2 2 2
(%o12) grad (z + y + x)
(%i13) express (%);
    d 2 2 2 d 2 2 2 d 2 2 2
(%o13) \[- (z + y + x), -- (z + y + x), -- (z + y + x)\]
      dx dy dz
(%i14) ev (% , diff);
    2 2 2
(%o14) [2 x, 2 y, 2 z]
(%i15) div ([x^2, y^2, z^2]);
    2 2 2
(%o15) div [x , y , z]
(%i16) express (%);
    d 2 2 2 d 2 2 2
(%o16) -- (z ) + -- (y ) + -- (x )
      dz dy dx
(%i17) ev (% , diff);
(%o17) 2 z + 2 y + 2 x
(%i18) curl ([x^2, y^2, z^2]);
    2 2 2
(%o18) curl [x , y , z]
(%i19) express (%);
    d 2 2 2 d 2 2 2 d 2 2 2 d 2 2 2
(%o19) \[- (z ) - -- (y ), -- (x ) - -- (z ), -- (y ) - -- (x )\]
      dy dz dx dx dy
(%i20) ev (% , diff);
(%o20) [0, 0, 0]
(%i21) laplacian (x^2 * y^2 * z^2);
    2 2 2
(%o21) laplacian (x y z)
(%i22) express (%);
    d 2 2 2 d 2 2 2 d 2 2 2 d 2 2 2
(%o22) \[- (x y z) + -- (x y z) + -- (x y z)\]
      2 2 2 2
      dz dy dx
```
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(%i13) ev (%, diff);
  2 2 2 2 2
(%o13) 2 y z + 2 x z + 2 x y

(%i14) [a, b, c] ~ [x, y, z];
(%o14) [a, b, c] ~ [x, y, z]

(%i15) express (%);
(%o15) [b z - c y, c x - a z, a y - b x]

gradeff

gradeff (f(x_1, ..., x_n), g_1, ..., g_m)
gradeff (a, x, expr)

Defines the partial derivatives (i.e., the components of the gradient) of the function 
f or variable a.

gradeff (f(x_1, ..., x_n), g_1, ..., g_m) defines df/dx_i as g_i, where g_i is an-expression; g_i may be a function call, but not the name of a function. The number of partial derivatives m may be less than the number of arguments n, in which case derivatives are defined with respect to x_1 through x_m only.

gradeff (a, x, expr) defines the derivative of variable a with respect to x as expr. This also establishes the dependence of a on x (via depends (a, x)).

The first argument f(x_1, ..., x_n) or a is quoted, but the remaining arguments g_1, ..., g_m are evaluated. gradeff returns the function or variable for which the partial derivatives are defined.

gradeff can redefine the derivatives of Maxima’s built-in functions. For example, gradeff (sin(x), sqrt (1 - sin(x)^2)) redefines the derivative of sin.

gradeff cannot define partial derivatives for a subscripted function.

printprops ([f_1, ..., f_n], gradeff) displays the partial derivatives of the functions f_1, ..., f_n, as defined by gradeff.

printprops ([a_n, ..., a_n], atomgrad) displays the partial derivatives of the variables a_n, ..., a_n, as defined by gradeff.

gradeffs is the list of the functions for which partial derivatives have been defined by gradeff. gradeffs does not include any variables for which partial derivatives have been defined by gradeff.

Gradients are needed when, for example, a function is not known explicitly but its first derivatives are and it is desired to obtain higher order derivatives.

gradeffs

[System variable]

Default value: []

gradeffs is the list of the functions for which partial derivatives have been defined by gradeff. gradeffs does not include any variables for which partial derivatives have been defined by gradeff.
19 Integration

19.1 Introduction to Integration
Maxima has several routines for handling integration. The `integrate` function makes use of most of them. There is also the `antid` package, which handles an unspecified function (and its derivatives, of course). For numerical uses, there is a set of adaptive integrators from QUADPACK, named `quad_qag`, `quad_qags`, etc., which are described under the heading QUADPACK. Hypergeometric functions are being worked on, see `specint` for details. Generally speaking, Maxima only handles integrals which are integrable in terms of the "elementary functions" (rational functions, trigonometrics, logs, exponentials, radicals, etc.) and a few extensions (error function, dilogarithm). It does not handle integrals in terms of unknown functions such as \( g(x) \) and \( h(x) \).

19.2 Functions and Variables for Integration

`changevar (expr, f(x,y), y, x)`  
Makes the change of variable given by \( f(x,y) = 0 \) in all integrals occurring in `expr` with integration with respect to \( x \). The new variable is \( y \).

The change of variable can also be written \( f(x) = g(y) \).

```
(%i1) assume(a > 0)$
(%i2) 'integrate (%e**sqrt(a*y), y, 0, 4);
    4
     / sqrt(a) sqrt(y)
   4 [- %e dy]
     /
   0

(%o2)
(%i3) changevar (%o2, y-z^2/a, z, y);

   0
     [ abs(z)
   2 [ z %e dz]
     ]

   - 2 sqrt(a)
   -------------------------------
   a
```

An expression containing a noun form, such as the instances of `integrate` above, may be evaluated by `ev` with the `nouns` flag. For example, the expression returned by `changevar` above may be evaluated by `ev (%o3, nouns)`.

`changevar` may also be used to make changes in the indices of a sum or product. However, it must be realized that when a change is made in a sum or product, this change must be a shift, i.e., \( i = j + \ldots \), not a higher degree function. E.g.,
(%i4) sum (a[i]*x^(i-2), i, 0, inf);
    inf
   ====
  \    i - 2
(%o4) > a x
     /    i
    =====
   i = 0

(%i5) changevar (%i, i-2-n, n, i);
    inf
   ====
  \    n
(%o5) > a x
     /    n + 2
    =====
   n = - 2

dblint (f, r, s, a, b)

A double-integral routine which was written in top-level Maxima and then translated and compiled to machine code. Use load ("dblint") to access this package. It uses the Simpson’s rule method in both the x and y directions to calculate

\[ \int_a^b \left( \int_{r(x)}^{s(x)} f(x, y) \, dy \right) \, dx. \]

The function \( f \) must be a translated or compiled function of two variables, and \( r \) and \( s \) must each be a translated or compiled function of one variable, while \( a \) and \( b \) must be floating point numbers. The routine has two global variables which determine the number of divisions of the x and y intervals: \texttt{dblint\_x} and \texttt{dblint\_y}, both of which are initially 10, and can be changed independently to other integer values (there are \( 2 \times \texttt{dblint\_x} + 1 \) points computed in the x direction, and \( 2 \times \texttt{dblint\_y} + 1 \) in the y direction). The routine subdivides the X axis and then for each value of X it first computes \( r(x) \) and \( s(x) \); then the Y axis between \( r(x) \) and \( s(x) \) is subdivided and the integral along the Y axis is performed using Simpson’s rule; then the integral along the X axis is done using Simpson’s rule with the function values being the Y-integrals. This procedure may be numerically unstable for a great variety of reasons, but is reasonably fast: avoid using it on highly oscillatory functions and functions with singularities (poles or branch points in the region). The Y integrals depend on how far apart \( r(x) \) and \( s(x) \) are, so if the distance \( s(x) - r(x) \) varies rapidly with X, there may be substantial errors arising from truncation with different step-sizes in the various Y integrals. One can increase \texttt{dblint\_x} and \texttt{dblint\_y} in an effort to improve the coverage of the region, at the expense of computation time. The function values are not saved, so if the function is very time-consuming, you will have to wait for re-computation if you change anything (sorry). It is required that the functions \( f, r, \) and \( s \) be either translated or compiled prior to calling \texttt{dblint}. This will result in orders of magnitude speed improvement over interpreted code in many cases!

demo ("dblint") executes a demonstration of \texttt{dblint} applied to an example problem.
defint (expr, x, a, b)  [Function]
Attempts to compute a definite integral. defint is called by integrate when limits of integration are specified, i.e., when integrate is called as integrate (expr, x, a, b). Thus from the user’s point of view, it is sufficient to call integrate.

defint returns a symbolic expression, either the computed integral or the noun form of the integral. See quad_qag and related functions for numerical approximation of definite integrals.

erfflag  [Option variable]
Default value: true

When erfflag is false, prevents risch from introducing the erf function in the answer if there were none in the integrand to begin with.

ilt (expr, s, t)  [Function]
Computes the inverse Laplace transform of expr with respect to s and parameter t. expr must be a ratio of polynomials whose denominator has only linear and quadratic factors; there is an extension of ilt, called pwilt (Piece-Wise Inverse Laplace Transform) that handles several other cases where ilt fails.

By using the functions laplace and ilt together with the solve or linsolve functions the user can solve a single differential or convolution integral equation or a set of them.

(%i1) 'integrate (sinh(a*x)*f(t-x), x, 0, t) + b*f(t) = t**2;
   t
/            2
[          ]
(%o1) I f(t - x) sinh(a x) dx + b f(t) = t
     /
     0
(%i2) laplace (%t, t, s);
   a laplace(f(t), t, s) 2
(%o2) b laplace(f(t), t, s) + --------------------- = --
          2 2 3
         2 2
   s - a s
(%i3) linsolve ([%], [laplace(f(t), t, s)]);
  2 2
2 s - 2 a
(%o3) [laplace(f(t), t, s) = ------------------]
      5 2 3
             b s + (a - a b) s
(%i4) ilt (rhs (first (%)), s, t);
Is a b (a b - 1) positive, negative, or zero?
pos;
sqrt(a b (a b - 1)) t
2 cosh(---------------------) 2
b a t
(%o4) - ----------------------------- + -------
3 2 2 a b - 1
a b - 2 a b + a
+ ---------------------
3 2 2 a b - 2 a b + a
2

intanalysis

[Option variable]
Default value: true

When true, definite integration tries to find poles in the integrand in the interval of integration. If there are, then the integral is evaluated appropriately as a principal value integral. If intanalysis is false, this check is not performed and integration is done assuming there are no poles.

See also ldefint.

Examples:
Maxima can solve the following integrals, when intanalysis is set to false:

(%i1) integrate(1/(sqrt(x)+1),x,0,1);
1
[ 1
I ----------- dx
] sqrt(x) + 1
0
(%o1)

(%i2) integrate(1/(sqrt(x)+1),x,0,1),intanalysis:false;
2 - 2 \log(2)
(%o2)

(%i3) integrate(cos(a)/sqrt((tan(a))^2+1),a,-%pi/2,%pi/2);
The number 1 isn't in the domain of atanh
-- an error. To debug this try: debugmode(true);

(%i4) intanalysis:false$
(%i5) integrate(cos(a)/sqrt((tan(a))^2+1),a,-%pi/2,%pi/2);
%pi
---
2
(%o5)
integrate [Function]

integrate (expr, x)
integrate (expr, x, a, b)

Attempts to symbolically compute the integral of expr with respect to x. integrate (expr, x) is an indefinite integral, while integrate (expr, x, a, b) is a definite integral, with limits of integration a and b. The limits should not contain x, although integrate does not enforce this restriction. a need not be less than b. If b is equal to a, integrate returns zero.

See quad_qag and related functions for numerical approximation of definite integrals. See residue for computation of residues (complex integration). See antid for an alternative means of computing indefinite integrals.

The integral (an expression free of integrate) is returned if integrate succeeds. Otherwise the return value is the noun form of the integral (the quoted operator 'integrate) or an expression containing one or more noun forms. The noun form of integrate is displayed with an integral sign.

In some circumstances it is useful to construct a noun form by hand, by quoting integrate with a single quote, e.g., 'integrate (expr, x). For example, the integral may depend on some parameters which are not yet computed. The noun may be applied to its arguments by ev (i, nouns) where i is the noun form of interest.

integrate handles definite integrals separately from indefinite, and employs a range of heuristics to handle each case. Special cases of definite integrals include limits of integration equal to zero or infinity (inf or minf), trigonometric functions with limits of integration equal to zero and %pi or 2 %pi, rational functions, integrals related to the definitions of the beta and psi functions, and some logarithmic and trigonometric integrals. Processing rational functions may include computation of residues. If an applicable special case is not found, an attempt will be made to compute the indefinite integral and evaluate it at the limits of integration. This may include taking a limit as a limit of integration goes to infinity or negative infinity; see also ldefint.

Special cases of indefinite integrals include trigonometric functions, exponential and logarithmic functions, and rational functions. integrate may also make use of a short table of elementary integrals.

integrate may carry out a change of variable if the integrand has the form f(g(x)) * diff(g(x), x). integrate attempts to find a subexpression g(x) such that the derivative of g(x) divides the integrand. This search may make use of derivatives defined by the gradef function. See also changefvar and antid.

If none of the preceding heuristics find the indefinite integral, the Risch algorithm is executed. The flag risch may be set as an evflag, in a call to ev or on the command line, e.g., ev (integrate (expr, x), risch) or integrate (expr, x), risch. If risch is present, integrate calls the risch function without attempting heuristics first. See also risch.

integrate works only with functional relations represented explicitly with the f(x) notation. integrate does not respect implicit dependencies established by the depends function.

integrate may need to know some property of a parameter in the integrand. integrate will first consult the assume database, and, if the variable of interest
is not there, \texttt{integrate} will ask the user. Depending on the question, suitable responses are \texttt{yes}; or \texttt{no}; or \texttt{pos};, \texttt{zero};, or \texttt{neg};.

\texttt{integrate} is not, by default, declared to be linear. See \texttt{declare} and \texttt{linear}.

\texttt{integrate} attempts integration by parts only in a few special cases.

Examples:

- Elementary indefinite and definite integrals.
  
  \begin{verbatim}
  (%i1) integrate (sin(x)^3, x);
  3
  cos (x)  
  (%o1) --------- - cos(x)
          3
  (%i2) integrate (x/ sqrt (b^2 - x^2), x);
          2  2
  (%o2) - sqrt(b - x )
  (%i3) integrate (cos(x)^2 * exp(x), x, 0, %pi);
    %pi
  (%o3) ------- - -
          5  5
  (%i4) integrate (x^2 * exp(-x^2), x, minf, inf);
  sqrt(%pi)
  (%o4)  ---------
          2
  \end{verbatim}

- Use of \texttt{assume} and interactive query.

  \begin{verbatim}
  (%i1) assume (a > 1)$
  (%i2) integrate (x**a/(x+1)**(5/2), x, 0, inf);
  2 a + 2
  Is ------ an integer?
  5
  no;
  Is 2 a - 3 positive, negative, or zero?
  neg;
  (%o2) beta(a + 1, - - a)
         2
  \end{verbatim}

- Change of variable. There are two changes of variable in this example: one using a derivative established by \texttt{gradef}, and one using the derivation \texttt{diff(r(x))} of an unspecified function \(r(x)\).

  \begin{verbatim}
  (%i3) gradef (q(x), sin(x**2));
  (%o3) q(x)
  \end{verbatim}
(%i4) diff (log (q (r (x))), x);
   2
  (--- (r(x))) sin(r (x))
  dx
---
q(r(x))

(%i5) integrate (% , x);
(%o5) log(q(r(x)))

Return value contains the 'integrate noun form. In this example, Maxima can extract one factor of the denominator of a rational function, but cannot factor the remainder or otherwise find its integral. grind shows the noun form 'integrate in the result. See also integrate_use_rootsof for more on integrals of rational functions.

(%i1) expand ((x-4) * (x^3+2*x+1));
   4 3 2
(%o1) x - 4 x + 2 x - 7 x - 4

(%i2) integrate (1/%, x);
     2
/  x + 4 x + 18
[ ------------- dx
   3
] 3

log(x - 4) / x + 2 x + 1
---------- - ----------
73 73

(%i3) grind (%);
log(x-4)/73-('integrate((x^2+4*x+18)/(x^3+2*x+1),x))/73$

Defining a function in terms of an integral. The body of a function is not evaluated when the function is defined. Thus the body of f_1 in this example contains the noun form of integrate. The quote-quote operator '' causes the integral to be evaluated, and the result becomes the body of f_2.

(%i1) f_1 (a) := integrate (x^3, x, 1, a);
3
(%o1) f_1(a) := integrate(x , x, 1, a)

(%i2) ev (f_1 (7), nouns);
(%o2) 600

(%i3) /* Note parentheses around integrate(...) here */
f_2 (a) := ''(integrate (x^3, x, 1, a));
4
(%o3) f_2(a) := -- - -
   4 4

(%i4) f_2 (7);
(%o4) 600

integration_constant [System variable]
Default value: %c
When a constant of integration is introduced by indefinite integration of an equation, the name of the constant is constructed by concatenating `integration_constant` and `integration_constant_counter`.

`integration_constant` may be assigned any symbol.

Examples:

```
(%i1) integrate (x^2 = 1, x);
   3
  -- = x + %c1
  3
(%o1)

(%i2) integration_constant : 'k;
  k
(%o2)

(%i3) integrate (x^2 = 1, x);
   3
  -- = x + k
  3
(%o3)
```

`integration_constant_counter` [System variable]

Default value: 0

When a constant of integration is introduced by indefinite integration of an equation, the name of the constant is constructed by concatenating `integration_constant` and `integration_constant_counter`.

`integration_constant_counter` is incremented before constructing the next integration constant.

Examples:

```
(%i1) integrate (x^2 = 1, x);
   3
  -- = x + %c1
  3
(%o1)

(%i2) integrate (x^2 = 1, x);
   3
  -- = x + %c2
  3
(%o2)

(%i3) integrate (x^2 = 1, x);
   3
  -- = x + %c3
  3
(%o3)

(%i4) reset (integration_constant_counter);
(%o4) [integration_constant_counter]
```
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(%i5) integrate (x^2 = 1, x);
3
x
(%o5) -- = x + %c1
3

\texttt{integrate}\textunderscore\texttt{use}\textunderscore\texttt{rootsof} \\
\hspace{1cm}[\text{Option variable}]

Default value: false

When \texttt{integrate}\textunderscore\texttt{use}\textunderscore\texttt{rootsof} is true and the denominator of a rational function cannot be factored, \texttt{integrate} returns the integral in a form which is a sum over the roots (not yet known) of the denominator.

For example, with \texttt{integrate}\textunderscore\texttt{use}\textunderscore\texttt{rootsof} set to false, \texttt{integrate} returns an unsolved integral of a rational function in noun form:

(%i1) integrate\textunderscore\textunderscore\textunderscore use\textunderscore\textunderscore\textunderscore rootsof: false$

(%i2) integrate (1/(1+x+x^5), x);
\frac{\text{\textstyle } 2 x + 1 - 4 \text{\textstyle } x + 5}{\text{\textstyle } 3 \text{\textstyle } 2}
\frac{\text{\textstyle } 2 x + 1}{\text{\textstyle } 3 \text{\textstyle } 2}
\frac{\text{\textstyle } 5 \text{\textstyle } \text{\texttt{atan}}}{}
\frac{\text{\textstyle } \text{\texttt{log}}}{}
\frac{\text{\textstyle } \sqrt{3}}{}
\frac{\text{\textstyle } 7}{7}
\frac{\text{\textstyle } 14}{14}
\frac{\text{\textstyle } 7 \text{\textstyle } \sqrt{3}}{7 

Now we set the flag to be true and the unsolved part of the integral will be expressed as a summation over the roots of the denominator of the rational function:

(%i3) integrate\textunderscore\textunderscore\textunderscore use\textunderscore\textunderscore\textunderscore rootsof: true$

(%i4) integrate (1/(1+x+x^5), x);
\frac{\text{\textstyle } 2 x + 1 - 4 \text{\textstyle } x + 5}{\text{\textstyle } 3 \text{\textstyle } 2}
\frac{\text{\textstyle } 2 x + 1}{\text{\textstyle } 3 \text{\textstyle } 2}
\frac{\text{\textstyle } 5 \text{\textstyle } \text{\texttt{atan}}}{}
\frac{\text{\textstyle } \text{\texttt{log}}}{}
\frac{\text{\textstyle } \sqrt{3}}{}
\frac{\text{\textstyle } 7}{7}
\frac{\text{\textstyle } 14}{14}
\frac{\text{\textstyle } 7 \text{\textstyle } \sqrt{3}}{7 

Alternatively the user may compute the roots of the denominator separately, and then express the integrand in terms of these roots, e.g., $1/((x - a)*(x - b)*(x - c))$ or $1/((x^2 - (a+b)*x + a*b)*(x - c))$ if the denominator is a cubic polynomial. Sometimes this will help Maxima obtain a more useful result.
laplace (expr, t, s)

Attempts to compute the Laplace transform of expr with respect to the variable t and transform parameter s. The Laplace transform of the function f(t) is the one-sided transform defined by

\[ F(s) = \int_{0}^{\infty} f(t)e^{-st}dt \]

where \( F(s) \) is the transform of \( f(t) \), represented by expr.

`laplace` recognizes in expr the functions `delta`, `exp`, `log`, `sin`, `cos`, `sinh`, `cosh`, and `erf`, as well as `derivative`, `integrate`, `sum`, and `ilt`. If `laplace` fails to find a transform the function `specint` is called. `specint` can find the laplace transform for expressions with special functions like the bessel functions `bessel_j`, `bessel_i`, … and can handle the `unit_step` function. See also `specint`.

If `specint` cannot find a solution too, a noun `laplace` is returned.

expr may also be a linear, constant coefficient differential equation in which case `atvalue` of the dependent variable is used. The required atvalue may be supplied either before or after the transform is computed. Since the initial conditions must be specified at zero, if one has boundary conditions imposed elsewhere he can impose these on the general solution and eliminate the constants by solving the general solution for them and substituting their values back.

`laplace` recognizes convolution integrals of the form

\[ \int_{0}^{t} f(x)g(t-x)dx \]

Other kinds of convolutions are not recognized.

Functional relations must be explicitly represented in expr; implicit relations, established by `depends`, are not recognized. That is, if \( f \) depends on \( x \) and \( y \), \( f(x, y) \) must appear in expr.

See also `ilt`, the inverse Laplace transform.

Examples:

```lisp
(%i1) laplace (exp (2*t + a) * sin(t) * t, t, s);
                   a
%e (2 s - 4)
(%o1) ---------------
         2             2
           (s - 4 s + 5)
(%i2) laplace ('diff (f (x), x), x, s);
(%o2) s laplace(f(x), x, s) - f(0)
(%i3) diff (diff (delta (t), t), t);
               d
(%o3) --- (delta(t))
               2
               dt
(%i4) laplace (%o3, t, s);
```

\[
\begin{align*}
\frac{d}{dt} \left( -\frac{2}{s} \delta(t) + s - \delta(0)s \right) = 0 \\
\text{assume}(a>0) \\
laplace(\text{gamma incomplete}(a,t),t,s), \text{gamma expand: true}; \\
- a - 1 \\
\begin{align*}
\text{gamma}(a) & \quad \text{gamma}(a) s \\
\frac{1}{s} & \quad a \\
(- + 1) & \\
s & \\
\text{factor}(\text{laplace}(\text{gamma incomplete}(1/2,t),t,s)); \\
\begin{align*}
\frac{s + 1}{\sqrt{\pi}} (\sqrt{s} \sqrt{\frac{\pi}{s}} - 1) & \\
\frac{s}{3/2 s + 1} & \\
\text{factor}(%) \\
\frac{\pi s}{\pi s} & \quad \frac{\text{e}}{\text{e}} \\
\frac{(s - \text{i}) (s + \text{i}) (\text{e} - 1)}{(s + \text{i}) (1 - \text{e} \text{i}) (s - \text{i}) (1 + \text{e} \text{i})} & \\
\frac{2}{(s + \text{i}) (1 - \text{e} \text{i}) (s - \text{i}) (1 + \text{e} \text{i})} & \\
\end{align*}
\end{align*}
\]

\text{ldefint} (expr, x, a, b) [Function]
Attempts to compute the definite integral of expr by using \text{limit} to evaluate the indefinite integral of expr with respect to x at the upper limit b and at the lower limit a. If it fails to compute the definite integral, \text{ldefint} returns an expression containing limits as noun forms.

\text{ldefint} is not called from \text{integrate}, so executing \text{ldefint} (expr, x, a, b) may yield a different result than \text{integrate} (expr, x, a, b). \text{ldefint} always uses the same method to evaluate the definite integral, while \text{integrate} may employ various heuristics and may recognize some special cases.
pwilt (expr, s, t) [Function]
Computes the inverse Laplace transform of expr with respect to s and parameter t. Unlike ilt, pwilt is able to return piece-wise and periodic functions and can also handle some cases with polynomials of degree greater than 3 in the denominator.

Two examples where ilt fails:

(%i1) pwilt (exp(-s)*s/(s^3-2*s-s+2), s, t);
   t - 1   2 (t - 1)
   ------- - -------
   3       9
(%o1) hstep(t - 1) (------------------ - ------------------)

(%i2) pwilt ((s^2+2)/(s^2-1), s, t);
   t - t
   3 %e 3 %e
   ------- -------
   2 2
(%o2) delta(t) + ------- - -------

potential (givengradient) [Function]
The calculation makes use of the global variable potentialzeroloc[0] which must be nonlist or of the form

[indeterminatej=expressionj, indeterminatek=expressionk, ...]
the former being equivalent to the nonlist expression for all right-hand sides in the latter. The indicated right-hand sides are used as the lower limit of integration. The success of the integrations may depend upon their values and order. potentialzeroloc is initially set to 0.

residue (expr, z, z_0) [Function]
Computes the residue in the complex plane of the expression expr when the variable z assumes the value z_0. The residue is the coefficient of (z - z_0)^(-1) in the Laurent series for expr.

(%i1) residue (s/(s**2+a**2), s, a*%i);
1
(%o1) -
2
(%i2) residue (sin(a*x)/x**4, x, 0);
3
(%o2) a

risch (expr, x) [Function]
Integrates expr with respect to x using the transcendental case of the Risch algorithm. (The algebraic case of the Risch algorithm has not been implemented.) This currently handles the cases of nested exponentials and logarithms which the main part of integrate can’t do. integrate will automatically apply risch if given these cases.
erfflag, if false, prevents risch from introducing the erf function in the answer if there were none in the integrand to begin with.

(%i1) risch (x^2*erf(x), x);
   3 2
  %pi x erf(x) + (sqrt(%pi) x + sqrt(%pi)) e
(%o1) -------------------------------
      3 %pi

(%i2) diff(%o1, x), ratsimp;
   2
2
  x erf(x)
(%o2) -------------------------------------------------

specint (exp(- s*t) * expr, t) [Function]
Compute the Laplace transform of expr with respect to the variable t. The integrand expr may contain special functions. The parameter s maybe be named something else; it is determined automatically, as the examples below show where p is used in some places.

The following special functions are handled by specint: incomplete gamma function, error functions (but not the error function erfi, it is easy to transform erfi e.g. to the error function erf), exponential integrals, bessel functions (including products of bessel functions), hankel functions, hermite and the laguerre polynomials.

Furthermore, specint can handle the hypergeometric function %f[p,q]([],[],z), the Whittaker function of the first kind %m[u,k](z) and of the second kind %w[u,k](z).

The result may be in terms of special functions and can include unsimplified hypergeometric functions.

When laplace fails to find a Laplace transform, specint is called. Because laplace knows more general rules for Laplace transforms, it is preferable to use laplace and not specint.

demo("hypgeo") displays several examples of Laplace transforms computed by specint.

Examples:

(%i1) assume (p > 0, a > 0)$
(%i2) specint (t^(1/2) * exp(-a*t/4) * exp(-p*t), t);
   sqrt(%pi)
   a 3/2
2 (p + -)
  4
(%o2) ------------------------

(%i3) specint (t^(1/2) * bessel_j(1, 2 * a^(1/2) * t^(1/2))
  * exp(-p*t), t);
   - a/p
   sqrt(a) %e
2
(%o3) ------------------------
  p
Examples for exponential integrals:

(%i4) assume(s>0,a>0,s-a>0)$
(%i5) ratsimp(specint(%e^(-a*t)
   *(log(a)+expintegral_e1(a*t))*%e^(-s*t),t));
   log(s)
(%o5) -------
   s - a
(%i6) logarc:true$
(%i7) gamma_expand:true$
radcan(specint((cos(t)*expintegral_si(t)
   -sin(t)*expintegral_ci(t))*%e^(-s*t),t));
   log(s)
(%o8) -------
   2
   s + 1
ratsimp(specint((2*t*log(a)+2/a*sin(a*t)
   -2*t*expintegral_ci(a*t))*%e^(-s*t),t));
   log(s+a)
(%o9) -----------------
   2
   s

Results when using the expansion of \texttt{gamma_incomplete} and when changing the representation to \texttt{expintegral_e1}:

(%i10) assume(s>0)$
(%i11) specint(1/sqrt(%pi*t)*unit_step(t-k)*%e^(-s*t),t);
   1
   gamma_incomplete(-, k s)
(%o11) ------------------------
   2
sqrt(%pi) sqrt(s)
(%i12) gamma_expand:true$
(%i13) specint(1/sqrt(%pi*t)*unit_step(t-k)*%e^(-s*t),t);
   erfc(sqrt(k) sqrt(s))
(%o13) ---------------------
   sqrt(s)
(%i14) expintrep:expintegral_e1$
(%i15) ratsimp(specint(1/(t+a)^2*%e^(-s*t),t));
   a s
   a s %e expintegral_e1(a s) - 1
(%o15) ----------------------------------
   a
tldefint (expr, x, a, b)  [Function]

Equivalent to ldefint with tlimswitch set to true.

19.3 Introduction to QUADPACK

QUADPACK is a collection of functions for the numerical computation of one-dimensional definite integrals. It originated from a joint project of R. Piessens\(^1\), E. de Doncker\(^2\), C. Ueberhuber\(^3\), and D. Kahaner\(^4\).

The QUADPACK library included in Maxima is an automatic translation (via the program f2cl\(^5\)) of the Fortran source code of QUADPACK as it appears in the SLATEC Common Mathematical Library, Version 4.1\(^5\). The SLATEC library is dated July 1993, but the QUADPACK functions were written some years before. There is another version of QUADPACK at Netlib\(^6\); it is not clear how that version differs from the SLATEC version.

The QUADPACK functions included in Maxima are all automatic, in the sense that these functions attempt to compute a result to a specified accuracy, requiring an unspecified number of function evaluations. Maxima’s Lisp translation of QUADPACK also includes some non-automatic functions, but they are not exposed at the Maxima level.

Further information about QUADPACK can be found in the QUADPACK book\(^7\).

19.3.1 Overview

\textbf{quad_qag} Integration of a general function over a finite interval. \textbf{quad_qag} implements a simple globally adaptive integrator using the strategy of Aind (Piessens, 1973). The caller may choose among 6 pairs of Gauss-Kronrod quadrature formulae for the rule evaluation component. The high-degree rules are suitable for strongly oscillating integrands.

\textbf{quad_qags} Integration of a general function over a finite interval. \textbf{quad_qags} implements globally adaptive interval subdivision with extrapolation (de Doncker, 1978) by the Epsilon algorithm (Wynn, 1956).

\textbf{quad_qagi} Integration of a general function over an infinite or semi-infinite interval. The interval is mapped onto a finite interval and then the same strategy as in \textbf{quad_qags} is applied.

\textbf{quad_qawo} Integration of \(\cos(\omega x)f(x)\) or \(\sin(\omega x)f(x)\) over a finite interval, where \(\omega\) is a constant. The rule evaluation component is based on the modified Clenshaw-

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\(^1\) Applied Mathematics and Programming Division, K.U. Leuven
\(^2\) Applied Mathematics and Programming Division, K.U. Leuven
\(^3\) Institut für Mathematik, T.U. Wien
\(^4\) National Bureau of Standards, Washington, D.C., U.S.A
\(^5\) \url{https://www.netlib.org/slatec}
\(^6\) \url{https://www.netlib.org/quadpack}
Curtis technique. `quad_qawo` applies adaptive subdivision with extrapolation, similar to `quad_qags`.

`quad_qawf`  
Calculates a Fourier cosine or Fourier sine transform on a semi-infinite interval. The same approach as in `quad_qawo` is applied on successive finite intervals, and convergence acceleration by means of the Epsilon algorithm (Wynn, 1956) is applied to the series of the integral contributions.

`quad_qaws`  
Integration of \( w(x)f(x) \) over a finite interval \([a,b]\), where \( w \) is a function of the form \((x - a)^\alpha(b - x)^\beta v(x) \) and \( v(x) \) is 1 or \( \log(x - a) \) or \( \log(b - x) \) or \( \log(x - a) \log(b - x) \), and \( \alpha > -1 \) and \( \beta > -1 \).
A globally adaptive subdivision strategy is applied, with modified Clenshaw-Curtis integration on the subintervals which contain \( a \) or \( b \).

`quad_qawc`  
Computes the Cauchy principal value of \( f(x)/(x - c) \) over a finite interval \((a,b)\) and specified \( c \). The strategy is globally adaptive, and modified Clenshaw-Curtis integration is used on the subranges which contain the point \( x = c \).

`quad_qagp`  
Basically the same as `quad_qags` but points of singularity or discontinuity of the integrand must be supplied. This makes it easier for the integrator to produce a good solution.

### 19.4 Functions and Variables for QUADPACK

`quad_qag`  
Integration of a general function over a finite interval. `quad_qag` implements a simple globally adaptive integrator using the strategy of Aind (Piessens, 1973). The caller may choose among 6 pairs of Gauss-Kronrod quadrature formulae for the rule evaluation component. The high-degree rules are suitable for strongly oscillating integrands.

`quad_qag` computes the integral

\[
\int_{a}^{b} f(x) \, dx
\]

The function to be integrated is \( f(x) \), with dependent variable \( x \), and the function is to be integrated between the limits \( a \) and \( b \). \( key \) is the integrator to be used and should be an integer between 1 and 6, inclusive. The value of \( key \) selects the order of the Gauss-Kronrod integration rule. High-order rules are suitable for strongly oscillating integrands.

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

The numerical integration is done adaptively by subdividing the integration region into sub-intervals until the desired accuracy is achieved.
The keyword arguments are optional and may be specified in any order. They all take the form `key=val`. The keyword arguments are:

- **epsrel** Desired relative error of approximation. Default is $1 \times 10^{-8}$.
- **epsabs** Desired absolute error of approximation. Default is 0.
- **limit** Size of internal work array. `limit` is the maximum number of subintervals to use. Default is 200.

`quad_qag` returns a list of four elements:
- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

- 0 if no problems were encountered;
- 1 if too many sub-intervals were done;
- 2 if excessive roundoff error is detected;
- 3 if extremely bad integrand behavior occurs;
- 6 if the input is invalid.

Examples:

```maxima
(%i1) quad_qag (x^(1/2)*log(1/x), x, 0, 1, 3, 'epsrel=5d-8);
(%o1) [.444444444492108, 3.1700968502883E-9, 961, 0]
(%i2) integrate (x^(1/2)*log(1/x), x, 0, 1);
     4
(%o2) -
     9
```

`quad_qags` [Function]

\[
\text{quad_qags}\ (f(x), x, a, b, [\text{epsrel}, \text{epsabs}, \text{limit}])
\]

Integration of a general function over a finite interval. `quad_qags` implements globally adaptive interval subdivision with extrapolation (de Doncker, 1978) by the Epsilon algorithm (Wynn, 1956).

`quad_qags` computes the integral

\[
\int_{a}^{b} f(x) \, dx
\]

The function to be integrated is \( f(x) \), with dependent variable \( x \), and the function is to be integrated between the limits \( a \) and \( b \).

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.
The keyword arguments are optional and may be specified in any order. They all take the form `key=val`. The keyword arguments are:

- **epsrel**: Desired relative error of approximation. Default is 1d-8.
- **epsabs**: Desired absolute error of approximation. Default is 0.
- **limit**: Size of internal work array. `limit` is the maximum number of subintervals to use. Default is 200.

`quad_qags` returns a list of four elements:
- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

0: no problems were encountered;
1: too many sub-intervals were done;
2: excessive roundoff error is detected;
3: extremely bad integrand behavior occurs;
4: failed to converge
5: integral is probably divergent or slowly convergent
6: if the input is invalid.

Examples:

```
(%i1) quad_qags (x^(1/2)*log(1/x), x, 0, 1, 'epsrel=1d-10);
(%o1) [0.4444444444444448, 1.11022302462516E-15, 315, 0]
```

Note that `quad_qags` is more accurate and efficient than `quad_qag` for this integrand.

`quad_qagi` evaluates one of the following integrals

\[ \int_{a}^{\infty} f(x) \, dx \]
\[ \int_{-\infty}^{a} f(x) \, dx \]
\[ \int_{a}^{-\infty} f(x) \, dx \]
\[ \int_{-\infty}^{\infty} f(x) \, dx \]
using the Quadpack QAGI routine. The function to be integrated is \( f(x) \), with
dependent variable \( x \), and the function is to be integrated over an infinite range.
The integrand may be specified as the name of a Maxima or Lisp function or operator,
a Maxima lambda expression, or a general Maxima expression.

One of the limits of integration must be infinity. If not, then \texttt{quad_qagi} will just
return the noun form.

The keyword arguments are optional and may be specified in any order. They all
take the form \texttt{key=val}. The keyword arguments are:
\begin{itemize}
  \item \texttt{epsrel}  Desired relative error of approximation. Default is 1d-8.
  \item \texttt{epsabs}  Desired absolute error of approximation. Default is 0.
  \item \texttt{limit}   Size of internal work array. \texttt{limit} is the maximum number of subintervals
to use. Default is 200.
\end{itemize}

\texttt{quad_qagi} returns a list of four elements:
\begin{itemize}
  \item an approximation to the integral,
  \item the estimated absolute error of the approximation,
  \item the number integrand evaluations,
  \item an error code.
\end{itemize}

The error code (fourth element of the return value) can have the values:
\begin{itemize}
  \item 0  no problems were encountered;
  \item 1  too many sub-intervals were done;
  \item 2  excessive roundoff error is detected;
  \item 3  extremely bad integrand behavior occurs;
  \item 4  failed to converge
  \item 5  integral is probably divergent or slowly convergent
  \item 6  if the input is invalid.
\end{itemize}

Examples:
\begin{verbatim}
(%i1) quad_qagi (x^2*exp(-4*x), x, 0, inf, 'epsrel=1d-8);
(%o1) [0.03125, 2.95916102995002E-11, 105, 0]
(%i2) integrate (x^2*exp(-4*x), x, 0, inf);
     1
(%o2) --
     32
\end{verbatim}

\texttt{quad_qawc} \hspace{1cm} [Function]
\begin{verbatim}
quad_qawc (f(x), x, c, a, b, [epsrel, epsabs, limit])
quad_qawc (f, x, c, a, b, [epsrel, epsabs, limit])
\end{verbatim}
Computes the Cauchy principal value of \( f(x)/(x - c) \) over a finite interval. The
strategy is globally adaptive, and modified Clenshaw-Curtis integration is used on
the subranges which contain the point \( x = c \).
quad_qawc computes the Cauchy principal value of

\[
\int_a^b \frac{f(x)}{x - c} \, dx
\]

using the Quadpack QAWC routine. The function to be integrated is \(f(x)/(x - c)\), with dependent variable \(x\), and the function is to be integrated over the interval \(a\) to \(b\).

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

The keyword arguments are optional and may be specified in any order. They all take the form \texttt{key=val}. The keyword arguments are:

- \texttt{epsrel} Desired relative error of approximation. Default is 1d-8.
- \texttt{epsabs} Desired absolute error of approximation. Default is 0.
- \texttt{limit} Size of internal work array. \texttt{limit} is the maximum number of subintervals to use. Default is 200.

\texttt{quad_qawc} returns a list of four elements:

- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

- 0 no problems were encountered;
- 1 too many sub-intervals were done;
- 2 excessive roundoff error is detected;
- 3 extremely bad integrand behavior occurs;
- 6 if the input is invalid.

Examples:

\begin{verbatim}
(%i1) quad_qawc (2^(-5)*((x-1)^2+4^(-5))^(-1), x, 2, 0, 5, 'epsrel=1d-7);
(%o1) [- 3.130120337415925, 1.306830140249558E-8, 495, 0]
\end{verbatim}
(\%i2) integrate (2^(-alpha)*(((x-1)^2 + 4^(-alpha))*(x-2))^(-1), x, 0, 5);

Principal Value

\[
\begin{align*}
\text{alpha} & \quad 9 \\
4 & \quad \log\left(\frac{1}{\alpha} + \frac{1}{4}\right) \\
\text{alpha} & \quad \alpha \\
64 & \quad 4 + 4 \\
4 & \quad 64 + 4 \\
\end{align*}
\]

(\%o2) \frac{\alpha}{2} + 2

\[
\begin{align*}
\frac{3\alpha}{2} & \quad \alpha \\
\alpha/2 & \quad \alpha \\
2 & \quad \alpha \\
\end{align*}
\]

\[
\begin{align*}
2 & \quad \alpha \\
4 & \quad \alpha \\
\end{align*}
\]

\[
\begin{align*}
\frac{3\alpha}{2} & \quad \alpha \\
\alpha/2 & \quad \alpha \\
2 & \quad \alpha \\
\end{align*}
\]

\[
\begin{align*}
\frac{3\alpha}{2} & \quad \alpha \\
\alpha/2 & \quad \alpha \\
2 & \quad \alpha \\
\end{align*}
\]

(\%i3) ev (\%, alpha=5, numer);

(\%o3) - 3.130120337415917

quad_qawf

quad_qawf (f(x), x, a, omega, trig, [epsabs, limit, maxp1, limlst])

quad_qawf (f, x, a, omega, trig, [epsabs, limit, maxp1, limlst])

Calculates a Fourier cosine or Fourier sine transform on a semi-infinite interval using the Quadpack QAWF function. The same approach as in quad_qawo is applied on successive finite intervals, and convergence acceleration by means of the Epsilon algorithm (Wynn, 1956) is applied to the series of the integral contributions.

quad_qawf computes the integral

\[
\int_{-\infty}^{\infty} f(x) \, w(x) \, dx
\]

The weight function \( w \) is selected by \( \text{trig} \):

\[
\text{cos} \quad w(x) = \cos \omega x
\]

\[
\text{sin} \quad w(x) = \sin \omega x
\]

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

The keyword arguments are optional and may be specified in any order. They all take the form \text{key=val}. The keyword arguments are:

\text{epsabs} \quad \text{Desired absolute error of approximation. Default is 1d-10.}
limit  Size of internal work array. \((\text{limit} - \text{limlst})/2\) is the maximum number of subintervals to use. Default is 200.

maxp1  Maximum number of Chebyshev moments. Must be greater than 0. Default is 100.

limlst  Upper bound on the number of cycles. Must be greater than or equal to 3. Default is 10.

\texttt{quad_qawf} returns a list of four elements:
- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:
- 0: no problems were encountered;
- 1: too many sub-intervals were done;
- 2: excessive roundoff error is detected;
- 3: extremely bad integrand behavior occurs;
- 6: if the input is invalid.

Examples:
\begin{verbatim}
(%i1) quad_qawf (exp(-x^2), x, 0, 1, 'cos, 'epsabs=1d-9);
(%o1) [.6901942235215714, 2.84846300257552E-11, 215, 0]
(%i2) integrate (exp(-x^2)*cos(x), x, 0, inf);
   - 1/4
(%o2) -----------------
       2
            %e    sqrt(%pi)
(%i3) ev (%o2, numer);
(%o3) .6901942235215714
\end{verbatim}

\texttt{quad_qawo}\quad [Function]
\begin{verbatim}
quad_qawo (f(x), x, a, b, omega, trig, [epsrel, epsabs, limit, maxp1, limlst])
quad_qawo (f, x, a, b, omega, trig, [epsrel, epsabs, limit, maxp1, limlst])
\end{verbatim}

Integration of \(\cos(\omega x)f(x)\) or \(\sin(\omega x)\) over a finite interval, where \(\omega\) is a constant. The rule evaluation component is based on the modified Clenshaw-Curtis technique. \texttt{quad_qawo} applies adaptive subdivision with extrapolation, similar to \texttt{quad_qags}. \texttt{quad_qawo} computes the integral using the Quadpack QAWO routine:

\[
\int_{a}^{b} f(x) w(x) \, dx
\]
Chapter 19: Integration

The weight function \( w \) is selected by \( \text{trig} \):

\[
\begin{align*}
\text{cos} & \quad w(x) = \cos \omega x \\
\text{sin} & \quad w(x) = \sin \omega x
\end{align*}
\]

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression. The keyword arguments are optional and may be specified in any order. They all take the form \texttt{key=val}. The keyword arguments are:

- **epsrel**: Desired relative error of approximation. Default is \( 1d-8 \).
- **epsabs**: Desired absolute error of approximation. Default is 0.
- **limit**: Size of internal work array. \( \text{limit}/2 \) is the maximum number of subintervals to use. Default is 200.
- **maxp1**: Maximum number of Chebyshev moments. Must be greater than 0. Default is 100.
- **limlst**: Upper bound on the number of cycles. Must be greater than or equal to 3. Default is 10.

\texttt{quad_qawo} returns a list of four elements:

- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

- 0: no problems were encountered;
- 1: too many sub-intervals were done;
- 2: excessive roundoff error is detected;
- 3: extremely bad integrand behavior occurs;
- 6: if the input is invalid.

Examples:

\begin{verbatim}
(%i1) quad_qawo (x^(-1/2)*exp(-2^(-2)*x), x, 1d-8, 20*2^2, 1, cos);
(%o1) [1.376043389877692, 4.72710759424899E-11, 765, 0]
(%i2) rectform (integrate (x^(-1/2)*exp(-2^(-alpha)*x) * cos(x),
               x, 0, inf));
           2 alpha
alpha/2 - 1/2       2 alpha
  sqrt(%pi) 2    sqrt(sqrt(2 + 1) + 1)
(%o2) ----------------------------------------
            2 alpha
                sqrt(2 + 1)
(%i3) ev (%, alpha=2, numer);
(%o3) 1.376043390090716
\end{verbatim}
quad_qaws\[Function\]\nquad_qaws(f(x), x, a, b, alpha, beta, wfun, [epsrel, epsabs, limit])
quad_qaws(f, x, a, b, alpha, beta, wfun, [epsrel, epsabs, limit])
Integration of \(w(x)f(x)\) over a finite interval, where \(w(x)\) is a certain algebraic or logarithmic function. A globally adaptive subdivision strategy is applied, with modified Clenshaw-Curtis integration on the subintervals which contain the endpoints of the interval of integration.
quad_qaws computes the integral using the Quadpack QAWS routine:
\[
\int_{a}^{b} f(x)w(x) \, dx
\]
The weight function \(w\) is selected by \(wfun\):
1. \(w(x) = (x - a)^\alpha(b - x)^\beta\)
2. \(w(x) = (x - a)^\alpha(b - x)^\beta \log(x - a)\)
3. \(w(x) = (x - a)^\alpha(b - x)^\beta \log(b - x)\)
4. \(w(x) = (x - a)^\alpha(b - x)^\beta \log(x - a) \log(b - x)\)
The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.
The keyword arguments are optional and may be specified in any order. They all take the form \texttt{key=val}. The keyword arguments are:
- \texttt{epsrel} Desired relative error of approximation. Default is 1d-8.
- \texttt{epsabs} Desired absolute error of approximation. Default is 0.
- \texttt{limit} Size of internal work array. \texttt{limit} is the maximum number of subintervals to use. Default is 200.
quad_qaws returns a list of four elements:
- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.
The error code (fourth element of the return value) can have the values:
0 no problems were encountered;
1 too many sub-intervals were done;
2 excessive roundoff error is detected;
3 extremely bad integrand behavior occurs;
Examples:

(\%i1) \quad \text{quad_qaws} \left(\frac{1}{(x+1+2^{-4})}, x, -1, 1, -0.5, -0.5, 1, \ \text{epsabs}=1d-9\right);
(\%o1) \quad [8.750097361672832, 1.24321522715422E-10, 170, 0]

(\%i2) \quad \text{integrate} \left(\frac{(1-x^2)^{-1/2}}{(x+1+2^{-\alpha})}, x, -1, 1\right);
\alpha

\text{pos};

\frac{\alpha}{2} \quad \text{alpha} \quad \frac{\sqrt{2} \quad \text{alpha}}{2} \quad + \quad 1
\quad \alpha
\quad \frac{\text{alpha}}{4} \quad + \quad 2

(\%o1) \quad \text{ev} \left(%, \ \text{alpha}=4, \ \text{numer}\right);
(\%o2) \quad 8.750097361672829

\text{quad_qagp}

\text{quad_qagp} \left(f(x), x, a, b, \text{points}, [\text{epsrel, epsabs, limit}]\right)
\text{quad_qagp} \left(f, x, a, b, \text{points}, [\text{epsrel, epsabs, limit}]\right)

Integration of a general function over a finite interval. \text{quad_qagp} implements globally adaptive interval subdivision with extrapolation (de Doncker, 1978) by the Epsilon algorithm (Wynn, 1956).

\text{quad_qagp} \text{ computes the integral}

\int_{a}^{b} f(x) \, dx

The function to be integrated is \( f(x) \), with dependent variable \( x \), and the function is to be integrated between the limits \( a \) and \( b \).

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

To help the integrator, the user must supply a list of points where the integrand is singular or discontinuous.

The keyword arguments are optional and may be specified in any order. They all take the form \text{key=val}. The keyword arguments are:

\text{epsrel} \quad \text{Desired relative error of approximation. Default is 1d-8.}

\text{epsabs} \quad \text{Desired absolute error of approximation. Default is 0.}

\text{limit} \quad \text{Size of internal work array. limit is the maximum number of subintervals to use. Default is 200.}

\text{quad_qagp} \text{ returns a list of four elements:}

\textbullet \quad \text{an approximation to the integral,}

\textbullet \quad \text{the estimated absolute error of the approximation,}
• the number integrand evaluations,
• an error code.

The error code (fourth element of the return value) can have the values:
0  no problems were encountered;
1  too many sub-intervals were done;
2  excessive roundoff error is detected;
3  extremely bad integrand behavior occurs;
4  failed to converge
5  integral is probably divergent or slowly convergent
6  if the input is invalid.

Examples:
(%i1) quad_qagp(x^3*log(abs((x^2-1)*(x^2-2))),x,0,3,[1,sqrt(2)]);
(%o1) [52.74074838347143, 2.6247632689546663e-7, 1029, 0]
(%i2) quad_qags(x^3*log(abs((x^2-1)*(x^2-2))), x, 0, 3);
(%o2) [52.74074847951494, 4.088443219529836e-7, 1869, 0]

The integrand has singularities at 1 and sqrt(2) so we supply these points to `quad_qagp`. We also note that `quad_qagp` is more accurate and more efficient than `quad_qags`.

quad_control (parameter, [value]) [Function]
Control error handling for quadpack. The parameter should be one of the following symbols:

current_error
  The current error number

control
  Controls if messages are printed or not. If it is set to zero or less, messages
  are suppressed.

max_message
  The maximum number of times any message is to be printed.

If value is not given, then the current value of the parameter is returned. If value is
given, the value of parameter is set to the given value.
20 Equations

20.1 Functions and Variables for Equations

\%rnum

[System variable]

Default value: 0

\%rnum is the counter for the \%r variables introduced in solutions by solve and algsys.

The next \%r variable is numbered \%rnum+1.

See also \%rnum_list.

\%rnum_list

[System variable]

Default value: []

\%rnum_list is the list of variables introduced in solutions by solve and algsys. \%r variables are added to \%rnum_list in the order they are created. This is convenient for doing substitutions into the solution later on.

See also \%rnum.

It’s recommended to use this list rather than doing concat ('\%r, j).

(%i1) solve ([x + y = 3], [x, y]);
(%o1) [[x = 3 - \%r1, y = \%r1]]

(%i2) \%rnum_list;
(%o2) []

(%i3) sol : solve ([x + 2*y + 3*z = 4], [x,y,z]);
(%o3) [[x = - 2 \%r3 - 3 \%r2 + 4, y = \%r3, z = \%r2]]

(%i4) \%rnum_list;
(%o4) []

(%i5) for i : 1 thru length (\%rnum_list) do
   sol : subst (t[i], \%rnum_list[i], sol)$
(%i6) sol;
(%o6) [[x = - 2 t - 3 t + 4, y = t , z = t ]]
     2  1  2  1

algepsilon

[Option variable]

Default value: 10^-8

algepsilon is used by algsys.

algexact

[Option variable]

Default value: false

algexact affects the behavior of algsys as follows:

If algexact is true, algsys always calls solve and then uses realroots on solve’s failures.

If algexact is false, solve is called only if the eliminant was not univariate, or if it was a quadratic or biquadratic.

Thus algexact: true does not guarantee only exact solutions, just that algsys will first try as hard as it can to give exact solutions, and only yield approximations when all else fails.
algsys

algsys ([expr_1, ..., expr_m], [x_1, ..., x_n])
algsys ([eqn_1, ..., eqn_m], [x_1, ..., x_n])

Solves the simultaneous polynomials expr_1, ..., expr_m or polynomial equations eqn_1, ..., eqn_m for the variables x_1, ..., x_n. An expression expr is equivalent to an equation expr = 0. There may be more equations than variables or vice versa.

algsys returns a list of solutions, with each solution given as a list of equations stating values of the variables x_1, ..., x_n which satisfy the system of equations. If algsys cannot find a solution, an empty list [] is returned.

The symbols %r1, %r2, ..., are introduced as needed to represent arbitrary parameters in the solution; these variables are also appended to the list %rnum_list.

The method is as follows:

1. First the equations are factored and split into subsystems.
2. For each subsystem S_i, an equation E and a variable x are selected. The variable is chosen to have lowest nonzero degree. Then the resultant of E and E_j with respect to x is computed for each of the remaining equations E_j in the subsystem S_i. This yields a new subsystem S_i' in one fewer variables, as x has been eliminated. The process now returns to (1).
3. Eventually, a subsystem consisting of a single equation is obtained. If the equation is multivariate and no approximations in the form of floating point numbers have been introduced, then solve is called to find an exact solution.

In some cases, solve is not be able to find a solution, or if it does the solution may be a very large expression.

If the equation is univariate and is either linear, quadratic, or biquadratic, then again solve is called if no approximations have been introduced. If approximations have been introduced or the equation is not univariate and neither linear, quadratic, or biquadratic, then if the switch realonly is true, the function realroots is called to find the real-valued solutions. If realonly is false, then allroots is called which looks for real and complex-valued solutions.

If algsys produces a solution which has fewer significant digits than required, the user can change the value of algepsilon to a higher value.

If algexact is set to true, solve will always be called.
4. Finally, the solutions obtained in step (3) are substituted into previous levels and the solution process returns to (1).

When algsys encounters a multivariate equation which contains floating point approximations (usually due to its failing to find exact solutions at an earlier stage), then it does not attempt to apply exact methods to such equations and instead prints the message: "algsys cannot solve - system too complicated."

Interactions with radcan can produce large or complicated expressions. In that case, it may be possible to isolate parts of the result with pickapart or reveal.

Occasionally, radcan may introduce an imaginary unit %i into a solution which is actually real-valued.

Examples:

(%i1) e1: 2*x*(1 - a1) - 2*(x - 1)*a2;
\%(o1) 2 (1 - a1) x - 2 a2 (x - 1)
\%(i2) e2: a2 - a1;
\%(o2) a2 - a1
\%(i3) e3: a1*(-y - x^2 + 1);
\%(o3) a1 (- y - x + 1)
\%(i4) e4: a2*(y - (x - 1)^2);
\%(o4) a2 (y - (x - 1) )
\%(i5) algsys ([e1, e2, e3, e4], [x, y, a1, a2]);
\%(o5) [[x = 0, y = %r1, a1 = 0, a2 = 0],
[x = 1, y = 0, a1 = 1, a2 = 1]]
\%(i6) e1: x^2 - y^2;
\%(o6) x - y
\%(i7) e2: -1 - y + 2*y^2 - x + x^2;
\%(o7) 2 y - y + x - x - 1
\%(i8) algsys ([e1, e2], [x, y]);
\%(o8) [[x = - -------, y = -------],
      sqrt(3)      sqrt(3)
1 1
[x = - -------, y = - -------], [x = - -, y = - -], [x = 1, y = 1]]
      sqrt(3)      sqrt(3)
1 1
3 3

\textbf{allroots} \hspace{1em} \textbf{[Function]}
\textbf{allroots (expr)}
\textbf{allroots (eqn)}

Computes numerical approximations of the real and complex roots of the polynomial expr or polynomial equation eqn of one variable.

The flag \texttt{polyfactor} when true causes \texttt{allroots} to factor the polynomial over the real numbers if the polynomial is real, or over the complex numbers, if the polynomial is complex.

\texttt{allroots} may give inaccurate results in case of multiple roots. If the polynomial is real, \texttt{allroots ( \%i*p)} may yield more accurate approximations than \texttt{allroots (p)}, as \texttt{allroots} invokes a different algorithm in that case.

\texttt{allroots} rejects non-polynomials. It requires that the numerator after \texttt{rat’ing} should be a polynomial, and it requires that the denominator be at most a complex number. As a result of this \texttt{allroots} will always return an equivalent (but factored) expression, if \texttt{polyfactor} is true.

Examples:

\(\%i1\)

\[
\begin{align*}
(1 + 2x)^3 & = 13.5(1 + x^5); \\
(2x + 1)^3 & = 13.5(x + 1)
\end{align*}
\]

\(\%o1\)

\[
(2x + 1)^3 = 13.5(x + 1)
\]

\(\%i2\)

\[
\text{soln: allroots (eqn)};
\]

\(\%o2\)

\[
x = 0.9659625152196369 + 0.4069597231924075i,
\]

\[
x = -0.9659625152196369 - 0.4069597231924075i,
\]

\[
x = 0.8296749902129361, x = -1.01575543828121,
\]

\(\%i3\)

\[
\text{for e in soln}
\]

\[
do (e2: subst (e, eqn), disp (expand (lhs(e2) - rhs(e2))));
\]

\[
-3.5527136788005E-15
\]

\[
-5.32907051820075E-15
\]

\[
4.44089209850063E-15 + 4.88498130835069E-15i
\]

\[
-4.44089209850063E-15 - 4.88498130835069E-15i
\]

\[
3.5527136788005E-15
\]

\(\%o3\)

\[
done
\]

\(\%i4\)

polyfactor: true$

\(\%i5\)

\[
\text{allroots (eqn)};
\]

\(\%o5\)

\[
-13.5(x - 1.0) (x - 0.8296749902129361)
\]

\[
2
\]

\[
(x + 1.01575543828121) (x + 0.8139194463848151)
\]

\[
+ 1.098699797110288
\]

bfallroots [{Function}]

bfallroots (expr)

bfallroots (eqn)

Computes numerical approximations of the real and complex roots of the polynomial expr or polynomial equation eqn of one variable.

In all respects, bfallroots is identical to allroots except that bfallroots computes the roots using bigfloats. See allroots for more information.

backsubst [{Option variable}]

Default value: true

When backsubst is false, prevents back substitution in linsolve after the equations have been triangularized. This may be helpful in very big problems where back substitution would cause the generation of extremely large expressions.

\(\%i11\)

\[
eq 1 : x + y + z = 6$
\]

\(\%i12\)

\[
eq 2 : x - y + z = 2$
\]
breakup

Default value: true

When breakup is true, solve expresses solutions of cubic and quartic equations in terms of common subexpressions, which are assigned to intermediate expression labels (\( \%t1, \%t2 \), etc.). Otherwise, common subexpressions are not identified.

breakup: true has an effect only when programmode is false.

Examples:

(%i1) programmode: false$
(%i2) breakup: true$
(%i3) solve (x^3 + x^2 - 1);

sqrt(23) 25 1/3
(--------- + --)
6 sqrt(3) 54

Solution:

sqrt(3) %i 1
---------- - -
2 2

sqrt(3) %i 1
2 2 2 1
x = (- ------------- - -) %t3 + ------------- - -
2 2 9 %t3 3
---------- - -
2 sqrt(3) %i 1

sqrt(3) %i 1
2 2 2 1
x = (------------- - -) %t3 + ------------- - -
2 2 9 %t3 3
---------- - -

1 1
x = %t3 + ----- - -
9 %t3 3

(%o6) [\%t4, \%t5, \%t6]
(%i6) breakup: false$
(%i7) solve (x^3 + x^2 - 1);

Solution:

sqrt(3) %i 1
---------- - -
\[
\frac{2}{\sqrt{23}} + \frac{25}{6^{1/3}} + \frac{1}{9}
\]

\[
\frac{\sqrt{3}}{6} + \frac{\sqrt{3}}{54} + \frac{\sqrt{3}}{6} - \frac{\sqrt{3}}{54}
\]

\[
\frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2}
\]

\[
\frac{\sqrt{3}}{6} + \frac{\sqrt{3}}{54} + \frac{\sqrt{3}}{6} - \frac{\sqrt{3}}{54}
\]

\[
\text{dimension}
\]

\[
\text{dimension}(\text{eqn})
\]

\[
\text{dimension}(\text{eqn}_1, \ldots, \text{eqn}_n)
\]

dimen is a package for dimensional analysis. load("dimen") loads this package. demo("dimen") displays a short demonstration.

dispflag

[Option variable]

Default value: true

If set to false within a block will inhibit the display of output generated by the solve functions called from within the block. Termination of the block with a dollar sign, $, sets dispflag to false.

funcsolve (eqn, g(t))

[Function]

Returns \([g(t) = \ldots] or [], depending on whether or not there exists a rational function \(g(t)\) satisfying eqn, which must be a first order, linear polynomial in \(f(t)\) and \(f(t+1)\)

\[
\text{eqn}: (n + 1) f(n) - (n + 3) f(n + 1) / (n + 1) =
\]

\[
(n - 1) / (n + 2);
\]

\[
(n + 3) f(n + 1) / n - 1
\]

\[
(n + 1) f(n) - \ldots = \ldots
\]
\( \frac{n + 1}{n + 2} \)

(%i2) funcsolve (eqn, f(n));

Dependent equations eliminated: (4 3)

\( f(n) = \frac{n}{(n + 1)(n + 2)} \)

Warning: this is a very rudimentary implementation – many safety checks and obvious generalizations are missing.

globalsolve

[Option variable]

Default value: \texttt{false}

When \texttt{globalsolve} is \texttt{true}, solved-for variables are assigned the solution values found by \texttt{linsolve}, and by \texttt{solve} when solving two or more linear equations.

When \texttt{globalsolve} is \texttt{false}, solutions found by \texttt{linsolve} and by \texttt{solve} when solving two or more linear equations are expressed as equations, and the solved-for variables are not assigned.

When solving anything other than two or more linear equations, \texttt{solve} ignores \texttt{globalsolve}. Other functions which solve equations (e.g., \texttt{algsys}) always ignore \texttt{globalsolve}.

Examples:

(%i1) globalsolve: true$

(%i2) solve ([x + 3*y = 2, 2*x - y = 5], [x, y]);

Solution

\[
\begin{align*}
\text{17} \\
\text{x : --} \\
\text{7} \\
\text{1} \\
\text{y : - -} \\
\text{7} \\
\text{[[\text{t2}, \text{t3}]]} \\
\text{17} \\
\text{x;} \\
\text{1} \\
\text{--} \\
\text{7} \\
\text{y;} \\
\text{1} \\
\text{--} \\
\text{7} \\
\end{align*}
\]

(%i3) x;

(%i4) y;

(%i5) globalsolve: false$

(%i6) kill (x, y)$

(%i7) solve ([x + 3*y = 2, 2*x - y = 5], [x, y]);

Solution
\[
\begin{align*}
(\%t7) & \quad x = \frac{1}{7} \\
(\%t8) & \quad y = -\frac{1}{7} \\
(\%o8) & \quad [\{\%t7, \%t8\}] \\
(\%i8) & \quad x; \\
(\%o8) & \quad x \\
(\%i9) & \quad y; \\
(\%o9) & \quad y
\end{align*}
\]

\texttt{ieqn (ie, unk, tech, n, guess)} [Function]

\texttt{ieqn} is a package for solving integral equations. \texttt{load("ieqn")} loads this package.

\textit{ie} is the integral equation; \textit{unk} is the unknown function; \textit{tech} is the technique to be tried from those given in the lists below; (\texttt{tech = first} means: try the first technique which finds a solution; \texttt{tech = all} means: try all applicable techniques); \textit{n} is the maximum number of terms to take for \texttt{taylor}, \texttt{neumann}, \texttt{firstkindseries}, or \texttt{fredseries} (it is also the maximum depth of recursion for the differentiation method); \textit{guess} is the initial guess for \texttt{neumann} or \texttt{firstkindseries}.

Two types of equations are considered. A second-kind equation of the following form,

\[
p(x) = q\left(x, p(x), \int_{a(x)}^{b(x)} w(x, u, p(x), p(u)) \, du\right)
\]

and a first-kind equation with the form

\[
f(x) = \int_{a(x)}^{b(x)} w(x, u, p(u)) \, du
\]

The different solution techniques used require particular forms of the expressions \texttt{q} and \texttt{w}. The techniques available are the following:

\textbf{Second-kind equations}

- \texttt{flfrnk2nd}: For fixed-limit, finite-rank integrands.
- \texttt{vlfrnk}: For variable-limit, finite-rank integrands.
- \texttt{transform}: Laplace transform for convolution types.
- \texttt{fredseries}: Fredholm-Carleman series for linear equations.
- \texttt{tailor}: Taylor series for quasi-linear variable-limit equations.
- \texttt{neumann}: Neumann series for quasi-second kind equations.
- \texttt{collocate}: Collocation using a power series form for \texttt{p(x)} evaluated at equally spaced points.

\textbf{First-kind equations}

- \texttt{flfrnk1st}: For fixed-limit, finite-rank integrands.
• \textit{vlfrnk}: For variable-limit, finite-rank integrands.
• \textit{abel}: For singular integrands
• \textit{transform}: See above
• \textit{collocate}: See above
• \textit{firstkindseries}: Iteration technique similar to neumann series.

The default values for the 2nd thru 5th parameters in the calling form are:

\textit{unk}: \( p(x) \), where \( p \) is the first function encountered in an integrand which is unknown to Maxima and \( x \) is the variable which occurs as an argument to the first occurrence of \( p \) found outside of an integral in the case of \textit{secondkind} equations, or is the only other variable besides the variable of integration in \textit{firstkind} equations. If the attempt to search for \( x \) fails, the user will be asked to supply the independent variable. \textit{tech}: \textit{first}. \( u \): 1. \textit{guess}: \textit{none} which will cause \textit{neumann} and \textit{firstkindseries} to use \( f(x) \) as an initial guess.

Examples:

\begin{verbatim}
(%i1) load("inteqn")$
(%i2) e: p(x) - 1 - x + \cos(x) + \integrate(\cos(x-u)*p(u),u,0,x)$
(%i3) ieqn(e, p(x), 'transform);
default 4th arg, number of iterations or coll. parms.: 1
default 5th arg, initial guess: none

(%t3) [x, transform]
(%o3) [%t3]

(%i4) e: 2*\integrate(p(x*sin(u)), u, 0, %pi/2) - a*x - b$
(%i5) ieqn(e, p(x), 'firstkindseries);
default 4th arg, number of iterations or coll. parms.: 1
default 5th arg, initial guess: none

(%t5) [2 a x + %pi b, firstkindseries, 1, approximate]
(%o5) [%t5]
\end{verbatim}

\textbf{ieqnprint}

Default value: \texttt{true}

\texttt{ieqnprint} governs the behavior of the result returned by the \texttt{ieqn} command. When \texttt{ieqnprint} is \texttt{false}, the lists returned by the \texttt{ieqn} function are of the form

\[[\texttt{solution, technique used, nterms, flag}]\]

where \texttt{flag} is absent if the solution is exact.

Otherwise, it is the word \texttt{approximate} or \texttt{incomplete} corresponding to an inexact or non-closed form solution, respectively. If a series method was used, \texttt{nterms} gives the number of terms taken (which could be less than the \texttt{n} given to \texttt{ieqn} if an error prevented generation of further terms).
**lhs (expr)**

Returns the left-hand side (that is, the first argument) of the expression `expr`, when the operator of `expr` is one of the relational operators `< <= = # equal notequal >= >`, one of the assignment operators `:= ::= := :=` or a user-defined binary infix operator, as declared by `infix`.

When `expr` is an atom or its operator is something other than the ones listed above, `lhs` returns `expr`.

See also `rhs`.

Examples:

```latex
(%i1) e: a + b = c;
(%o1) b + a = c
(%i2) lhs (e);
(%o2) b + a
(%i3) rhs (e);
(%o3) c
(%i4) [lhs (a < b), lhs (a <= b), lhs (a >= b),
    lhs (a > b)];
(%o4) [a, a, a, a]
(%i5) [lhs (a = b), lhs (a # b), lhs (equal (a, b)),
    lhs (notequal (a, b))];
(%o5) [a, a, a, a]
(%i6) e1: '(foo(x) := 2*x);
(%o6) foo(x) := 2 x
(%i7) e2: '(bar(y) ::= 3*y);
(%o7) bar(y) ::= 3 y
(%i8) e3: '(x : y);
(%o8) x : y
(%i9) e4: '(x :: y);
(%o9) x :: y
(%i10) [lhs (e1), lhs (e2), lhs (e3), lhs (e4)];
(%o10) [foo(x), bar(y), x, x]
(%i11) infix ("["");
(%o11) ][
(%i12) lhs (a ][ b);
(%o12) a
```

**linsolve ([expr_1, ..., expr_m], [x_1, ..., x_n])**

Solves the list of simultaneous linear equations for the list of variables. The expressions must each be polynomials in the variables and may be equations. If the length of the list of variables doesn’t match the number of linearly-independent equations to solve the result will be an empty list.

When `globalsolve` is `true`, each solved-for variable is bound to its value in the solution of the equations.

When `backsubst` is `false`, `linsolve` does not carry out back substitution after the equations have been triangularized. This may be necessary in very big problems where back substitution would cause the generation of extremely large expressions.
When \texttt{linsolve\_params} is \texttt{true}, \texttt{linsolve} also generates the \%r symbols used to represent arbitrary parameters described in the manual under \texttt{algsys}. Otherwise, \texttt{linsolve} solves an under-determined system of equations with some variables expressed in terms of others.

When \texttt{programmode} is \texttt{false}, \texttt{linsolve} displays the solution with intermediate expression (\%t) labels, and returns the list of labels.

See also \texttt{algsys}, \texttt{eliminate}. and \texttt{solve}.

Examples:

```
(%i1) e1: x + z = y;
(%o1) z + x = y
(%i2) e2: 2*a*x - y = 2*a^2;
     2
(%o2) 2 a x - y = 2 a
(%i3) e3: y - 2*z = 2;
(%o3) y - 2 z = 2
(%i4) [globalsolve: false, programmode: true];
(%o4) [false, true]
(%i5) linsolve ([e1, e2, e3], [x, y, z]);
(%i6) [globalsolve: false, programmode: false];
(%i7) linsolve ([e1, e2, e3], [x, y, z]);
(%o7) \%t7, \%t8, \%t9
(%i8) ''%;
(%o8) [z = a - 1, y = 2 a, x = a + 1]
(%i9) [globalsolve: true, programmode: false];
(%o9) [true, false]
(%i10) linsolve ([e1, e2, e3], [x, y, z]);
(%i11) ''%;
(%o11) [z : a - 1, y : 2 a, x : a + 1]
```

Solution

```
(%o12) z = a - 1

(%o13) y = 2 a

(%o14) x = a + 1
```

```
(%o15) [%t7, %t8, %t9]

(%o16) [%t11, %t12, %t13]
```

```
(%i17) [x, y, z];
```
(%i14) [a + 1, 2 a, a - 1]
(%i15) [globalsolve: true, programmode: true];
(%o15) [true, true]
(%i16) linsolve ([e1, e2, e3], '[x, y, z]);
(%o16) [x : a + 1, y : 2 a, z : a - 1]
(%i17) [x, y, z];
(%o17) [a + 1, 2 a, a - 1]

linsolvewarn [Option variable]
Default value: true

When linsolvewarn is true, linsolve prints a message "Dependent equations eliminated".

linsolve_params [Option variable]
Default value: true

When linsolve_params is true, linsolve also generates the %r symbols used to represent arbitrary parameters described in the manual under algsys. Otherwise, linsolve solves an under-determined system of equations with some variables expressed in terms of others.

multiplicities [System variable]
Default value: not_set_yet

multiplicities is set to a list of the multiplicities of the individual solutions returned by solve or realroots.

nroots (p, low, high) [Function]
Returns the number of real roots of the real univariate polynomial p in the half-open interval (low, high]. The endpoints of the interval may be minf or inf.

nroots uses the method of Sturm sequences.

(%i11) p: x^10 - 2*x^4 + 1/2$
(%i12) nroots (p, -6, 9.1);
(%o12) 4

nthroot (p, n) [Function]
where p is a polynomial with integer coefficients and n is a positive integer returns q, a polynomial over the integers, such that q^n = p or prints an error message indicating that p is not a perfect nth power. This routine is much faster than factor or even sqfr.

polyfactor [Option variable]
Default value: false

The option variable polyfactor when true causes allroots and bfallroots to factor the polynomial over the real numbers if the polynomial is real, or over the complex numbers, if the polynomial is complex.

See allroots for an example.
**progammode**  
[Option variable]  
Default value: true  
When programmode is true, solve, realroots, allroots, and linsolve return solutions as elements in a list. (Except when backsubst is set to false, in which case programmode: false is assumed.)  
When programmode is false, solve, etc. create intermediate expression labels %t1, %t2, etc., and assign the solutions to them.  

**realonly**  
[Option variable]  
Default value: false  
When realonly is true, algsys returns only those solutions which are free of %i.  

**realroots**  
[Function]  
realroots (expr, bound)  
realroots (eqn, bound)  
realroots (expr)  
realroots (eqn)  
Computes rational approximations of the real roots of the polynomial expr or polynomial equation eqn of one variable, to within a tolerance of bound. Coefficients of expr or eqn must be literal numbers; symbol constants such as %pi are rejected.  
realroots assigns the multiplicities of the roots it finds to the global variable multiplicities.  
realroots constructs a Sturm sequence to bracket each root, and then applies bisection to refine the approximations. All coefficients are converted to rational equivalents before searching for roots, and computations are carried out by exact rational arithmetic. Even if some coefficients are floating-point numbers, the results are rational (unless coerced to floats by the float or numer flags).  
When bound is less than 1, all integer roots are found exactly. When bound is unspecified, it is assumed equal to the global variable rootsepsilon.  
When the global variable programmode is true, realroots returns a list of the form [x = x_1, x = x_2, ...]. When programmode is false, realroots creates intermediate expression labels %t1, %t2, ..., assigns the results to them, and returns the list of labels.  
See also allroots, bfallroots, guess_exact_value, and lhs.  
Examples:  
(%i1) realroots (-1 - x + x^5, 5e-6);  
612003  
(%o1) [x = -------]  
524288  
(%i2) ev (%[1], float);  
(%o2) x = 1.167303085327148  
(%i3) ev (-1 - x + x^5, %);  
(%o3) - 7.396496210176905E-6  
(%i1) realroots (expand ((1 - x)^5 * (2 - x)^3 * (3 - x)), 1e-20);  
(%o1) [x = 1, x = 2, x = 3]
(\%i2) multiplicities;                               [5, 3, 1]

rhs (expr)                                      [Function]
Returns the right-hand side (that is, the second argument) of the expression expr, when the operator of expr is one of the relational operators \(<\leq\ =\ #\ \texttt{equal}\ \texttt{notequal}\ >\), one of the assignment operators \(\:=\ :=\ :=\ :=\ :=\), or a user-defined binary infix operator, as declared by \texttt{infix}.

When expr is an atom or its operator is something other than the ones listed above, rhs returns 0.

See also \texttt{lhs}.

Examples:

(\%i1) e: aa + bb = cc;                     \%o1
(\%i2) lhs (e);                            \%o2
(\%i3) rhs (e);                            \%o3
(\%i4) \[rhs (aa < bb), rhs (aa <= bb), rhs (aa >= bb), rhs (aa > bb)]\; \%o4
(\%i5) \[rhs (aa = bb), rhs (aa # bb), rhs (\texttt{equal} (aa, bb)), rhs (\texttt{notequal} (aa, bb))\]; \%o5
(\%i6) e1: \'(foo(x) := 2*x);               \%o6
(\%i7) e2: \'(bar(y) ::= 3*y);               \%o7
(\%i8) e3: \'(x : y);                       \%o8
(\%i9) e4: \'(x :: y);                      \%o9
(\%i10) \[rhs (e1), rhs (e2), rhs (e3), rhs (e4)]; \%o10
(\%i11) infix (\"\[\");                    \%o11
(\%i12) rhs (aa )[ bb);                    \%o12

rootsconmode                                      [Option variable]
Default value: \texttt{true}

\texttt{rootsconmode} governs the behavior of the \texttt{rootscontract} command. See \texttt{rootscontract} for details.

rootscontract (expr)                              [Function]
Converts products of roots into roots of products. For example, \texttt{rootscontract (sqrt(x)*y^(3/2))} yields \texttt{sqrt(x*y^3)}.
When `radexpand` is `true` and `domain` is `real`, `rootscontract` converts `abs` into `sqrt`, e.g., `rootscontract (abs(x)*sqrt(y))` yields `sqrt(x^2*y)`.

There is an option `rootsconmode` affecting `rootscontract` as follows:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Value of <code>rootsconmode</code></th>
<th>Result of applying <code>rootscontract</code></th>
</tr>
</thead>
<tbody>
<tr>
<td><code>x^(1/2)*y^(3/2)</code></td>
<td>false</td>
<td><code>(x*y^3)^(1/2)</code></td>
</tr>
<tr>
<td><code>x^(1/2)*y^(1/4)</code></td>
<td>false</td>
<td><code>x^(1/2)*y^(1/4)</code></td>
</tr>
<tr>
<td><code>x^(1/2)*y^(1/4)</code></td>
<td>true</td>
<td><code>(x*y^(1/2))^(1/2)</code></td>
</tr>
<tr>
<td><code>x^(1/2)*y^(1/3)</code></td>
<td>true</td>
<td><code>x^(1/2)*y^(1/3)</code></td>
</tr>
<tr>
<td><code>x^(1/2)*y^(1/4)</code></td>
<td>all</td>
<td><code>(x^2*y)^(1/4)</code></td>
</tr>
<tr>
<td><code>x^(1/2)*y^(1/3)</code></td>
<td>all</td>
<td><code>(x^3*y^2)^(1/6)</code></td>
</tr>
</tbody>
</table>

When `rootsconmode` is `false`, `rootscontract` contracts only with respect to rational number exponents whose denominators are the same. The key to the `rootsconmode`: `true` examples is simply that 2 divides into 4 but not into 3. `rootsconmode`: `all` involves taking the least common multiple of the denominators of the exponents.

`rootscontract` uses `ratsimp` in a manner similar to `logcontract`.

Examples:

```math
(%i1) rootsconmode: false$
(%i2) rootscontract (x^(1/2)*y^(3/2));
   
3
(%o2) sqrt(x y )
   
(%i3) rootscontract (x^(1/2)*y^(1/4));
   
1/4
(%o3) sqrt(x) y
   
(%i4) rootsconmode: true$
(%i5) rootscontract (x^(1/2)*y^(1/4));
   
(%o5) sqrt(x sqrt(y))
   
(%i6) rootscontract (x^(1/2)*y^(1/3));
   
1/3
(%o6) sqrt(x) y
   
(%i7) rootsconmode: all$
(%i8) rootscontract (x^(1/2)*y^(1/4));
   
2 1/4
(%o8) (x y)
   
(%i9) rootscontract (x^(1/2)*y^(1/3));
   
3 2 1/6
(%o9) (x y )
   
(%i10) rootsconmode: false$
(%i11) rootscontract (sqrt(sqrt(x) + sqrt(1 + x))
   *
sqrt(sqrt(1 + x) - sqrt(x)));
   
1
(%o11)
   
(%i12) rootsconmode: true$
(%i13) rootscontract (sqrt(5+sqrt(5)) - 5^(1/4)*sqrt(1+sqrt(5)));
   
0
(%o13)
```
rootsepsilon

Default value: 1.0e-7

rootsepsilon is the tolerance which establishes the confidence interval for the roots found by the realroots function.

solve

solve (expr, x)
solve (expr)
solve ([eqn_1, ..., eqn_n], [x_1, ..., x_n])

Solves the algebraic equation expr for the variable x and returns a list of solution equations in x. If expr is not an equation, the equation expr = 0 is assumed in its place. x may be a function (e.g. f(x)), or other non-atomic expression except a sum or product. x may be omitted if expr contains only one variable. expr may be a rational expression, and may contain trigonometric functions, exponentials, etc.

The following method is used:

Let E be the expression and X be the variable. If E is linear in X then it is trivially solved for X. Otherwise if E is of the form A*X^N + B then the result is (-B/A)^1/N times the N'th roots of unity.

If E is not linear in X then the gcd of the exponents of X in E (say N) is divided into the exponents and the multiplicity of the roots is multiplied by N. Then solve is called again on the result. If E factors then solve is called on each of the factors. Finally solve will use the quadratic, cubic, or quartic formulas where necessary.

In the case where E is a polynomial in some function of the variable to be solved for, say F(X), then it is first solved for F(X) (call the result C), then the equation F(X)=C can be solved for X provided the inverse of the function F is known.

breakup if false will cause solve to express the solutions of cubic or quartic equations as single expressions rather than as made up of several common subexpressions which is the default.

multiplicities - will be set to a list of the multiplicities of the individual solutions returned by solve, realroots, or allroots. Try apropos (solve) for the switches which affect solve. describe may then by used on the individual switch names if their purpose is not clear.

solve ([eqn_1, ..., eqn_n], [x_1, ..., x_n]) solves a system of simultaneous (linear or non-linear) polynomial equations by calling linsolve or algsys and returns a list of the solution lists in the variables. In the case of linsolve this list would contain a single list of solutions. It takes two lists as arguments. The first list represents the equations to be solved; the second list is a list of the unknowns to be determined. If the total number of variables in the equations is equal to the number of equations, the second argument-list may be omitted.

When programmode is false, solve displays solutions with intermediate expression (%t) labels, and returns the list of labels.

When globalsolve is true and the problem is to solve two or more linear equations, each solved-for variable is bound to its value in the solution of the equations.

Examples:

(%i1) solve (asin (cos (3*x))*(f(x) - 1), x);
solve: using arc-trig functions to get a solution.
Some solutions will be lost.

(%i1) \(x = \frac{\pi}{6}, f(x) = 1\)

(%i2) \(\text{ev (solve (5^f(x) = 125, f(x)), solveradcan)}\);

(%o2) \(f(x) = \frac{\log(125)}{\log(5)}\)

(%i3) \([4x^2 - y^2 = 12, xy - x = 2];

(%o3) \([4 x - y = 12, x y - x = 2]\)

(%i4) \(\text{solve (\%, \{x, y\});}\)

(%o4) \([x = 2, y = 2], [x = 0.5202594388652008 \text{ i} - 0.1331240357358706, y = 0.07678378523787788 + 3.608003221870287 \text{ i}], [x = -0.5202594388652008 \text{ i} - 0.1331240357358706, y = 3.608003221870287 \text{ i} + 0.07678378523787788], [x = -1.733751846381093, y = -0.1535675710019696]]\)

(%i5) \(\text{solve (1 + ax + x^3, x)};\)

(%o5) \(\frac{\sqrt{3} \text{ i}}{2} \left( - \frac{\sqrt{4 a + 27}}{6 \sqrt{3}} \frac{1}{2} \right) \left( - \frac{\sqrt{3} \text{ i}}{2} \frac{1}{2} \right)\)

\(- \frac{\sqrt{3} \text{ i}}{2} \frac{1}{2} \frac{1}{2} a\)

\(- \frac{\sqrt{3} \text{ i}}{2} \frac{1}{2} \)
\[
\frac{3}{\sqrt{4a + 27}} 1 \frac{1}{3} \left(\frac{2}{6\sqrt{3}} - \frac{a}{2}\right)
\]

\[
\frac{3}{\sqrt{4a + 27}} 1 \frac{1}{3} \left(\frac{2}{6\sqrt{3}} - \frac{a}{2}\right) - \frac{3}{\sqrt{4a + 27}} 1 \frac{1}{3}
\]

\[
\text{(iii) solve } (x^3 - 1);
\]

\[
\text{solve: dependent equations eliminated: (2)}
\]

\[
\text{solve: dependent equations eliminated: (2)}
\]

The symbols \%r are used to denote arbitrary constants in a solution.

See \texttt{algsys} and \texttt{\%rnum_list} for more information.
When solvedecomposes is true, solve calls polydecomp if asked to solve polynomials.

solveexplicit  [Option variable]
  Default value: false
  When solveexplicit is true, inhibits solve from returning implicit solutions, that is, solutions of the form \( F(x) = 0 \) where \( F \) is some function.

solvefactors  [Option variable]
  Default value: true
  When solvefactors is false, solve does not try to factor the expression. The false setting may be desired in some cases where factoring is not necessary.

solvenullwarn  [Option variable]
  Default value: true
  When solvenullwarn is true, solve prints a warning message if called with either a null equation list or a null variable list. For example, solve ([], []) would print two warning messages and return [].

solveradcan  [Option variable]
  Default value: false
  When solveradcan is true, solve calls radcan which makes solve slower but will allow certain problems containing exponentials and logarithms to be solved.

solvetrigwarn  [Option variable]
  Default value: true
  When solvetrigwarn is true, solve may print a message saying that it is using inverse trigonometric functions to solve the equation, and thereby losing solutions.
21 Differential Equations

21.1 Introduction to Differential Equations

This section describes the functions available in Maxima to obtain analytic solutions for some specific types of first and second-order equations. To obtain a numerical solution for a system of differential equations, see the additional package dynamics. For graphical representations in phase space, see the additional package plotdf.

21.2 Functions and Variables for Differential Equations

bc2 \((\text{solution}, \text{xval1}, \text{yval1}, \text{xval2}, \text{yval2})\)  
Solves a boundary value problem for a second order differential equation. Here: \(\text{solution}\) is a general solution to the equation, as found by \text{ode2}; \text{xval1} specifies the value of the independent variable in a first point, in the form \(x = x_1\), and \text{yval1} gives the value of the dependent variable in that point, in the form \(y = y_1\). The expressions \text{xval2} and \text{yval2} give the values for these variables at a second point, using the same form.

See \text{ode2} for an example of its usage.

desolve \((\text{eqn}, \text{y})\)  
The function \text{desolve} solves systems of linear ordinary differential equations using Laplace transform. Here the \text{eqn}'s are differential equations in the dependent variables \(y_1, ..., y_n\). The functional dependence of \(y_1, ..., y_n\) on an independent variable, for instance \(x\), must be explicitly indicated in the variables and its derivatives. For example, the correct way to define the differential equations would be:

\[
\begin{align*}
\text{eqn}_1: \quad & \text{'diff(f(x),x,2)} = \sin(x) + \text{'diff(g(x),x)}; \\
\text{eqn}_2: \quad & \text{'diff(f(x),x) + x}^2 - \text{f(x)} = 2*\text{'diff(g(x),x,2)};
\end{align*}
\]

The call to the function \text{desolve} would then be:

\[
\text{desolve([eqn}_1, \text{ eqn}_2], \text{ [f(x),g(x)]});
\]

If initial conditions at \(x=0\) are known, they can be supplied before calling \text{desolve} by using \text{atvalue}.

\[
\begin{align*}
(\%i1) & \quad \text{'diff(f(x),x_='diff(g(x),x)+sin(x);} \\
(\%o1) & \quad \text{d} \quad \text{d} \\
(\%i2) & \quad \text{-- (f(x)) = -- (g(x)) + sin(x)} \\
& \quad \text{dx} \quad \text{dx} \\
(\%i3) & \quad \text{'diff(g(x),x,2)='diff(f(x),x)-cos(x);} \\
& \quad \text{2} \quad \text{d} \quad \text{d} \\
& \quad \text{dx} \\
(\%o2) & \quad \text{-- (g(x)) = -- (f(x)) - cos(x)} \\
& \quad \text{2} \quad \text{dx} \\
(\%i3) & \quad \text{atvalue('diff(g(x),x),x=0,a);} \\
\end{align*}
\]
(%o3) a
(%i4) atvalue(f(x),x=0,1);
(%o4) 1
(%i5) desolve([%o1,%o2],[f(x),g(x)]);
(%o5) [f(x) = a %e - a + 1, g(x) =
         x
         cos(x) + a %e - a + g(0) - 1]
(%i6) [%o1,%o2],%o5,diff;
(%o6) [a %e = a %e , a %e - cos(x) = a %e - cos(x)]

If `desolve` cannot obtain a solution, it returns `false`.
See also `ode2`, `drawdf` and `rk`.

`ic1 (solution, xval, yval)`  
Solves initial value problems for first order differential equations. Here `solution` is a general solution to the equation, as found by `ode2`, `xval` gives an initial value for the independent variable in the form `x = x0`, and `yval` gives the initial value for the dependent variable in the form `y = y0`.
See `ode2` for an example of its usage.

`ic2 (solution, xval, yval, dval)`  
Solves initial value problems for second-order differential equations. Here `solution` is a general solution to the equation, as found by `ode2`, `xval` gives the initial value for the independent variable in the form `x = x0`, `yval` gives the initial value of the dependent variable in the form `y = y0`, and `dval` gives the initial value for the first derivative of the dependent variable with respect to independent variable, in the form `diff(y,x) = dy0 (diff does not have to be quoted)`.
See `ode2` for an example of its usage.

`ode2 (eqn, dvar, ivar)`  
The function `ode2` solves an ordinary differential equation (ODE) of first or second order. It takes three arguments: an ODE given by `eqn`, the dependent variable `dvar`, and the independent variable `ivar`. When successful, it returns either an explicit or implicit solution for the dependent variable. `%c` is used to represent the integration constant in the case of first-order equations, and `%k1` and `%k2` the constants for second-order equations. The dependence of the dependent variable on the independent variable does not have to be written explicitly, as in the case of `desolve`, but the independent variable must always be given as the third argument.
If `ode2` cannot obtain a solution for whatever reason, it returns `false`, after perhaps printing out an error message. The methods implemented for first order equations in the order in which they are tested are: linear, separable, exact - perhaps requiring an integrating factor, homogeneous, Bernoulli’s equation, and a generalized homogeneous method. The types of second-order equations which can be solved are: constant coefficients, exact, linear homogeneous with non-constant coefficients which can be
transformed to constant coefficients, the Euler or equi-dimensional equation, equations solvable by the method of variation of parameters, and equations which are free of either the independent or of the dependent variable so that they can be reduced to two first order linear equations to be solved sequentially.

In the course of solving ODE’s, several variables are set purely for informational purposes: method denotes the method of solution used (e.g., linear), intfactor denotes any integrating factor used, odeindex denotes the index for Bernoulli’s method or for the generalized homogeneous method, and yp denotes the particular solution for the variation of parameters technique.

In order to solve initial value problems (IVP) functions ic1 and ic2 are available for first and second order equations, and to solve second-order boundary value problems (BVP) the function bc2 can be used.

See also desolve, drawdf and rk.

Example:

\[
\begin{align*}
(\%i1) & \quad x^2 \cdot \text{diff}(y,x) + 3y \cdot x = \text{sin}(x)/x; \\
(\%o1) & \quad \text{x \quad dy \quad sin(x)} \\
(\%i2) & \quad \text{ode2(\%,y,x);} \\
(\%o2) & \quad \text{\%c - cos(x)} \\
(\%i3) & \quad \text{ic1(\%o2,x=\%pi,y=0);} \\
(\%o3) & \quad \text{cos(x) + 1} \\
(\%i4) & \quad \text{'diff(y,x,2) + y*\text{diff}(y,x)^3 = 0;} \\
(\%o4) & \quad \text{\text{2 \quad dy \quad 3}} \\
(\%i5) & \quad \text{ode2(\%,y,x);} \\
(\%o5) & \quad \text{\%k1 \quad y} \\
(\%i6) & \quad \text{ratsimp(ic2(\%o5,x=0,y=0,'\text{diff}(y,x)=2));} \\
(\%o6) & \quad \text{\text{2 \quad y \quad 3 \quad y}} \\
(\%i7) & \quad \text{bc2(\%o5,x=0,y=1,x=1,y=3);} \\
(\%o7) & \quad \text{\text{6 \quad y \quad 3 \quad y}}
\end{align*}
\]
\[ \frac{y - 10}{6} y^3 = \frac{x - \cdot}{2} \]

**method**  
The variable `method` is set by `ode2` to the successful solution method.

**%c**  
`%c` is the integration constant in the solutions of first order ODEs returned from `ode2`.

**%k1**  
`%k1` is the first integration constant in the solutions of second order ODEs returned from `ode2`.

**%k2**  
`%k2` is the second integration constant in the solutions of second order ODEs returned from `ode2`.

**yp**  
`yp` is the particular solution of an ODE found by `ode2` when using the variation of parameters technique.
22 Numerical

22.1 Introduction to fast Fourier transform

The \texttt{fft} package comprises functions for the numerical (not symbolic) computation of the fast Fourier transform. This is limited to sequences with length that is a power of two. For more general lengths, consider the \texttt{fftpack5} package that supports sequences of any length, but is most efficient if the length is a product of small primes.

22.2 Functions and Variables for \texttt{fft}

\begin{itemize}
\item \texttt{polartorect (r, t)} \hspace{1cm} [Function]
  \begin{itemize}
  \item Translates complex values of the form $r \, \text{e}^{-\text{i}t}$ to the form $a + b \, \text{i}$, where $r$ is the magnitude and $t$ is the phase. $r$ and $t$ are 1-dimensional arrays of the same size. The array size need not be a power of 2.
  \item The original values of the input arrays are replaced by the real and imaginary parts, $a$ and $b$, on return. The outputs are calculated as
    \begin{align*}
    a &= r \cos(t) \\
    b &= r \sin(t)
    \end{align*}
  \end{itemize}
  \texttt{polartorect} is the inverse function of \texttt{recttopolar}.
  \texttt{load("fft")} loads this function. See also \texttt{fft}.
\item \texttt{recttopolar (a, b)} \hspace{1cm} [Function]
  \begin{itemize}
  \item Translates complex values of the form $a + b \, \text{i}$ to the form $r \, \text{e}^{-\text{i}t}$, where $a$ is the real part and $b$ is the imaginary part. $a$ and $b$ are 1-dimensional arrays of the same size. The array size need not be a power of 2.
  \item The original values of the input arrays are replaced by the magnitude and angle, $r$ and $t$, on return. The outputs are calculated as
    \begin{align*}
    r &= \sqrt{a^2 + b^2} \\
    t &= \text{atan2}(b, a)
    \end{align*}
  \end{itemize}
  \texttt{recttopolar} is the inverse function of \texttt{polartorect}.
  \texttt{load("fft")} loads this function. See also \texttt{fft}.
\item \texttt{inverse_fft (y)} \hspace{1cm} [Function]
  \begin{itemize}
  \item Computes the inverse complex fast Fourier transform. \texttt{y} is a list or array (named or unnamed) which contains the data to transform. The number of elements must be a power of 2. The elements must be literal numbers (integers, rationals, floats, or bigfloats) or symbolic constants, or expressions $a + b \times \text{i}$ where $a$ and $b$ are literal numbers or symbolic constants.
  \item \texttt{inverse_fft} returns a new object of the same type as \texttt{y}, which is not modified.
  \item Results are always computed as floats or expressions $a + b \times \text{i}$ where $a$ and $b$ are floats.
  \item If bigfloat precision is needed the function \texttt{bf_inverse_fft} can be used instead as a drop-in replacement of \texttt{inverse_fft} that is slower, but supports bfloats.
  \end{itemize}
\end{itemize}
The inverse discrete Fourier transform is defined as follows. Let \( x \) be the output of the inverse transform. Then for \( j \) from 0 through \( n - 1 \),

\[
x[j] = \sum y[k] \exp(-2 \%i \pi j k / n), \quad k, 0, n - 1
\]

As there are various sign and normalization conventions possible, this definition of the transform may differ from that used by other mathematical software.

\texttt{load("fft")} loads this function.

See also \texttt{fft} (forward transform), \texttt{recttopolar}, and \texttt{polartorect}.

Examples:

Real data.

\begin{verbatim}
(%i1) load ("fft")$
(%i2) fpprintprec : 4$
(%i3) L : [1, 2, 3, 4, -1, -2, -3, -4]$
(%i4) L1 : inverse_fft (L);
(%o4) [0.0, 14.49 %i - .8284, 0.0, 2.485 %i + 4.828, 0.0, 4.828 - 2.485 %i, 0.0, -14.49 %i - .8284]
(%i5) L2 : fft (L1);
(%o5) [1.0, 2.0 - 2.168L-19 %i, 3.0 - 7.525L-20 %i, 4.0 - 4.256L-19 %i, - 1.0, 2.168L-19 %i - 2.0, 7.525L-20 %i - 3.0, 4.256L-19 %i - 4.0]
(%i6) lmax (abs (L2 - L));
(%o6) 3.545L-16
\end{verbatim}

Complex data.

\begin{verbatim}
(%i1) load ("fft")$
(%i2) fpprintprec : 4$
(%i3) L : [1, 1 + %i, 1 - %i, -1, -1, 1 - %i, 1 + %i, 1]$
(%i4) L1 : inverse_fft (L);
(%o4) [4.0, 2.711L-19 %i + 4.0, 2.0 %i - 2.0, - 2.828 %i - 2.828, 0.0, 5.421L-20 %i + 4.0, - 2.0 %i - 2.0, 2.828 %i + 2.828]
(%i5) L2 : fft (L1);
(%o5) [4.066E-20 %i + 1.0, 1.0 %i + 1.0, 1.0 - 1.0 %i, 1.55L-19 %i - 1.0, - 4.066E-20 %i - 1.0, 1.0 - 1.0 %i, 1.0 %i + 1.0, 1.0 - 7.368L-20 %i]
(%i6) lmax (abs (L2 - L));
(%o6) 6.841L-17
\end{verbatim}

\texttt{fft (x)} \textbf{[Function]}

Computes the complex fast Fourier transform. \( x \) is a list or array (named or unnamed) which contains the data to transform. The number of elements must be a power of 2. The elements must be literal numbers (integers, rationals, floats, or bigfloats) or symbolic constants, or expressions \( a + b*\%i \) where \( a \) and \( b \) are literal numbers or symbolic constants.

\texttt{fft} returns a new object of the same type as \( x \), which is not modified. Results are always computed as floats or expressions \( a + b*\%i \) where \( a \) and \( b \) are floats. If bigfloat precision is needed the function \texttt{bf_fft} can be used instead as a drop-in replacement.
of \texttt{fft} that is slower, but supports bfloats. In addition if it is known that the input consists of only real values (no imaginary parts), \texttt{real_fft} can be used which is potentially faster.

The discrete Fourier transform is defined as follows. Let $y$ be the output of the transform. Then for $k$ from 0 through $n - 1$,

$$y[k] = \frac{1}{n} \sum_{j=0}^{n-1} x[j] \exp(\pm 2 \pi i j k / n),$$

As there are various sign and normalization conventions possible, this definition of the transform may differ from that used by other mathematical software.

When the data $x$ are real, real coefficients $a$ and $b$ can be computed such that

$$x[j] = \sum a[k] \cos(2\pi j k / n) + b[k] \sin(2\pi j k / n),$$

with

$$a[0] = \text{realpart}(y[0]),$$
$$b[0] = 0$$

and, for $k$ from 1 through $n/2 - 1$,

$$a[k] = \text{realpart}(y[k] + y[n - k]),$$
$$b[k] = \text{imagpart}(y[n - k] - y[k])$$

and

$$a[n/2] = \text{realpart}(y[n/2]),$$
$$b[n/2] = 0$$

\texttt{load("fft")} loads this function.

See also \texttt{inverse_fft} (inverse transform), \texttt{recttopolar}, and \texttt{polartorect}.. See \texttt{real_fft} for FFTs of a real-valued input, and \texttt{bf_fft} and \texttt{bf_real_fft} for operations on bigfloat values. Finally, for transforms of any size (but limited to float values), see \texttt{fftpack5_fft} and \texttt{fftpack5_real_fft}.

Examples:

**Real data.**

\begin{verbatim}
(%i1) load ("fft") $
(%i2) fpprintprec : 4 $ 
(%i3) L : [1, 2, 3, 4, -1, -2, -3, -4] $ 
(%i4) L1 : fft (L); 
(%o4) [0.0, 1.811 %i - .1036, 0.0, 0.3107 %i + .6036, 0.0, 
    0.6036 - 0.3107 %i, 0.0, (- 1.811 %i) - 0.1036] 
(%i5) L2 : inverse_fft (L1); 
(%o5) [1.0, 2.168L-19 %i + 2.0, 7.525L-20 %i + 3.0, 
    4.256L-19 %i + 4.0, - 1.0, - 2.168L-19 %i - 2.0, 
    - 7.525L-20 %i - 3.0, - 4.256L-19 %i - 4.0] 
(%i6) lmax (abs (L2 - L)); 
(%o6) 3.545L-16
\end{verbatim}

**Complex data.**

\begin{verbatim}
(%i1) load ("fft") $ 
(%i2) fpprintprec : 4 $ 
(%i3) L : [1, 1 + %i, 1 - %i, -1, -1, 1 - %i, 1 + %i, 1] $ 
(%i4) L1 : fft (L); 
\end{verbatim}
Computation of sine and cosine coefficients.

(%i1) load ("fft")$
(%i2) fpprintprec : 4$
(%i3) L : [1, 2, 3, 4, 5, 6, 7, 8]$
(%i4) n : length (L)$
(%i5) x : make_array (any, n)$
(%i6) fillarray (x, L)$
(%i7) y : fft (x)$
(%i8) a : make_array (any, n/2 + 1)$
(%i9) b : make_array (any, n/2 + 1)$
(%i10) a[0] : realpart (y[0])$
(%i11) b[0] : 0$
(%i12) for k : 1 thru n/2 - 1 do$
(a[k] : realpart (y[k] + y[n - k]),$
   b[k] : imagpart (y[n - k] - y[k]));
(%o12) done
(%i13) a[n/2] : y[n/2]$
(%i14) b[n/2] : 0$
(%i15) listarray (a);$
(%o15) [4.5, - 1.0, - 1.0, - 1.0, - 0.5]
(%i16) listarray (b);$
(%o16) [0, - 2.414, - 1.0, - .4142, 0]
(%i17) f(j) := sum (a[k]*cos(2*%pi*j*k/n) + b[k]*sin(2*%pi*j*k/n),$
   k, 0, n/2)$
(%i18) makelist (float (f (j)), j, 0, n - 1);$
(%o18) [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0]

real_fft (x)
[Function]
Computes the fast Fourier transform of a real-valued sequence x. This is equivalent
to performing fft(x), except that only the first N/2+1 results are returned, where N
is the length of x. N must be power of two.

No check is made that x contains only real values.

The symmetry properties of the Fourier transform of real sequences to reduce he complexity. In particular the first and last output values of real_fft are purely real.
For larger sequences, real_fft may be computed more quickly than fft.

Since the output length is short, the normal inverse_fft cannot be directly used. Use inverse_real_fft to compute the inverse.
inverse_real_fft (y)          [Function]
Computes the inverse Fourier transform of y, which must have a length of \(N/2+1\)
where \(N\) is a power of two. That is, the input \(x\) is expected to be the output of
real_fft.
No check is made to ensure that the input has the correct format. (The first and last
elements must be purely real.)

bf_inverse_fft (y)             [Function]
Computes the inverse complex fast Fourier transform. This is the bigfloat version of
inverse_fft that converts the input to bigfloats and returns a bigfloat result.

bf_fft (y)                     [Function]
Computes the forward complex fast Fourier transform. This is the bigfloat version of
fft that converts the input to bigfloats and returns a bigfloat result.

bf_real_fft (x)                [Function]
Computes the forward fast Fourier transform of a real-valued input returning a
bigfloat result. This is the bigfloat version of real_fft.

bf_inverse_real_fft (y)        [Function]
Computes the inverse fast Fourier transform with a real-valued bigfloat output. This
is the bigfloat version of inverse_real_fft.

22.3 Functions and Variables for FFTPACK5

FFTPACK5 provides several routines to compute Fourier transforms for both real and complex
sequences and their inverses. The forward transform is defined the same as for fft. The
major difference is the length of the sequence is not constrained to be a power of two. In
fact, any length is supported, but it is most efficient when the length has the form \(2^r \times 3^s \times 5^t\).

load("fftpack5") loads this function.

fftpack5_fft (x)               [Function]
Like fft (fft), this computes the fast Fourier transform of a complex sequence.
However, the length of \(x\) is not limited to a power of 2.
load("fftpack5") loads this function.

Examples:
Real data.

(\%i1) load("fftpack5") $
(\%i2) fpprintprec : 4 $
(\%i3) L : [1, 2, 3, 4, -1, -2, -3, -4] $
(\%i4) L1 : fftpack5_fft(L);
(\%o4) [0.0, 1.811 \%i - 0.1036, 0.0, 0.3107 \%i + 0.6036, 0.0,
       0.6036 - 0.3107 \%i, 0.0, (- 1.811 \%i) - 0.1036]
(\%i5) L2 : fftpack5_inverse_fft(L1);
(\%o5) [1.0, 4.441e-16 \%i + 2.0, 1.837e-16 \%i + 3.0, 4.0 - 4.441e-16 \%i,
       - 1.0, (- 4.441e-16 \%i) - 2.0, (- 1.837e-16 \%i) - 3.0, 4.441e-16
       \%i - 4.0]
(\%i6) lmax (abs (L2-L));
(%o6) 4.441e-16
(%i7) L : [1, 2, 3, 4, 5, 6]
(%i8) L1 : fftpack5_fft(L);
(%o8) [3.5, (- 0.866 %i) - 0.5, (- 0.2887 %i) - 0.5, (- 1.48e-16 %i) - 0.5,
   0.2887 %i - 0.5, 0.866 %i - 0.5]
(%i9) L2 : fftpack5_inverse_fft (L1);
(%o9) [1.0 - 1.48e-16 %i, 3.701e-17 %i + 2.0, 3.0 - 1.48e-16 %i,
   4.0 - 1.811e-16 %i, 5.0 - 1.48e-16 %i, 5.881e-16 %i + 6.0]
(%i10) lmax(abs(L2-L));
(%o10) 9.064e-16

Complex data.
(%i1) load("fftpack5")$
(%i2) fpprintprec : 4$
(%i3) L : [1, 1 + %i, 1 - %i, -1, -1, 1 + %i, 1 + %i, 1]$
(%i4) L1 : fftpack5_inverse_fft (L);
(%o4) [4.0, 2.828 %i + 2.828, (- 2.0 %i) - 2.0, 4.0, 0.0,
   (- 2.828 %i) - 2.828, 2.0 %i - 2.0, 4.0]
(%i5) L2 : fftpack5_fft(L1);
(%o5) [1.0, 1.0 %i + 1.0, 1.0 - 1.0 %i, (- 2.776e-17 %i) - 1.0, - 1.0,
   1.0 - 1.0 %i, 1.0 %i + 1.0, 1.0 - 2.776e-17 %i]
(%i6) lmax(abs(L2-L));
(%o6) 1.11e-16

fftpack5_inverse_fft (y) [Function]
Computes the inverse complex Fourier transform, like inverse_fft, but is not
constrained to be a power of two.

fftpack5_real_fft (x) [Function]
Computes the fast Fourier transform of a real-valued sequence x, just like real_fft,
except the length is not constrained to be a power of two.

Examples:
(%i1) fpprintprec : 4$
(%i2) L : [1, 2, 3, 4, 5, 6]$
(%i3) L1 : fftpack5_real_fft(L);
(%o3) [3.5, (- 0.866 %i) - 0.5, (- 0.2887 %i) - 0.5, - 0.5]
(%i4) L2 : fftpack5_inverse_real_fft(L1, 6);
(%o4) [1.0, 2.0, 3.0, 4.0, 5.0, 6.0]
(%i5) lmax(abs(L2-L));
(%o5) 1.332e-15
(%i6) fftpack5_inverse_real_fft(L1, 7);
(%o6) [0.5, 2.083, 2.562, 3.7, 4.3, 5.438, 5.917]
The last example shows how important it to set the length correctly for fftpack5_inverse_real_fft.
fftpack5_inverse_real_fft (y, n)  
Computes the inverse Fourier transform of y, which must have a length of \( \text{floor}(n/2) + 1 \). The length of sequence produced by the inverse transform must be specified by \( n \). This is required because the length of \( y \) does not uniquely determine \( n \). The last element of \( y \) is always real if \( n \) is even, but it can be complex when \( n \) is odd.

### 22.4 Functions for numerical solution of equations

**horner**

\[
\text{horner}(\text{expr}, x) \\
\text{horner}(\text{expr})
\]

Returns a rearranged representation of \( \text{expr} \) as in Horner’s rule, using \( x \) as the main variable if it is specified. \( x \) may be omitted in which case the main variable of the canonical rational expression form of \( \text{expr} \) is used.

**find_root**

\[
\text{find_root}(\text{expr}, x, a, b, [\text{abserr}, \text{relerr}]) \\
\text{find_root}(f, a, b, [\text{abserr}, \text{relerr}])
\]

Finds a root of the expression \( \text{expr} \) or the function \( f \) over the closed interval \([a, b]\). The expression \( \text{expr} \) may be an equation, in which case \( \text{find_root} \) seeks a root of \( \text{lhs}(\text{expr}) - \text{rhs}(\text{expr}) \).

Given that Maxima can evaluate \( \text{expr} \) or \( f \) over \([a, b]\) and that \( \text{expr} \) or \( f \) is continuous, \( \text{find_root} \) is guaranteed to find the root, or one of the roots if there is more than one.

\( \text{find_root} \) initially applies binary search. If the function in question appears to be smooth enough, \( \text{find_root} \) applies linear interpolation instead.
\texttt{bf\_find\_root} is a bigfloat version of \texttt{find\_root}. The function is computed using bigfloat arithmetic and a bigfloat result is returned. Otherwise, \texttt{bf\_find\_root} is identical to \texttt{find\_root}, and the following description is equally applicable to \texttt{bf\_find\_root}.

The accuracy of \texttt{find\_root} is governed by \texttt{abserr} and \texttt{relerr}, which are optional keyword arguments to \texttt{find\_root}. These keyword arguments take the form \texttt{key=val}. The keyword arguments are

\begin{itemize}
  \item \texttt{abserr} \hspace{1cm} Desired absolute error of function value at root. Default is \texttt{find\_root\_abs}.
  \item \texttt{relerr} \hspace{1cm} Desired relative error of root. Default is \texttt{find\_root\_rel}.
\end{itemize}

\texttt{find\_root} stops when the function in question evaluates to something less than or equal to \texttt{abserr}, or if successive approximants \(x_0\), \(x_1\) differ by no more than \texttt{relerr} \* \texttt{max(abs(x_0), abs(x_1))}. The default values of \texttt{find\_root\_abs} and \texttt{find\_root\_rel} are both zero.

\texttt{find\_root} expects the function in question to have a different sign at the endpoints of the search interval. When the function evaluates to a number at both endpoints and these numbers have the same sign, the behavior of \texttt{find\_root} is governed by \texttt{find\_root\_error}. When \texttt{find\_root\_error} is \texttt{true}, \texttt{find\_root} prints an error message. Otherwise \texttt{find\_root} returns the value of \texttt{find\_root\_error}. The default value of \texttt{find\_root\_error} is \texttt{true}.

If \(f\) evaluates to something other than a number at any step in the search algorithm, \texttt{find\_root} returns a partially-evaluated \texttt{find\_root} expression.

The order of \(a\) and \(b\) is ignored; the region in which a root is sought is \([\min(a, b), \max(a, b)]\).

Examples:

\begin{verbatim}
(\%i1) f(x) := sin(x) - x/2;
\%o1) f(x) := sin(x) - \-
     \hspace{1cm} x
     \hspace{1cm} 2
(\%i2) find_root (sin(x) - x/2, x, 0.1, %pi);
\%o2) 1.895494267033981
(\%i3) find_root (sin(x) = x/2, x, 0.1, %pi);
\%o3) 1.895494267033981
(\%i4) find_root (f(x), x, 0.1, %pi);
\%o4) 1.895494267033981
(\%i5) find_root (f, 0.1, %pi);
\%o5) 1.895494267033981
(\%i6) find_root (exp(x) = y, x, 0, 100);
\%o6) find_root(%e = y, x, 0.0, 100.0)
(\%i7) find_root (exp(x) = y, x, 0, 100), y = 10;
\%o7) 2.302585092994046
(\%i8) log (10.0);
\%o8) 2.302585092994046
\end{verbatim}
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(%i9) fpprec:32;
(%o9) 32
(%i10) bf_find_root (exp(x) = y, x, 0, 100), y = 10;
(%o10) 2.3025850929940456840179914546844b0
(%i11) log(10b0);
(%o11) 2.3025850929940456840179914546844b0

newton (expr, x, x_0, eps) [Function]
Returns an approximate solution of \( expr = 0 \) by Newton’s method, considering \( expr \) to be a function of one variable, \( x \). The search begins with \( x = x_0 \) and proceeds until \( \text{abs}(expr) < \text{eps} \) (with \( expr \) evaluated at the current value of \( x \)).
newton allows undefined variables to appear in \( expr \), so long as the termination test \( \text{abs}(expr) < \text{eps} \) evaluates to \text{true} or \text{false}. Thus it is not necessary that \( expr \) evaluate to a number.
load("newton1") loads this function.
See also realroots, allroots, find_root and mnewton.
Examples:

(%i1) load ("newton1");
(%o1) /maxima/share/numeric/newton1.mac
(%i2) newton (cos (u), u, 1, 1/100);
(%o2) 1.570675277161251
(%i3) ev (cos (u), u = %);
(%o3) 1.2104963335033529e-4
(%i4) assume (a > 0);
(%o4) [a > 0]
(%i5) newton (x^2 - a^2, x, a/2, a^2/100);
(%o5) 1.00030487804878 a
(%i6) ev (x^2 - a^2, x = %);
(%o6) 6.098490481853958e-4 a

22.5 Introduction to numerical solution of differential equations

The Ordinary Differential Equations (ODE) solved by the functions in this section should have the form,
\[
\frac{dy}{dx} = F(x, y)
\]
which is a first-order ODE. Higher order differential equations of order \( n \) must be written as a system of \( n \) first-order equations of that kind. For instance, a second-order ODE should be written as a system of two equations
\[
\frac{dx}{dt} = G(x, y, t) \quad \frac{dy}{dt} = F(x, y, t)
\]
The first argument in the functions will be a list with the expressions on the right-side of the ODE’s. The variables whose derivatives are represented by those expressions should
be given in a second list. In the case above those variables are \( x \) and \( y \). The independent variable, \( t \) in the examples above, might be given in a separated option. If the expressions given do not depend on that independent variable, the system is called autonomous.

### 22.6 Functions for numerical solution of differential equations

**plotdf**

```maxima
plotdf (\textit{dydx}, \textit{options}.)
plotdf (\textit{dvdu}, [\textit{u}, \textit{v}], \textit{options}.)
plotdf ([\textit{dxdt}, \textit{cdydt}], \textit{options}.)
plotdf ([\textit{dudt}, \textit{cdvdt}], [\textit{u}, \textit{cv}], \textit{options}.)
```

The function `plotdf` creates a two-dimensional plot of the direction field (also called slope field) for a first-order Ordinary Differential Equation (ODE) or a system of two autonomous first-order ODE’s.

Plotdf requires Xmaxima, even if its run from a Maxima session in a console, since the plot will be created by the Tk scripts in Xmaxima. If Xmaxima is not installed plotdf will not work.

`dydx`, `dxdt` and `dydt` are expressions that depend on \( x \) and \( y \). `dvdu`, `dudt` and `dvdt` are expressions that depend on \( u \) and \( v \). In addition to those two variables, the expressions can also depend on a set of parameters, with numerical values given with the `parameters` option (the option syntax is given below), or with a range of allowed values specified by a `sliders` option.

Several other options can be given within the command, or selected in the menu. Integral curves can be obtained by clicking on the plot, or with the option `trajectory_at`. The direction of the integration can be controlled with the `direction` option, which can have values of `forward`, `backward` or `both`. The number of integration steps is given by `nsteps`; at each integration step the time increment will be adjusted automatically to produce displacements much smaller than the size of the plot window. The numerical method used is 4th order Runge-Kutta with variable time steps.

**Plot window menu:**

The menu bar of the plot window has the following seven icons:

- An X. Can be used to close the plot window.
- A wrench and a screwdriver. Opens the configuration menu with several fields that show the ODE(s) in use and various other settings. If a pair of coordinates are entered in the field `Trajectory at` and the enter key is pressed, a new integral curve will be shown, in addition to the ones already shown.
- Two arrows following a circle. Replots the direction field with the new settings defined in the configuration menu and replots only the last integral curve that was previously plotted.
- Hard disk drive with an arrow. Used to save a copy of the plot, in Postscript format, in the file specified in a field of the box that appears when that icon is clicked.
- Magnifying glass with a plus sign. Zooms in the plot.
- Magnifying glass with a minus sign. Zooms out the plot. The plot can be displaced by holding down the right mouse button while the mouse is moved.
Icon of a plot. Opens another window with a plot of the two variables in terms of time, for the last integral curve that was plotted.

**Plot options:**

Options can also be given within the `plotdf` itself, each one being a list of two or more elements. The first element in each option is the name of the option, and the remainder is the value or values assigned to the option.

The options which are recognized by `plotdf` are the following:

- `nsteps` defines the number of steps that will be used for the independent variable, to compute an integral curve. The default value is 100.
- `direction` defines the direction of the independent variable that will be followed to compute an integral curve. Possible values are `forward`, to make the independent variable increase `nsteps` times, with increments `tstep`, `backward`, to make the independent variable decrease, or `both` that will lead to an integral curve that extends `nsteps` forward, and `nsteps` backward. The keywords `right` and `left` can be used as synonyms for `forward` and `backward`. The default value is `both`.
- `tinitial` defines the initial value of variable `t` used to compute integral curves. Since the differential equations are autonomous, that setting will only appear in the plot of the curves as functions of `t`. The default value is 0.
- `versus_t` is used to create a second plot window, with a plot of an integral curve, as two functions `x`, `y`, of the independent variable `t`. If `versus_t` is given any value different from 0, the second plot window will be displayed. The second plot window includes another menu, similar to the menu of the main plot window. The default value is 0.
- `trajectory_at` defines the coordinates `xinitial` and `yinitial` for the starting point of an integral curve. The option is empty by default.
- `parameters` defines a list of parameters, and their numerical values, used in the definition of the differential equations. The name and values of the parameters must be given in a string with a comma-separated sequence of pairs `name=value`.
- `sliders` defines a list of parameters that will be changed interactively using slider buttons, and the range of variation of those parameters. The names and ranges of the parameters must be given in a string with a comma-separated sequence of elements `name:min:max`.
- `xfun` defines a string with semi-colon-separated sequence of functions of `x` to be displayed, on top of the direction field. Those functions will be parsed by Tcl and not by Maxima.
- `x` should be followed by two numbers, which will set up the minimum and maximum values shown on the horizontal axis. If the variable on the horizontal axis is not `x`, then this option should have the name of the variable on the horizontal axis. The default horizontal range is from -10 to 10.
- `y` should be followed by two numbers, which will set up the minimum and maximum values shown on the vertical axis. If the variable on the vertical axis is not `y`, then this option should have the name of the variable on the vertical axis. The default vertical range is from -10 to 10.
- `xaxislabel` will be used to identify the horizontal axis. Its default value is the name of the first state variable.

- `yaxislabel` will be used to identify the vertical axis. Its default value is the name of the second state variable.

- `number_of_arrows` should be set to a square number and defines the approximate density of the arrows being drawn. The default value is 225.

Examples:

- To show the direction field of the differential equation \( y' = \exp(-x) + y \) and the solution that goes through (2, -0.1):

  ```maxima
  (%i1) plotdf(exp(-x)+y,[trajectory_at,2,-0.1])$
  ```

- To obtain the direction field for the equation \( \text{diff}(y, x) = x - y^2 \) and the solution with initial condition \( y(-1) = 3 \), we can use the command:

  ```maxima
  (%i1) plotdf(x-y^2,[xfun,"sqrt(x);-sqrt(x)", [trajectory_at,-1,3], [direction,forward], [y,-5,5], [x,-4,16])$
  ```

  The graph also shows the function \( y = \sqrt{x} \).
• The following example shows the direction field of a harmonic oscillator, defined by the two equations \( \frac{dz}{dt} = v \) and \( \frac{dv}{dt} = -k \cdot z/m \), and the integral curve through \((z,v) = (6,0)\), with a slider that will allow you to change the value of \( m \) interactively \((k\) is fixed at 2):

\[
(\%i1) \text{plotdf([v,-k*z/m], [z,v], [parameters,"m=2,k=2"],}
\text{[sliders,"m=1:5"], [trajectory_at,6,0])}$

• To plot the direction field of the Duffing equation, \( m \cdot x'' + c \cdot x' + k \cdot x + b \cdot x^3 = 0 \), we introduce the variable \( y = x' \) and use:

\[
(\%i1) \text{plotdf([y,-(k*x + c*y + b*x^3)/m],}
\text{[parameters,"k=-1,m=1.0,c=0,b=1"],}
\text{[sliders,"k=-2:2,m=-1:1"],[tstep,0.1])}$

• The direction field for a damped pendulum, including the solution for the given initial conditions, with a slider that can be used to change the value of the mass \( m \), and with a plot of the two state variables as a function of time:

\[
\text{(%i1) \quad} \text{plotdf([w,-g*sin(a)/l - b*w/m/l], [a,w],}
\text{ [parameters,}\, \text{"g=9.8,l=0.5,m=0.3,b=0.05"},]
\text{ [trajectory_at,1.05,-9],[tstep,0.01],}
\text{ [a,-10,2], [w,-14,14], [direction,forward],}
\text{ [nsteps,300], [sliders,}\, \text{"m=0.1:1"}, [versus_t,1])}
\]
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ploteq (exp, ...options...)  [Function]
Plots equipotential curves for exp, which should be an expression depending on two variables. The curves are obtained by integrating the differential equation that define the orthogonal trajectories to the solutions of the autonomous system obtained from the gradient of the expression given. The plot can also show the integral curves for that gradient system (option fieldlines).

This program also requires Xmaxima, even if its run from a Maxima session in a console, since the plot will be created by the Tk scripts in Xmaxima. By default, the plot region will be empty until the user clicks in a point (or gives its coordinate with in the set-up menu or via the trajectory.at option).

Most options accepted by plotdf can also be used for ploteq and the plot interface is the same that was described in plotdf.

Example:

(%i1) V: 900/((x+1)^2+y^2)^(1/2)-900/((x-1)^2+y^2)^(1/2)$
(%i2) ploteq(V, [x, -2, 2], [y, -2, 2], [fieldlines, "blue"])$

Clicking on a point will plot the equipotential curve that passes by that point (in red) and the orthogonal trajectory (in blue).

rk  [Function]

rk (ODE, var, initial, domain)
rk ([ODE1, ..., ODEm], [v1, ..., vm], [init1, ..., initm], domain)
The first form solves numerically one first-order ordinary differential equation, and the second form solves a system of m of those equations, using the 4th order Runge-Kutta method. var represents the dependent variable. ODE must be an expression that depends only on the independent and dependent variables and defines the derivative of the dependent variable with respect to the independent variable.

The independent variable is specified with domain, which must be a list of four elements as, for instance:

[t, 0, 10, 0.1]
the first element of the list identifies the independent variable, the second and third elements are the initial and final values for that variable, and the last element sets the increments that should be used within that interval.

If \( m \) equations are going to be solved, there should be \( m \) dependent variables \( v_1, v_2, ..., v_m \). The initial values for those variables will be \( init1, init2, ..., initm \). There will still be just one independent variable defined by \texttt{domain}, as in the previous case. \( ODE1, ..., ODEm \) are the expressions that define the derivatives of each dependent variable in terms of the independent variable. The only variables that may appear in those expressions are the independent variable and any of the dependent variables. It is important to give the derivatives \( ODE1, ..., ODEm \) in the list in exactly the same order used for the dependent variables; for instance, the third element in the list will be interpreted as the derivative of the third dependent variable.

The program will try to integrate the equations from the initial value of the independent variable until its last value, using constant increments. If at some step one of the dependent variables takes an absolute value too large, the integration will be interrupted at that point. The result will be a list with as many elements as the number of iterations made. Each element in the results list is itself another list with \( m+1 \) elements: the value of the independent variable, followed by the values of the dependent variables corresponding to that point.

See also \texttt{drawdf}, \texttt{desolve} and \texttt{ode2}.

Examples:

To solve numerically the differential equation

\[
\frac{dx}{dt} = t - x^2
\]

With initial value \( x(t=0) = 1 \), in the interval of \( t \) from 0 to 8 and with increments of 0.1 for \( t \), use:

\[
(%i1) \text{results: rk(t-x^2,x,1,[t,0,8,0.1])}
\]

\[
(%i2) \text{plot2d ([discrete, results])}
\]

the results will be saved in the list \texttt{results} and the plot will show the solution obtained, with \( t \) on the horizontal axis and \( x \) on the vertical axis.

To solve numerically the system:

\[
\begin{align*}
\frac{dx}{dt} &= 4 - x^2 - 4y^2 \\
\frac{dy}{dt} &= y^2 - x^2 + 1
\end{align*}
\]

for \( t \) between 0 and 4, and with values of -1.25 and 0.75 for \( x \) and \( y \) at \( t=0 \):

\[
(%i1) \text{sol: rk([4-x^2-4*y^2, y^2-x^2+1], [x, y], [-1.25, 0.75], [t, 0, 4, 0.02])}
\]

\[
(%i2) \text{plot2d([discrete, makelist([p[1], p[3]], p, sol]), [xlabel, "t"], [ylabel, "y")}
\]

The plot will show the solution for variable \( y \) as a function of \( t \).
23 Matrices and Linear Algebra

23.1 Introduction to Matrices and Linear Algebra

23.1.1 Dot

The operator . represents noncommutative multiplication and scalar product. When the operands are 1-column or 1-row matrices \( a \) and \( b \), the expression \( a \cdot b \) is equivalent to \( \text{sum} \ (a[i]*b[i], \ i, \ 1, \ \text{length}(a)) \). If \( a \) and \( b \) are not complex, this is the scalar product, also called the inner product or dot product, of \( a \) and \( b \). The scalar product is defined as \( \text{conjugate}(a) \cdot b \) when \( a \) and \( b \) are complex; \text{innerproduct} in the \text{eigen} package provides the complex scalar product.

When the operands are more general matrices, the product is the matrix product \( a \cdot b \). The number of rows of \( b \) must equal the number of columns of \( a \), and the result has number of rows equal to the number of rows of \( a \) and number of columns equal to the number of columns of \( b \).

To distinguish . as an arithmetic operator from the decimal point in a floating point number, it may be necessary to leave spaces on either side. For example, \( 5.e3 \) is 5000.0 but \( 5 . e3 \) is 5 times \( e3 \).

There are several flags which govern the simplification of expressions involving ., namely \text{dot0nscsimp}, \text{dot0simp}, \text{dot1simp}, \text{dotassoc}, \text{dotconstrules}, \text{dotdistrib}, \text{dotexptsimp}, \text{dotident}, and \text{dotscrules}.

23.1.2 Matrices

Matrices are handled with speed and memory-efficiency in mind. This means that assigning a matrix to a variable will create a reference to, not a copy of the matrix. If the matrix is modified all references to the matrix point to the modified object (See \text{copymatrix} for a way of avoiding this):

\begin{verbatim}
(%i1) M1: matrix([0,0],[0,0]);
[ 0 0 ]
(%o1) [ ]
[ 0 0 ]

(%i2) M2: M1;
[ 0 0 ]
(%o2) [ ]
[ 0 0 ]

(%i3) M1[1][1]: 2;
(%o3) 2

(%i4) M2;
[ 2 0 ]
(%o4) [ ]
[ 0 0 ]
\end{verbatim}

Converting a matrix to nested lists and vice versa works the following way:

\begin{verbatim}
(%i1) 1: [[1,2],[3,4]];
(%o1) [[1, 2], [3, 4]]
\end{verbatim}
(%i2) M1: apply('matrix,l);  
[ 1 2 ]
(%o2) [ 
[ 3 4 ]
(%i3) M2: transpose(M1);  
[ 1 3 ]
(%o3) [ 
[ 2 4 ]
(%i4) args(M2);
(%o4) [[[1, 3], [2, 4]]

23.1.3 Vectors

vect is a package of functions for vector analysis. load ("vect") loads this package, and demo ("vect") displays a demonstration.

The vector analysis package can combine and simplify symbolic expressions including dot products and cross products, together with the gradient, divergence, curl, and Laplacian operators. The distribution of these operators over sums or products is governed by several flags, as are various other expansions, including expansion into components in any specific orthogonal coordinate systems. There are also functions for deriving the scalar or vector potential of a field.

The vect package contains these functions: vectorsimp, scalefactors, express, potential, and vectorpotential.

By default the vect package does not declare the dot operator to be a commutative operator. To get a commutative dot operator ., the command declare(".", commutative) must be executed.

23.1.4 eigen

The package eigen contains several functions devoted to the symbolic computation of eigenvalues and eigenvectors. Maxima loads the package automatically if one of the functions eigenvalues or eigenvectors is invoked. The package may be loaded explicitly as load ("eigen").

demo ("eigen") displays a demonstration of the capabilities of this package. batch ("eigen") executes the same demonstration, but without the user prompt between successive computations.

The functions in the eigen package are: innerproduct, unitvector, columnvector, gramschmidt, eigenvalues, eigenvectors, uniteigenvectors, and similaritytransform.

23.2 Functions and Variables for Matrices and Linear Algebra

addcol (M, list_1, ..., list_n)  
[Function]  
Appends the column(s) given by the one or more lists (or matrices) onto the matrix M.

See also addrow and append.
Function

addrow \((M, \text{list}_1, \ldots, \text{list}_n)\)

Appends the row(s) given by the one or more lists (or matrices) onto the matrix \(M\).

See also \textbf{addcol} and \textbf{append}.

Function

\textbf{adjoint} \((M)\)

Returns the adjoint of the matrix \(M\). The adjoint matrix is the transpose of the matrix of cofactors of \(M\).

Function

\textbf{augcoefmatrix} \(([\text{eqn}_1, \ldots, \text{eqn}_m], [x_1, \ldots, x_n])\)

Returns the augmented coefficient matrix for the variables \(x_1, \ldots, x_n\) of the system of linear equations \(\text{eqn}_1, \ldots, \text{eqn}_m\). This is the coefficient matrix with a column adjoined for the constant terms in each equation (i.e., those terms not dependent upon \(x_1, \ldots, x_n\)).

\begin{verbatim}
(%i1) m: [2*x - (a - 1)*y = 5*b, c + b*y + a*x = 0]
(%i2) augcoefmatrix (m, [x, y]);
\end{verbatim}

\begin{verbatim}
[ 2 1 - a - 5 b ]
[ ]
[ a b c ]
\end{verbatim}

Function

\textbf{cauchy_matrix} \(([x_1, x_2, \ldots, x_m], [y_1, y_2, \ldots, y_n])\)

\textbf{cauchy_matrix} \(([x_1, x_2, \ldots, x_n])\)

Returns a \(n\) by \(m\) Cauchy matrix with the elements \(a[i,j] = 1/(x_i+y_i)\). The second argument of \textbf{cauchy_matrix} is optional. For this case the elements of the Cauchy matrix are \(a[i,j] = 1/(x_i+x_j)\).

Remark: In the literature the Cauchy matrix can be found defined in two forms. A second definition is \(a[i,j] = 1/(x_i-y_i)\).

Examples:

\begin{verbatim}
(%i1) cauchy_matrix([x1, x2], [y1, y2]);
\end{verbatim}

\begin{verbatim}
[ 1 1 ]
[ -------- -------- ]
[ y1 + x1 y2 + x1 ]
\end{verbatim}

\begin{verbatim}
(%o1) [ ]
\end{verbatim}

\begin{verbatim}
[ 1 1 ]
[ -------- -------- ]
[ y1 + x2 y2 + x2 ]
\end{verbatim}

\begin{verbatim}
(%i2) cauchy_matrix([x1, x2]);
\end{verbatim}

\begin{verbatim}
[ 1 1 ]
[ ---- ---- ]
[ 2 x1 x2 + x1 ]
\end{verbatim}

\begin{verbatim}
(%o2) [ ]
\end{verbatim}

\begin{verbatim}
[ 1 1 ]
[ ---- ---- ]
[ x2 + x1 2 x2 ]
\end{verbatim}
charpoly \((M, x)\)

Returns the characteristic polynomial for the matrix \(M\) with respect to variable \(x\). That is, \(\text{determinant}(M - \text{diagmatrix}(\text{length}(M), x))\).

\[(\%i1)\] \[a: \text{matrix}([[3, 1], [2, 4]]);\]
\[(\%o1)\] \[\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\]
\[(\%i2)\] \[\text{expand}(\text{charpoly}(a, \lambda));\]
\[(\%o2)\] \[
\begin{bmatrix} \lambda - 7 \lambda + 10 \\
2 \end{bmatrix}\]
\[(\%i3)\] \[(\text{programmode: true, solve(\%)})\];
\[(\%o3)\] \[\{\lambda = 5, \lambda = 2\}\]
\[(\%i4)\] \[\text{matrix}([x1], [x2]);\]
\[(\%o4)\] \[\begin{bmatrix} x1 \\
x2 \end{bmatrix}\]
\[(\%i5)\] \[\text{ev}(a \cdot \% - \lambda \% , \% \text{th}(2)[1]);\]
\[(\%o5)\] \[\begin{bmatrix} \lambda - 7 \lambda + 10 \\
2 \end{bmatrix}\]
\[(\%i6)\] \[\%[1, 1] = 0;\]
\[(\%o6)\] \[x2 - 2 x1 = 0\]
\[(\%i7)\] \[x2^2 + x1^2 = 1;\]
\[(\%o7)\] \[x2 + x1 = 1\]
\[(\%i8)\] \[\text{solve}\((\% \text{th}(2), \%, [x1, x2]);\]
\[(\%o8)\] \[\begin{bmatrix} x1 = \frac{-1}{\sqrt{5}}, x2 = \frac{2}{\sqrt{5}} \\
x1 = \frac{-2}{\sqrt{5}} \end{bmatrix}\]

coefmatrix \(([\text{eqn}_1, \ldots, \text{eqn}_m], [x_1, \ldots, x_n])\)

Returns the coefficient matrix for the variables \(x_1, \ldots, x_n\) of the system of linear equations \(\text{eqn}_1, \ldots, \text{eqn}_m\).

\[(\%i1)\] \[\text{coefmatrix}([2*x-(a-1)*y+5*b = 0, b*y+a*x = 3], [x, y]);\]
\[(\%o1)\] \[\begin{bmatrix} 2 & 1 - a \\
a & b \end{bmatrix}\]

col \((M, i)\)

Returns the \(i\)'th column of the matrix \(M\). The return value is a matrix.
columnvector \( (L) \)  \[\text{[Function]}\]
covect \( (L) \)  \[\text{[Function]}\]
Returns a matrix of one column and length \( L \) rows, containing the elements of the list \( L \).
covect \ is a synonym for columnvector.
load \("eigen"\) \ loads this function.
This is useful if you want to use parts of the outputs of the functions in this package in matrix calculations.
Example:
\[
(%i1) \text{load \("eigen")}$
\text{Warning - you are redefining the Macsyma function eigenvalues}
\text{Warning - you are redefining the Macsyma function eigenvectors}
(%i2) \text{columnvector ([aa, bb, cc, dd])};
\[ \begin{bmatrix} aa \\ bb \\ cc \\ dd \end{bmatrix} \]
(%o2)

copymatrix \( (M) \)  \[\text{[Function]}\]
Returns a copy of the matrix \( M \). This is the only way to make a copy aside from copying \( M \) element by element.
Note that an assignment of one matrix to another, as in \( m2: m1 \), does not copy \( m1 \). An assignment \( m2 \ [i,j]: x \) or \( \text{setelmx}(x, i, j, m2) \) also modifies \( m1 \ [i,j] \). Creating a copy with \text{copymatrix} and then using assignment creates a separate, modified copy.

determinant \( (M) \)  \[\text{[Function]}\]
Computes the determinant of \( M \) by a method similar to Gaussian elimination.
The form of the result depends upon the setting of the switch \text{ratmx}.
There is a special routine for computing sparse determinants which is called when the switches \text{ratmx} and \text{sparse} are both \text{true}.

detout  \[\text{[Option variable]}\]
Default value: \text{false}
When \text{detout} \ is \text{true}, the determinant of a matrix whose inverse is computed is factored out of the inverse.
For this switch to have an effect \text{doallmxops} and \text{doscmxops} should be \text{false} (see their descriptions). Alternatively this switch can be given to \text{ev} which causes the other two to be set correctly.
Example:
\[
(%i1) m: \text{matrix ([a, b], [c, d])};
\begin{bmatrix} a & b \\ c & d \end{bmatrix}
(%o1) \]
\[ \begin{bmatrix} c & d \end{bmatrix} \]

\( \text{(i2)} \) \text{detout: true}\$

\( \text{(i3)} \) \text{doallmxops: false}\$

\( \text{(i4)} \) \text{doscmxops: false}\$

\( \text{(i5)} \) \text{invert (m);} \\
\[ \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]

\( \text{(o5)} \) \\
\[ \frac{1}{a \ d - b \ c} \]

\text{diagmatrix}(n, x) \hspace{1cm} \text{[Function]} \]

Returns a diagonal matrix of size \( n \) by \( n \) with the diagonal elements all equal to \( x \). \text{diagmatrix}(n, 1) \text{ returns an identity matrix (same as ident(n)).} \\
\( n \) must evaluate to an integer, otherwise \text{diagmatrix} \text{ complains with an error message.} \\
\( x \) can be any kind of expression, including another matrix. If \( x \) is a matrix, it is not copied; all diagonal elements refer to the same instance, \( x \).

\text{doallmxops} \hspace{1cm} \text{[Option variable]} \\
 Default value: \text{true} \\
 When \text{doallmxops} \text{ is true, all operations relating to matrices are carried out. When it is false then the setting of the individual dot switches govern which operations are performed.}

\text{domxexpt} \hspace{1cm} \text{[Option variable]} \\
 Default value: \text{true} \\
 When \text{domxexpt} \text{ is true, a matrix exponential, exp(M) where M is a matrix, is interpreted as a matrix with element [i,j] equal to \text{exp}(m[i,j]). Otherwise exp(M) evaluates to exp(\text{ev}(M)).} \\
\text{domxexpt} \text{ affects all expressions of the form base}^\text{power} \text{ where base is an expression assumed scalar or constant, and power is a list or matrix.} \\
\text{Example:} \\
\( \text{(i1)} \) \text{m: matrix ([1, %i], [a+b, %pi]);} \\
\[ \begin{bmatrix} 1 & %i \\ b + a & %pi \end{bmatrix} \]

\( \text{(i2)} \) \text{domxexpt: false}\$

\( \text{(i3)} \) \text{(1 - c)}^\text{m;} \\
\[ \begin{bmatrix} 1 & %i \\ b + a & %pi \end{bmatrix} \]

\( \text{(o3)} \) \text{(1 - c)} \\
\( \text{(i4)} \) \text{domxexpt: true}\$

\( \text{(i5)} \) \text{(1 - c)}^\text{m;} \\
\[ \begin{bmatrix} 1 & %i \\ 1 - c & (1 - c) \end{bmatrix} \]
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(%o5) 

\[
\begin{bmatrix}
 b + a \\
 (1 - c)
\end{bmatrix}
\]

**domxmxops**

Default value: true

When `domxmxops` is true, all matrix-matrix or matrix-list operations are carried out (but not scalar-matrix operations); if this switch is false such operations are not carried out.

**domxnctimes**

Default value: false

When `domxnctimes` is true, non-commutative products of matrices are carried out.

**dontfactor**

Default value: []

dontfactor may be set to a list of variables with respect to which factoring is not to occur. (The list is initially empty.) Factoring also will not take place with respect to any variables which are less important, according the variable ordering assumed for canonical rational expression (CRE) form, than those on the dontfactor list.

**doscmxops**

Default value: false

When `doscmxops` is true, scalar-matrix operations are carried out.

**doscmxplus**

Default value: false

When `doscmxplus` is true, scalar-matrix operations yield a matrix result. This switch is not subsumed under `doallmxops`.

**dot0nscsimp**

Default value: true

When `dot0nscsimp` is true, a non-commutative product of zero and a nonscalar term is simplified to a commutative product.

**dot0simp**

Default value: true

When `dot0simp` is true, a non-commutative product of zero and a scalar term is simplified to a commutative product.

**dot1simp**

Default value: true

When `dot1simp` is true, a non-commutative product of one and another term is simplified to a commutative product.

**dotassoc**

Default value: true

When `dotassoc` is true, an expression \((A.B).C\) simplifies to \(A.(B.C)\).
dotconstrules
Default value: true
When dotconstrules is true, a non-commutative product of a constant and another
term is simplified to a commutative product. Turning on this flag effectively turns on
dot0simp, dot0nscsimp, and dot1simp as well.

dotdistrib
Default value: false
When dotdistrib is true, an expression A.(B + C) simplifies to A.B + A.C.

dotexptsimp
Default value: true
When dotexptsimp is true, an expression A.A simplifies to A^2.

dotident
Default value: 1
donident is the value returned by X^0.

dotscrules
Default value: false
When dotscrules is true, an expression A.SC or SC.A simplifies to SC*A and
A.(SC*B) simplifies to SC*(A.B).

echelon (M)
Returns the echelon form of the matrix M, as produced by Gaussian elimination. The
echelon form is computed from M by elementary row operations such that the first
non-zero element in each row in the resulting matrix is one and the column elements
under the first one in each row are all zero.
triangularize also carries out Gaussian elimination, but it does not normalize the
leading non-zero element in each row.
lu_factor and cholesky are other functions which yield triangularized matrices.

(%i1) M: matrix ([3, 7, aa, bb], [-1, 8, 5, 2], [9, 2, 11, 4]);
[ 3 7 aa bb ]
[ ]
(%o1) [ -1 8 5 2 ]
[ ]
[ 9 2 11 4 ]
(%i2) echelon (M);
[ 1 -8 -5 -2 ]
[ ]
[ 28 11 ]
[ ]
[ 0 1 -- -- ]
(%o2) [ 37 37 ]
[ ]
[ 37 bb - 119 ]
[ 0 0 1 ---------- ]
[ 37 aa - 313 ]
**eigenvalues** \((M)\)  
**eivals** \((M)\)  
Returns a list of two lists containing the eigenvalues of the matrix \(M\). The first sublist of the return value is the list of eigenvalues of the matrix, and the second sublist is the list of the multiplicities of the eigenvalues in the corresponding order.

**eivals** is a synonym for **eigenvalues**.

**eigenvalues** calls the function **solve** to find the roots of the characteristic polynomial of the matrix. Sometimes **solve** may not be able to find the roots of the polynomial; in that case some other functions in this package (except **innerproduct**, **unitvector**, **columnvector** and **gramschmidt**) will not work. Sometimes **solve** may find only a subset of the roots of the polynomial. This may happen when the factoring of the polynomial contains polynomials of degree 5 or more. In such cases a warning message is displayed and the only the roots found and their corresponding multiplicities are returned.

In some cases the eigenvalues found by **solve** may be complicated expressions. (This may happen when **solve** returns a not-so-obviously real expression for an eigenvalue which is known to be real.) It may be possible to simplify the eigenvalues using some other functions.

The package **eigen.mac** is loaded automatically when **eigenvalues** or **eigenvectors** is referenced. If **eigen.mac** is not already loaded, **load** ("eigen") loads it. After loading, all functions and variables in the package are available.

**eigenvectors** \((M)\)  
**eivects** \((M)\)  
Computes eigenvectors of the matrix \(M\). The return value is a list of two elements. The first is a list of the eigenvalues of \(M\) and a list of the multiplicities of the eigenvalues. The second is a list of lists of eigenvectors. There is one list of eigenvectors for each eigenvalue. There may be one or more eigenvectors in each list.

**eivects** is a synonym for **eigenvectors**.

The package **eigen.mac** is loaded automatically when **eigenvalues** or **eigenvectors** is referenced. If **eigen.mac** is not already loaded, **load** ("eigen") loads it. After loading, all functions and variables in the package are available.

Note that **eigenvectors** internally calls **eigenvalues** to obtain eigenvalues. So, when **eigenvalues** returns a subset of all the eigenvalues, the **eigenvectors** returns the corresponding subset of the all the eigenvectors, with the same warning displayed as **eigenvalues**.

The flags that affect this function are:

- **nondiagonalizable** is set to **true** or **false** depending on whether the matrix is nondiagonalizable or diagonalizable after **eigenvectors** returns.
- **hermitianmatrix** when **true**, causes the degenerate eigenvectors of the Hermitian matrix to be orthogonalized using the Gram-Schmidt algorithm.
- **knowneigvals** when **true** causes the **eigen** package to assume the eigenvalues of the matrix are known to the user and stored under the global name **listeigvals**. **listeigvals** should be set to a list similar to the output **eigenvalues**.
The function \texttt{algsys} is used here to solve for the eigenvectors. Sometimes if the eigenvalues are messy, \texttt{algsys} may not be able to find a solution. In some cases, it may be possible to simplify the eigenvalues by first finding them using \texttt{eigenvalues} command and then using other functions to reduce them to something simpler. Following simplification, \texttt{eigenvectors} can be called again with the \texttt{knowneigvals} flag set to \texttt{true}.

See also \texttt{eigenvalues}.

Examples:

A matrix which has just one eigenvector per eigenvalue.

\begin{verbatim}
(%i1) M1: matrix ([11, -1], [1, 7]);
   [ 11  - 1 ]
(%o1) [        ]
   [  1    7 ]
(%i2) [vals, vecs] : eigenvectors (M1);
(%o2) [[[9 - sqrt(3), sqrt(3) + 9], [1, 1]],
     [[[1, sqrt(3) + 2]], [[1, 2 - sqrt(3)]]]]
(%i3) for i thru length (vals[1]) do disp (val[i] = vals[1][i],
    mult[i] = vals[2][i], vec[i] = vecs[i]); val = 9 - sqrt(3) 1
    mult = 1 1
    vec = [[1, sqrt(3) + 2]] 1
val = sqrt(3) + 9 2
    mult = 1 2
    vec = [[1, 2 - sqrt(3)]] 2
(%o3) done
\end{verbatim}

A matrix which has two eigenvectors for one eigenvalue (namely 2).
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(%i1) M1: matrix ([0, 1, 0, 0], [0, 0, 0, 0], [0, 0, 2, 0], [0, 0, 0, 2]);
(%o1)
[ 0 1 0 0 ]
[ 0 0 0 0 ]
[ 0 0 2 0 ]
[ 0 0 0 2 ]

(%i2) [vals, vecs]: eigenvectors (M1);
(%o2) [[[0, 2], [2, 2]], [[[1, 0, 0, 0]], [[0, 0, 1, 0], [0, 0, 0, 1]]]]

(%i3) for i thru length (vals[1]) do disp (val[i] = vals[1][i], mult[i] = vals[2][i], vec[i] = vecs[i]);
val = 0
1
mult = 2
1
vec = [[1, 0, 0, 0]]
1

val = 2
2
mult = 2
2
vec = [[0, 0, 1, 0], [0, 0, 0, 1]]
2

(%o3) done

ematrix (m, n, x, i, j) [Function]
Returns an m by n matrix, all elements of which are zero except for the [i, j] element which is x.

entermatrix (m, n) [Function]
Returns an m by n matrix, reading the elements interactively.
If n is equal to m, Maxima prompts for the type of the matrix (diagonal, symmetric, antisymmetric, or general) and for each element. Each response is terminated by a semicolon ; or dollar sign $.
If n is not equal to m, Maxima prompts for each element.
The elements may be any expressions, which are evaluated. entermatrix evaluates its arguments.

(%i1) n: 3$
(%i2) m: entermatrix (n, n)$

Is the matrix 1. Diagonal 2. Symmetric 3. Antisymmetric
4. General

Answer 1, 2, 3 or 4 :
1$

Row 1 Column 1:
(a+b)^n$
Row 2 Column 2:
(a+b)^(n+1)$
Row 3 Column 3:
(a+b)^(n+2)$

Matrix entered.
(%i3) m;

[ 3 ]
( b + a 0 0 ]
[ ]
(%o3) [ 4 ]
[ 0 ( b + a ) 0 ]
[ ]
[ 5 ]
[ 0 0 ( b + a ) ]

Function
genmatrix
genmatrix
(genmatrix (a, i_2, j_2, i_1, j_1))
genmatrix (a, i_2, j_2, i_1)
genmatrix (a, i_2, j_2)

Returns a matrix generated from a, taking element a[i_1, j_1] as the upper-left element and a[i_2, j_2] as the lower-right element of the matrix. Here a is a declared array (created by array but not by make_array) or a hashed array, or a memoizing function, or a lambda expression of two arguments. (A memoizing function is created like other functions with := or define, but arguments are enclosed in square brackets instead of parentheses.)

If j_1 is omitted, it is assumed equal to i_1. If both j_1 and i_1 are omitted, both are assumed equal to 1.

If a selected element i, j of the array is undefined, the matrix will contain a symbolic element a[i, j].

Examples:

(%i1) h[i, j] := 1 / (i + j - 1);

(%o1) h := ---------
i, j i + j - 1
\begin{verbatim}
(%i2) genmatrix (h, 3, 3);
[ 1 1 ]
[ 1 - - ]
[ 2 3 ]
[ 1 1 1 ]
[ - - - ]
[ 2 3 4 ]
[ - - - ]
[ 1 1 1 ]
[ - - - ]
[ 3 4 5 ]
(%o2)
(%i3) array (a, fixnum, 2, 2);
(%o3) a
(%i4) a [1, 1] : %e;
(%o4) %e
(%i5) a [2, 2] : %pi;
(%o5) %pi
(%i6) genmatrix (a, 2, 2);
[ %e 0 ]
[ ]
[ 0 %pi ]
(%i7) genmatrix (lambda ([i, j], j - i), 3, 3);
[ 0 1 2 ]
[ ]
[ - 1 0 1 ]
[ ]
[ - 2 - 1 0 ]
(%i8) genmatrix (B, 2, 2);
[ B B ]
[ 1, 1 1, 2 ]
(%o8)
[ B B ]
[ 2, 1 2, 2 ]

gramschmidt

gramschmidt (x)
gramschmidt (x, F)
Carries out the Gram-Schmidt orthogonalization algorithm on x, which is either a matrix or a list of lists. x is not modified by gramschmidt. The inner product employed by gramschmidt is F, if present, otherwise the inner product is the function innerproduct.

If x is a matrix, the algorithm is applied to the rows of x. If x is a list of lists, the algorithm is applied to the sublists, which must have equal numbers of elements. In either case, the return value is a list of lists, the sublists of which are orthogonal and span the same space as x. If the dimension of the span of x is less than the number of rows or sublists, some sublists of the return value are zero.
\end{verbatim}
factor is called at each stage of the algorithm to simplify intermediate results. As a consequence, the return value may contain factored integers.

load("eigen") loads this function.

Example:

Gram-Schmidt algorithm using default inner product function.

```
(%i1) load("eigen")
(%i2) x: matrix([1, 2, 3], [9, 18, 30], [12, 48, 60]);
(%o2)
[ 1 2 3 ]
[    ]
[ 9 18 30 ]
[    ]
[ 12 48 60 ]
(%i3) y: gramschmidt(x);
(%o3) [[1, 2, 3], [- ---, - --, ---], [- ----, ----, 0]]
      2 7    7 5    2 7
(%i4) map(innerproduct,[y[1],y[2],y[3]], [y[2],y[3],y[1]]);
(%o4) [0, 0, 0]
```

Gram-Schmidt algorithm using a specified inner product function.

```
(%i1) load("eigen")
(%i2) ip (f, g) := integrate (f * g, u, a, b);
(%o2) ip(f, g) := integrate(f g, u, a, b)
(%i3) y: gramschmidt([1, sin(u), cos(u)], ip), a=-%pi/2, b=%pi/2;
(%o3) [1, sin(u), --------------]
             %pi
(%i4) map(ip, [y[1], y[2], y[3]], [y[2], y[3], y[1]]), a=-%pi/2, b=%pi/2;
(%o4) [0, 0, 0]
```

ident (n)

Returns an n by n identity matrix.

innerproduct (x, y) [Function]

inprod (x, y) [Function]

Returns the inner product (also called the scalar product or dot product) of x and y, which are lists of equal length, or both 1-column or 1-row matrices of equal length. The return value is conjugate(x) . y, where . is the noncommutative multiplication operator.

load ("eigen") loads this function.

inprod is a synonym for innerproduct.

invert_by_adjoint (M) [Function]

Returns the inverse of the matrix M. The inverse is computed by the adjoint method.

invert_by_adjoint honors the ratmx and detout flags, the same as invert.
invert \((M)\)  
Returns the inverse of the matrix \(M\). The inverse is computed via the LU decomposition.

When `ratmx` is `true`, elements of \(M\) are converted to canonical rational expressions (CRE), and the elements of the return value are also CRE.

When `ratmx` is `false`, elements of \(M\) are not converted to a common representation. In particular, float and bigfloat elements are not converted to rationals.

When `detout` is `true`, the determinant is factored out of the inverse. The global flags `doallmxops` and `doscmxops` must be `false` to prevent the determinant from being absorbed into the inverse. `xthru` can multiply the determinant into the inverse.

`invert` does not apply any simplifications to the elements of the inverse apart from the default arithmetic simplifications. `ratsimp` and `expand` can apply additional simplifications. In particular, when \(M\) has polynomial elements, `expand(invert(M))` might be preferable.

`invert(M)` is equivalent to \(M^{-1}\).

list_matrix_entries \((M)\)  
Returns a list containing the elements of the matrix \(M\).

Example:

```
(%i1) list_matrix_entries(matrix([a,b],[c,d]));
(%o1) [a, b, c, d]
```

lmxchar  
Default value: []

`lmxchar` is the character displayed as the left delimiter of a matrix. See also `rmxchar`.

Example:

```
(%i1) lmxchar: "|
(%i2) matrix ([a, b, c], [d, e, f], [g, h, i]);
   [ a  b  c ]
   [       ]
   [ d  e  f ]
   [       ]
   [ g  h  i ]
(%o2)
```

matrix \((\text{row}_1, \ldots, \text{row}_n)\)  
Returns a rectangular matrix which has the rows \(\text{row}_1, \ldots, \text{row}_n\). Each row is a list of expressions. All rows must be the same length.

The operations `+` (addition), `-` (subtraction), `*` (multiplication), and `/` (division), are carried out element by element when the operands are two matrices, a scalar and a matrix, or a matrix and a scalar. The operation `^` (exponentiation, equivalently `**`) is carried out element by element if the operands are a scalar and a matrix or a matrix and a scalar, but not if the operands are two matrices. All operations are normally carried out in full, including `.` (noncommutative multiplication).

Matrix multiplication is represented by the noncommutative multiplication operator `..` The corresponding noncommutative exponentiation operator is `^-`. For a matrix
\[ A, A.A = A^{-2} \text{ and } A^{-1} \text{ is the inverse of } A, \text{ if it exists. } A^{-1} \text{ is equivalent to } \text{invert}(A). \]

There are switches for controlling simplification of expressions involving dot and matrix-list operations. These are \texttt{doallmxops}, \texttt{domxexpt}, \texttt{domxmxops}, \texttt{doscmxops}, and \texttt{doscmxplus}.

There are additional options which are related to matrices. These are: \texttt{lmxchar}, \texttt{rmxchar}, \texttt{ratmx}, \texttt{listarith}, \texttt{detout}, \texttt{scalarmatrix} and \texttt{sparse}.

There are a number of functions which take matrices as arguments or yield matrices as return values. See \texttt{eigenvalues}, \texttt{eigenvectors}, \texttt{determinant}, \texttt{charpoly}, \texttt{genmatrix}, \texttt{addcol}, \texttt{addrow}, \texttt{copymatrix}, \texttt{transpose}, \texttt{echelon}, and \texttt{rank}.

Examples:

- Construction of matrices from lists.
  \( \text{E11) } x: \text{matrix ([[17, 3], [-8, 11]]); } \)
  \( \text{E12) } y: \text{matrix ([[}\pi, \text{e}], [a, b]); } \)

- Addition, element by element.
  \( \text{E13) } x + y; \)

- Subtraction, element by element.
  \( \text{E14) } x - y; \)

- Multiplication, element by element.
  \( \text{E15) } x \ast y; \)

- Division, element by element.
  \( \text{E16) } x / y; \)
Matrix to a scalar exponent, element by element.

```
(%i7) x ^ 3;
(%o7) 
```

Scalar base to a matrix exponent, element by element.

```
(%i8) exp(y);
(%o8) 
```

Matrix base to a matrix exponent. This is not carried out element by element. See also `matrixexp`.

```
(%i9) x ^ y;
(%o9) 
```

Noncommutative matrix multiplication.

```
(%i10) x . y;
(%o10) 
```

Noncommutative matrix exponentiation. A scalar base \( b \) to a matrix power \( M \) is carried out element by element and so \( b^{\mathbf{M}} \) is the same as \( b^m \).

```
(%i12) x ^^ 3;
(%o12) 
```

```
(%i13) %e ^^ y;
(%o13) 
```

A matrix raised to a -1 exponent with noncommutative exponentiation is the matrix inverse, if it exists.
(%i14) x ^^ -1;
[ 11  3  ]
[ --- - --- ]
[ 211 211 ]
(%o14)
[   ]
[ 8  17 ]
[ --- --- ]
[ 211 211 ]
(%i15) x . (x ^^ -1);
[ 1 0 ]
(%o15)
[   ]
[ 0 1 ]

matrixexp

Function

matrixexp (M)
matrixexp (M, n)
matrixexp (M, V)
Calculates the matrix exponential
\[ e^{MV} \]
. Instead of the vector \( V \) a number \( n \) can be specified as the second argument. If this
argument is omitted matrixexp replaces it by 1.
The matrix exponential of a matrix \( M \) can be expressed as a power series:
\[ e^M = \sum_{k=0}^{\infty} \left( \frac{M^k}{k!} \right) \]

matrixmap (f, M)

Function

Returns a matrix with element \( i,j \) equal to \( f(M[i,j]). \)
See also map, fullmap, fullmapl, and apply.

matrixexp (expr)

Function

Returns true if \( \text{expr} \) is a matrix, otherwise false.

Option variable

matrix_element_add

Default value: +

matrix_element_add is the operation invoked in place of addition in a matrix multi-
plcation. matrix_element_add can be assigned any n-ary operator (that is, a
function which handles any number of arguments). The assigned value may be the
name of an operator enclosed in quote marks, the name of a function, or a lambda
expression.
See also matrix_element_mult and matrix_element_transpose.

Example:

(%i1) matrix_element_add: "*"$
(%i2) matrix_element_mult: "^"$
(%i3) aa: matrix ([a, b, c], [d, e, f]);
    [ a  b  c ]
(%o3) \[
\begin{bmatrix}
d & e & f \\
x & y & z 
\end{bmatrix}
\]

(%i4) \text{bb: matrix ([u, v, w], [x, y, z]);}
\[
\begin{bmatrix}
u & v & w \\
x & y & z 
\end{bmatrix}
\]

(%o4) \[
\begin{bmatrix}
[ & & ]
[ & & ]
\end{bmatrix}
\]

(%i5) \text{aa . transpose (bb);} \\
\[
\begin{bmatrix}
u & v & w & x & y & z \\
a & b & c & a & b & c \\
[ & & ]
[ & & ]
\end{bmatrix}
\]

(%o5) \[
\begin{bmatrix}
[ & & ]
[ & & ]
\end{bmatrix}
\]

\text{matrix_element_mult} \quad [\text{Option variable}]

Default value: *

\text{matrix_element_mult} is the operation invoked in place of multiplication in a matrix multiplication. \text{matrix_element_mult} can be assigned any binary operator. The assigned value may be the name of an operator enclosed in quote marks, the name of a function, or a lambda expression.

The dot operator " is a useful choice in some contexts.

See also \text{matrix_element_add} and \text{matrix_element_transpose}.

Example:

(%i1) \text{matrix_element_add: lambda ([[x]], sqrt (apply (+, x)))}$

(%i2) \text{matrix_element_mult: lambda ([x, y], (x - y)^2)}$

(%i3) \text{[a, b, c] . [x, y, z];}$

\[
\begin{bmatrix}
2 & 2 & 2 \\
\text{sqrt((c - z) + (b - y) + (a - x))} \\
\end{bmatrix}
\]

(%i4) \text{aa: matrix ([a, b, c], [d, e, f]);}$

(%i4) \[
\begin{bmatrix}
a & b & c \\
[ & & ]
\end{bmatrix}
\]

(%i5) \text{bb: matrix ([u, v, w], [x, y, z]);}$

(%i5) \[
\begin{bmatrix}
u & v & w \\
x & y & z 
\end{bmatrix}
\]

(%i6) \text{aa . transpose (bb);} \\
\[
\begin{bmatrix}
2 & 2 & 2 \\
\text{sqrt((c - w) + (b - v) + (a - u))} \\
\end{bmatrix}
\]

(%o6) Col 1 = \[
\begin{bmatrix}
2 & 2 & 2 \\
\text{sqrt((f - w) + (e - v) + (d - u))} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 2 & 2 \\
\text{sqrt((c - z) + (b - y) + (a - x))} \\
\end{bmatrix}
\]

Col 2 = \[
\begin{bmatrix}
2 & 2 & 2 \\
\end{bmatrix}
\]
\[
\sqrt{(f - z) + (e - y) + (d - x)}
\]

**matrix_element_transpose**

Default value: `false`

`matrix_element_transpose` is the operation applied to each element of a matrix when it is transposed. `matrix_element_mult` can be assigned any unary operator. The assigned value may be the name of an operator enclosed in quote marks, the name of a function, or a lambda expression.

When `matrix_element_transpose` equals `transpose`, the `transpose` function is applied to every element. When `matrix_element_transpose` equals `nonscalars`, the `transpose` function is applied to every nonscalar element. If some element is an atom, the `nonscalars` option applies `transpose` only if the atom is declared nonscalar, while the `transpose` option always applies `transpose`.

The default value, `false`, means no operation is applied.

See also `matrix_element_add` and `matrix_element_mult`.

Examples:

```maxima
(%i1) declare (a, nonscalar)$
(%i2) transpose ([a, b]);
     [ transpose(a) ]
     [   ]
     [   b   ]
(%i3) matrix_element_transpose: nonscalars$
(%i4) transpose ([a, b]);
     [ transpose(a) ]
     [   ]
     [   b   ]
(%i5) matrix_element_transpose: transpose$
(%i6) transpose ([a, b]);
     [ transpose(a) ]
     [   ]
     [   transpose(b) ]
(%i7) matrix_element_transpose: lambda ([x], realpart(x) - %i*imagpart(x))$
(%i8) m: matrix ([[1 + 5*%i, 3 - 2*%i], [7*%i, 11]]);
     [ 5 %i + 1  3 - 2 %i ]
     [   ]
     [ 7 %i   11   ]
(%i9) transpose (m);
     [ 1 - 5 %i  7 %i ]
     [   ]
     [ 2 %i + 3  11   ]
```

**mattrace**

Returns the trace (that is, the sum of the elements on the main diagonal) of the square matrix `M`.

`mattrace` is called by `ncharpoly`, an alternative to Maxima’s `charpoly`.
load ("nchrpl") loads this function.

**minor (M, i, j)**  
Returns the $i, j$ minor of the matrix $M$. That is, $M$ with row $i$ and column $j$ removed.

**ncharpoly (M, x)**  
Returns the characteristic polynomial of the matrix $M$ with respect to $x$. This is an alternative to Maxima's `charpoly`. `ncharpoly` works by computing traces of powers of the given matrix, which are known to be equal to sums of powers of the roots of the characteristic polynomial. From these quantities the symmetric functions of the roots can be calculated, which are nothing more than the coefficients of the characteristic polynomial. `charpoly` works by forming the determinant of $x \cdot \text{ident}[n] - a$. Thus `ncharpoly` wins, for example, in the case of large dense matrices filled with integers, since it avoids polynomial arithmetic altogether.

load ("nchrpl") loads this file.

**newdet (M)**  
Computes the determinant of the matrix $M$ by the Johnson-Gentleman tree minor algorithm. `newdet` returns the result in CRE form.

**permanent (M)**  
Computes the permanent of the matrix $M$ by the Johnson-Gentleman tree minor algorithm. A permanent is like a determinant but with no sign changes. `permanent` returns the result in CRE form.

See also `newdet`.

**rank (M)**  
Computes the rank of the matrix $M$. That is, the order of the largest non-singular subdeterminant of $M$.

`rank` may return the wrong answer if it cannot determine that a matrix element that is equivalent to zero is indeed so.

**ratmx**  
Default value: `false`

When `ratmx` is `false`, determinant and matrix addition, subtraction, and multiplication are performed in the representation of the matrix elements and cause the result of matrix inversion to be left in general representation.

When `ratmx` is `true`, the 4 operations mentioned above are performed in CRE form and the result of matrix inverse is in CRE form. Note that this may cause the elements to be expanded (depending on the setting of `ratfac`) which might not always be desired.

**row (M, i)**  
Returns the $i$'th row of the matrix $M$. The return value is a matrix.

**rmxchar**  
Default value: `]`

`rmxchar` is the character drawn on the right-hand side of a matrix.

See also `lmxchar`. 
scalarmatrixp  [Option variable]
  Default value: true
  
  When scalarmatrixp is true, then whenever a 1 x 1 matrix is produced as a result of computing the dot product of matrices it is simplified to a scalar, namely the sole element of the matrix.

  When scalarmatrixp is all, then all 1 x 1 matrices are simplified to scalars.

  When scalarmatrixp is false, 1 x 1 matrices are not simplified to scalars.

scalefactors (coordinatetransform)  [Function]
  Here the argument coordinatetransform evaluates to the form [[expression1, expression2, ...], indeterminate1, indeterminate2, ...], where the variables indeterminate1, indeterminate2, etc. are the curvilinear coordinate variables and where a set of rectangular Cartesian components is given in terms of the curvilinear coordinates by [expression1, expression2, ...]. coordinates is set to the vector [indeterminate1, indeterminate2, ...], and dimension is set to the length of this vector. SF[1], SF[2], ..., SF[DIMENSION] are set to the coordinate scale factors, and sfprod is set to the product of these scale factors. Initially, coordinates is [X, Y, Z], dimension is 3, and SF[1]=SF[2]=SF[3]=SFPROD=1, corresponding to 3-dimensional rectangular Cartesian coordinates. To expand an expression into physical components in the current coordinate system, there is a function with usage of the form

setelmx (x, i, j, M)  [Function]
  Assigns x to the (i, j)'th element of the matrix M, and returns the altered matrix.
  M [i, j]: x has the same effect, but returns x instead of M.

similaritytransform (M)  [Function]
simtran (M)  [Function]
  similaritytransform computes a similarity transform of the matrix M. It returns a list which is the output of the uniteigenvectors command. In addition if the flag nondiagonalizable is false two global matrices leftmatrix and rightmatrix are computed. These matrices have the property that leftmatrix . M . rightmatrix is a diagonal matrix with the eigenvalues of M on the diagonal. If nondiagonalizable is true the left and right matrices are not computed.

  If the flag hermitianmatrix is true then leftmatrix is the complex conjugate of the transpose of rightmatrix. Otherwise leftmatrix is the inverse of rightmatrix. rightmatrix is the matrix the columns of which are the unit eigenvectors of M. The other flags (see eigenvalues and eigenvectors) have the same effects since similaritytransform calls the other functions in the package in order to be able to form rightmatrix.

load ("eigen") loads this function.
  simtran is a synonym for similaritytransform.

sparse  [Option variable]
  Default value: false
  
  When sparse is true, and if ratmx is true, then determinant will use special routines for computing sparse determinants.
submatrix [Function]

submatrix \((i_1, \ldots, i_m, M, j_1, \ldots, j_n)\)
submatrix \((i_1, \ldots, i_m, M)\)
submatrix \((M, j_1, \ldots, j_n)\)

Returns a new matrix composed of the matrix \(M\) with rows \(i_1, \ldots, i_m\) deleted, and columns \(j_1, \ldots, j_n\) deleted.

transpose \((M)\) [Function]

Returns the transpose of \(M\).

If \(M\) is a matrix, the return value is another matrix \(N\) such that \(N[i,j] = M[j,i]\).
If \(M\) is a list, the return value is a matrix \(N\) of length \(m\) rows and 1 column, such that \(N[i,1] = M[i]\).
Otherwise \(M\) is a symbol, and the return value is a noun expression 'transpose \((M)\).

triangularize \((M)\) [Function]

Returns the upper triangular form of the matrix \(M\), as produced by Gaussian elimination. The return value is the same as echelon, except that the leading nonzero coefficient in each row is not normalized to 1.

lu_factor and cholesky are other functions which yield triangularized matrices.

\[(%i1)\ M: \text{matrix}([[3, 7, aa, bb], [-1, 8, 5, 2], [9, 2, 11, 4]]);\]
\[
\begin{bmatrix}
3 & 7 & aa & bb \\
-1 & 8 & 5 & 2 \\
9 & 2 & 11 & 4
\end{bmatrix}
\]

\[(%o1)\]

\[(%i2)\ \text{triangularize} \((M)\);\]
\[
\begin{bmatrix}
-1 & 8 & 5 & 2 \\
0 & -74 & -56 & -22 \\
0 & 0 & 626 - 74 aa & 238 - 74 bb
\end{bmatrix}
\]

uniteigenvectors \((M)\) [Function]
ueivects \((M)\) [Function]

Computes unit eigenvectors of the matrix \(M\). The return value is a list of lists, the first sublist of which is the output of the eigenvectors command, and the other sublists of which are the unit eigenvectors of the matrix corresponding to those eigenvalues respectively.

The flags mentioned in the description of the eigenvectors command have the same effects in this one as well.

When knowneigvects is true, the eigen package assumes that the eigenvectors of the matrix are known to the user and are stored under the global name listeigvects. listeigvects should be set to a list similar to the output of the eigenvectors command.

If knowneigvects is set to true and the list of eigenvectors is given the setting of the flag nondiagonalizable may not be correct. If that is the case please set it to the
correct value. The author assumes that the user knows what he is doing and will not try to diagonalize a matrix the eigenvectors of which do not span the vector space of the appropriate dimension.

\texttt{load ("eigen")} loads this function.

\texttt{ueivects} is a synonym for \texttt{uniteigenvectors}.

\textbf{unitvector} \((x)\)  
\textbf{uvect} \((x)\)  
\begin{itemize}
  \item Returns \(x/norm(x)\); this is a unit vector in the same direction as \(x\).
\end{itemize}

\texttt{load ("eigen")} loads this function.

\texttt{uvect} is a synonym for \texttt{unitvector}.

\textbf{vectorpotential} \((\text{givencurl})\)  
\begin{itemize}
  \item Returns the vector potential of a given curl vector, in the current coordinate system.
\end{itemize}

\texttt{potentialzeroloc} has a similar role as for \texttt{potential}, but the order of the left-hand sides of the equations must be a cyclic permutation of the coordinate variables.

\textbf{vectorsimp} \((\text{expr})\)  
\begin{itemize}
  \item Applies simplifications and expansions according to the following global flags:
    \begin{itemize}
      \item \texttt{expandall}, \texttt{expanddot}, \texttt{expanddotplus}, \texttt{expandcross}, \texttt{expandcrossplus}, \texttt{expandcrosscross}, \texttt{expandgrad}, \texttt{expandgradplus}, \texttt{expandgradprod}, \texttt{expanddiv}, \texttt{expanddivplus}, \texttt{expanddivprod}, \texttt{expandcurl}, \texttt{expandcurlplus}, \texttt{expandcurlcurl}, \texttt{expandidlaplacian}, \texttt{expandidlaplacianplus}, and \texttt{expandidlaplacianprod}.
    \end{itemize}
  \end{itemize}

All these flags have default value \texttt{false}. The \texttt{plus} suffix refers to employing additivity or distributivity. The \texttt{prod} suffix refers to the expansion for an operand that is any kind of product.

\texttt{expandcrosscross}  
\begin{itemize}
  \item Simplifies \(p \sim (q \sim r)\) to \((p.r)q - (p.q)r\).
\end{itemize}

\texttt{expandcurlcurl}  
\begin{itemize}
  \item Simplifies curl curl \(p\) to grad div \(p + \text{div grad} \ p\).
\end{itemize}

\texttt{expandidlaplaciantodivgrad}  
\begin{itemize}
  \item Simplifies laplacian \(p\) to div grad \(p\).
\end{itemize}

\texttt{expandcross}  
\begin{itemize}
  \item Enables \texttt{expandcrossplus} and \texttt{expandcrosscross}.
\end{itemize}

\texttt{expandplus}  
\begin{itemize}
  \item Enables \texttt{expanddotplus}, \texttt{expandcrossplus}, \texttt{expandgradplus}, \texttt{expanddivplus}, \texttt{expandcurlplus}, and \texttt{expandidlaplacianplus}.
\end{itemize}

\texttt{expandprod}  
\begin{itemize}
  \item Enables \texttt{expandgradprod}, \texttt{expanddivprod}, and \texttt{expandidlaplacianprod}.
\end{itemize}

These flags have all been declared \texttt{evflag}.
vect_cross  [Option variable]
   Default value: false
   When vect_cross is true, it allows DIFF(X~Y,T) to work where ~ is defined in
   SHARE;VECT (where VECT_CROSS is set to true, anyway.)

zeromatrix (m, n)  [Function]
   Returns an m by n matrix, all elements of which are zero.
24 Affine

24.1 Introduction to Affine

affine is a package to work with groups of polynomials.

24.2 Functions and Variables for Affine

fast_linsolve ([expr_1, ..., expr_m], [x_1, ..., x_n])
[Function]
Solves the simultaneous linear equations expr_1, ..., expr_m for the variables x_1, ..., x_n. Each expr_i may be an equation or a general expression; if given as a general expression, it is treated as an equation of the form expr_i = 0.
The return value is a list of equations of the form [x_1 = a_1, ..., x_n = a_n] where a_1, ..., a_n are all free of x_1, ..., x_n.
fast_linsolve is faster than linsolve for system of equations which are sparse.
load("affine") loads this function.

grobner_basis ([expr_1, ..., expr_m])
[Function]
Returns a Groebner basis for the equations expr_1, ..., expr_m. The function polysimp can then be used to simplify other functions relative to the equations.
grobner_basis ([3*x^2+1, y*x])$
polysimp (y^2*x + x^3*9 + 2) ==> -3*x + 2
polysimp(f) yields 0 if and only if f is in the ideal generated by expr_1, ..., expr_m, that is, if and only if f is a polynomial combination of the elements of expr_1, ..., expr_m.
load("affine") loads this function.

set_up_dot_simplifications
[Function]
set_up_dot_simplifications (eqns, check_through_degree)
set_up_dot_simplifications (eqns)
The eqns are polynomial equations in non commutative variables. The value of current_variables is the list of variables used for computing degrees. The equations must be homogeneous, in order for the procedure to terminate.
If you have checked overlapping simplifications in dot_simplifications above the degree of f, then the following is true: dotsimp (f) yields 0 if and only if f is in the ideal generated by the equations, i.e., if and only if f is a polynomial combination of the elements of the equations.
The degree is that returned by nc_degree. This in turn is influenced by the weights of individual variables.
load("affine") loads this function.

declare_weights (x_1, w_1, ..., x_n, w_n)
[Function]
Assigns weights w_1, ..., w_n to x_1, ..., x_n, respectively. These are the weights used in computing nc_degree.
load("affine") loads this function.
nc_degree (p)
Returns the degree of a noncommutative polynomial p. See declare_weights.
load("affine") loads this function.

dotsimp (f)
Returns 0 if and only if f is in the ideal generated by the equations, i.e., if and only if f is a polynomial combination of the elements of the equations.
load("affine") loads this function.

fast_central_elements ([x_1, ..., x_n], n)
If set_up_dot_simplifications has been previously done, finds the central polynomials in the variables x_1, ..., x_n in the given degree, n.
For example:
set_up_dot_simplifications ([y.x + x.y], 3);
fast_central_elements ([x, y], 2);
[y.y, x.x];
load("affine") loads this function.

check_overlaps (n, add_to_simps)
Checks the overlaps thru degree n, making sure that you have sufficient simplification rules in each degree, for dotsimp to work correctly. This process can be speeded up if you know before hand what the dimension of the space of monomials is. If it is of finite global dimension, then hilbert should be used. If you don’t know the monomial dimensions, do not specify a rank_function. An optional third argument reset, false says don’t bother to query about resetting things.
load("affine") loads this function.

mono ([x_1, ..., x_n], n)
Returns the list of independent monomials relative to the current dot simplifications of degree n in the variables x_1, ..., x_n.
load("affine") loads this function.

monomial_dimensions (n)
Compute the Hilbert series through degree n for the current algebra.
load("affine") loads this function.

extract_linear_equations ([p_1, ..., p_n], [m_1, ..., m_n])
Makes a list of the coefficients of the noncommutative polynomials p_1, ..., p_n of the noncommutative monomials m_1, ..., m_n. The coefficients should be scalars. Use list_nc_monomials to build the list of monomials.
load("affine") loads this function.

list_nc_monomials
list_nc_monomials ([p_1, ..., p_n])
list_nc_monomials (p)
Returns a list of the non commutative monomials occurring in a polynomial p or a list of polynomials p_1, ..., p_n.
load("affine") loads this function.
all_dotsimp_denoms  [Option variable]
  Default value: false

  When all_dotsimp_denoms is a list, the denominators encountered by dotsimp are appended to the list. all_dotsimp_denoms may be initialized to an empty list [] before calling dotsimp.

  By default, denominators are not collected by dotsimp.
25 itensor

25.1 Introduction to itensor

Maxima implements symbolic tensor manipulation of two distinct types: component tensor manipulation (ctensor package) and indicial tensor manipulation (itensor package).

Nota bene: Please see the note on 'new tensor notation' below.

Component tensor manipulation means that geometrical tensor objects are represented as arrays or matrices. Tensor operations such as contraction or covariant differentiation are carried out by actually summing over repeated (dummy) indices with do statements. That is, one explicitly performs operations on the appropriate tensor components stored in an array or matrix.

Indicial tensor manipulation is implemented by representing tensors as functions of their covariant, contravariant and derivative indices. Tensor operations such as contraction or covariant differentiation are performed by manipulating the indices themselves rather than the components to which they correspond.

These two approaches to the treatment of differential, algebraic and analytic processes in the context of Riemannian geometry have various advantages and disadvantages which reveal themselves only through the particular nature and difficulty of the user’s problem. However, one should keep in mind the following characteristics of the two implementations:

The representation of tensors and tensor operations explicitly in terms of their components makes ctensor easy to use. Specification of the metric and the computation of the induced tensors and invariants is straightforward. Although all of Maxima’s powerful simplification capacity is at hand, a complex metric with intricate functional and coordinate dependencies can easily lead to expressions whose size is excessive and whose structure is hidden. In addition, many calculations involve intermediate expressions which swell causing programs to terminate before completion. Through experience, a user can avoid many of these difficulties.

Because of the special way in which tensors and tensor operations are represented in terms of symbolic operations on their indices, expressions which in the component representation would be unmanageable can sometimes be greatly simplified by using the special routines for symmetrical objects in itensor. In this way the structure of a large expression may be more transparent. On the other hand, because of the special indicial representation in itensor, in some cases the user may find difficulty with the specification of the metric, function definition, and the evaluation of differentiated "indexed" objects.

The itensor package can carry out differentiation with respect to an indexed variable, which allows one to use the package when dealing with Lagrangian and Hamiltonian formalisms. As it is possible to differentiate a field Lagrangian with respect to an (indexed) field variable, one can use Maxima to derive the corresponding Euler-Lagrange equations in indicial form. These equations can be translated into component tensor (ctensor) programs using the ic_convert function, allowing us to solve the field equations in a particular coordinate representation, or to recast the equations of motion in Hamiltonian form. See einhil.dem and bradic.dem for two comprehensive examples. The first, einhil.dem, uses the Einstein-Hilbert action to derive the Einstein field tensor in
the homogeneous and isotropic case (Friedmann equations) and the spherically symmetric, static case (Schwarzschild solution.) The second, bradic.dem, demonstrates how to compute the Friedmann equations from the action of Brans-Dicke gravity theory, and also derives the Hamiltonian associated with the theory’s scalar field.

### 25.1.1 New tensor notation

Earlier versions of the itensor package in Maxima used a notation that sometimes led to incorrect index ordering. Consider the following, for instance:

```maxima
(%i2) imetric(g);
(%o2) done
(%i3) ishow(g([],[j,k])*g([],[i,l])*a([i,j],[]))$
     i l j k
     g g a
     i j
(%i4) ishow(contract(%)$                      k l
     a
(%i5) ishow(g([-j,-k],[])\*g([-i,-l],[])\*a([i,j],[]))$
     i l j k
     g a g
     i j
(%i6) ishow(contract(%)$
     l k
     a
```

This result is incorrect unless a happens to be a symmetric tensor. The reason why this happens is that although itensor correctly maintains the order within the set of covariant and contravariant indices, once an index is raised or lowered, its position relative to the other set of indices is lost.

To avoid this problem, a new notation has been developed that remains fully compatible with the existing notation and can be used interchangeably. In this notation, contravariant indices are inserted in the appropriate positions in the covariant index list, but with a minus sign prepended. Functions like contract_Itensor and ishow are now aware of this new index notation and can process tensors appropriately.

In this new notation, the previous example yields a correct result:

```maxima
(%i5) ishow(g([-j,-k],[])\*g([-i,-l],[])\*a([i,j],[]))$
     i l j k
     g a g
     i j
(%i6) ishow(contract(%)$
     l k
     a
```

Presently, the only code that makes use of this notation is the lc2kdt function. Through this notation, it achieves consistent results as it applies the metric tensor to resolve Levi-Civita symbols without resorting to numeric indices.

Since this code is brand new, it probably contains bugs. While it has been tested to make sure that it doesn’t break anything using the "old" tensor notation, there is a considerable chance that "new" tensors will fail to interoperate with certain functions or features. These bugs will be fixed as they are encountered... until then, caveat emptor!

### 25.1.2 Indicial tensor manipulation

The indicial tensor manipulation package may be loaded by load("itensor"). Demos are also available: try demo("tensor").
In itensor a tensor is represented as an "indexed object". This is a function of 3 groups of indices which represent the covariant, contravariant and derivative indices. The covariant indices are specified by a list as the first argument to the indexed object, and the contravariant indices by a list as the second argument. If the indexed object lacks either of these groups of indices then the empty list [] is given as the corresponding argument. Thus, \(g([a,b], [c])\) represents an indexed object called \(g\) which has two covariant indices \((a,b)\), one contravariant index \((c)\) and no derivative indices.

The derivative indices, if they are present, are appended as additional arguments to the symbolic function representing the tensor. They can be explicitly specified by the user or be created in the process of differentiation with respect to some coordinate variable. Since ordinary differentiation is commutative, the derivative indices are sorted alphanumerically, unless iframe_flag is set to true, indicating that a frame metric is being used. This canonical ordering makes it possible for Maxima to recognize that, for example, \(t([a],[b],i,j)\) is the same as \(t([a],[b],j,i)\). Differentiation of an indexed object with respect to some coordinate whose index does not appear as an argument to the indexed object would normally yield zero. This is because Maxima would not know that the tensor represented by the indexed object might depend implicitly on the corresponding coordinate. By modifying the existing Maxima function diff in itensor, Maxima now assumes that all indexed objects depend on any variable of differentiation unless otherwise stated. This makes it possible for the summation convention to be extended to derivative indices. It should be noted that itensor does not possess the capabilities of raising derivative indices, and so they are always treated as covariant.

The following functions are available in the tensor package for manipulating indexed objects. At present, with respect to the simplification routines, it is assumed that indexed objects do not by default possess symmetry properties. This can be overridden by setting the variable allsym[false] to true, which will result in treating all indexed objects completely symmetric in their lists of covariant indices and symmetric in their lists of contravariant indices.

The itensor package generally treats tensors as opaque objects. Tensorial equations are manipulated based on algebraic rules, specifically symmetry and contraction rules. In addition, the itensor package understands covariant differentiation, curvature, and torsion. Calculations can be performed relative to a metric of moving frame, depending on the setting of the iframe_flag variable.

A sample session below demonstrates how to load the itensor package, specify the name of the metric, and perform some simple calculations.

```
(%i1) load("itensor");
(%o1) /share/tensor/itensor.lisp
(%i2) imetric(g);
(%o2) done
(%i3) components(g([i,j],[i])*p([i,j])*e([i],[i]))$
(%i4) ishow(g([k,l],[i]))$
(%i5) ishow(diff(v([i],[i]),t))$
(%i6) depends(v,t);
```
(%i6) [v(t)]

(%i7) ishow(diff(v([i],[]),t))$

\[ \frac{d}{dt} v \]

(%t7)

(%i8) ishow(idiff(v([i],[]),j))$

\[ v_{i,j} \]

(%t8)

(%i9) ishow(extdiff(v([i],[]),j))$

\[ v_{i} - v_{j,i} \]

(%t9)

(%o10) true

(%i14) ishow(covdiff(v([i],[]),j))$

\[ v \cdot v_{i,j} - v_{i,j} \cdot v \]

(%t14)

(%i15) ishow(ev(%,ichr2))$

\[ v \cdot v_{i,j} - \frac{1}{2} v_{i,j} \cdot v \]

(%t15)

(%i16) ishow(radcan(ev(%,ifc2,ifc1)))$

\[ v \cdot v_{i,j} - \frac{1}{2} v_{i,j} \cdot v \]

(%t16)

(%o13) true

(%i17) ishow(canform(s([i,j],[])-s([j,i])))$

\[ s_{i,j} - s_{j,i} \]

(%t17)
Chapter 25: itensor

25.2 Functions and Variables for itensor

25.2.1 Managing indexed objects

dispcon
  dispcon (tensor_1, tensor_2, ...)
dispcon (all)
Displays the contraction properties of its arguments as were given to defcon. dispcon (all) displays all the contraction properties which were defined.

entertensor (name)
is a function which, by prompting, allows one to create an indexed object called name with any number of tensorial and derivative indices. Either a single index or a list of indices (which may be null) is acceptable input (see the example under covdiff).

 changename (old, new, expr)
will change the name of all indexed objects called old to new in expr. old may be either a symbol or a list of the form [name, m, n] in which case only those indexed objects called name with m covariant and n contravariant indices will be renamed to new.

listoftens
Lists all tensors in a tensorial expression, complete with their indices. E.g.,

(%i6) ishow(a([[i,j], [k]])*b([[u], [], v]+c([[x, y], []])*d([[[]], []])*e)
     k
(%t6)    d e c + a b
         x y   i j u, v
(%i7) ishow(listoftens(%))
( %t7)   [a , b , c , d]
         i j  u, v  x y
ishow (expr) [Function]
displays expr with the indexed objects in it shown having their covariant indices
as subscripts and contravariant indices as superscripts. The derivative indices are
displayed as subscripts, separated from the covariant indices by a comma (see the
eamples throughout this document).

indices (expr) [Function]
Returns a list of two elements. The first is a list of the free indices in expr (those
that occur only once). The second is the list of the dummy indices in expr (those
that occur exactly twice) as the following example demonstrates.

\[
\begin{align*}
\text{(\%i1) load(\"itensor\");} \\
\text{(\%o1) /share/tensor/itensor.lisp} \\
\text{(\%i2) ishow(a([i,j],[k,l],m,n)*b([k,o],[j,m,p],q,r))} \\
\text{(\%t2) a \quad b} \\
\text{\quad i, j, m, n \quad k, o, q, r} \\
\text{(\%i3) indices(\%);} \\
\text{(\%o3) \quad [1, p, i, n, o, q, r], [k, j, m]} \\
\end{align*}
\]

A tensor product containing the same index more than twice is syntactically illegal.
indices attempts to deal with these expressions in a reasonable manner; however,
when it is called to operate upon such an illegal expression, its behavior should be
considered undefined.

rename [Function]

rename (expr)
rename (expr, count)
Returns an expression equivalent to expr but with the dummy indices in each term
chosen from the set \%1, \%2, ..., if the optional second argument is omitted. Other-
wise, the dummy indices are indexed beginning at the value of count. Each dummy
index in a product will be different. For a sum, rename will operate upon each term
in the sum resetting the counter with each term. In this way rename can serve as a
tensorial simplifier. In addition, the indices will be sorted alphanumerically (if allsym
is true) with respect to covariant or contravariant indices depending upon the value
of flipflag. If flipflag is false then the indices will be renamed according to
the order of the contravariant indices. If flipflag is true the renaming will occur
according to the order of the covariant indices. It often happens that the combined
effect of the two renamings will reduce an expression more than either one by itself.

\[
\begin{align*}
\text{(\%i1) load(\"itensor\");} \\
\text{(\%o1) /share/tensor/itensor.lisp} \\
\text{(\%i2) allsym: true;} \\
\text{(\%o2) true} \\
\text{(\%i3) g(\[], [%4, %5])*g(\[], [%6, %7])*ichr2([%1, %4], [%3])*} \\
\text{ichr2([%2, %3], [u])*ichr2([%5, %6], [%1])*ichr2([%7, r], [%2])-} \\
\text{g(\[], [%4, %5])*g(\[], [%6, %7])*ichr2([%1, %2], [u])*} \\
\end{align*}
\]
ichr2( [%3,%5], [%1] ) * ichr2( [%4,%6], [%3] ) * ichr2( [%7,r], [%2] ), noeval$

(expr:ishow( ))$

%4 %5 %6 %7 u %1 %2

%4 %5 %6 %7 %2 %3 %1 r

-chr2 ichr2 ichr2 ichr2 ichr2

%1 %2 %3 %5 %4 %6 %7 r

-g g ichr2 ichr2 ichr2 ichr2

%1 %2 %3 %4 %5 %6 %7 r

flipflag: true;

true

(ishow(rename(expr)))$

%1 %2 %3 %4 %5 %6 %7 u

-chr2 ichr2 ichr2 ichr2

%1 %2 %3 %4 %5 %6 %7 r

-g g ichr2 ichr2 ichr2 ichr2

%1 %2 %3 %4 %5 %6 %7 r

flipflag: false;

false

(rename(%th(2)));

0

(ishow(rename(expr)))$

%1 %2 %3 %4 %5 %6 %7 u

-chr2 ichr2 ichr2 ichr2

%1 %2 %3 %5 %4 %6 %7 r

-g g ichr2 ichr2 ichr2 ichr2

%1 %2 %3 %6 %5 %7 u

%1 %2 %3 %4 %6 %5 %7

flipflag

[Option variable]

Default value: false

If false then the indices will be renamed according to the order of the contravariant indices, otherwise according to the order of the covariant indices.

If flipflag is false then rename forms a list of the contravariant indices as they are encountered from left to right (if true then of the covariant indices). The first dummy index in the list is renamed to %1, the next to %2, etc. Then sorting occurs after the rename-ing (see the example under rename).

defcon

defcon (tensor_1)
defcon (tensor_1, tensor_2, tensor_3)
gives tensor_1 the property that the contraction of a product of tensor_1 and tensor_2 results in tensor_3 with the appropriate indices. If only one argument, tensor_1, is given, then the contraction of the product of tensor_1 with any indexed object having
the appropriate indices (say \texttt{my\_tensor}) will yield an indexed object with that name, i.e. \texttt{my\_tensor}, and with a new set of indices reflecting the contractions performed. For example, if \texttt{imetric:g}, then \texttt{defcon(g)} will implement the raising and lowering of indices through contraction with the metric tensor. More than one \texttt{defcon} can be given for the same indexed object; the latest one given which applies in a particular contraction will be used. \texttt{contractions} is a list of those indexed objects which have been given contraction properties with \texttt{defcon}.

\texttt{remcon} \hspace{1cm} \textbf{[Function]}

\begin{verbatim}
remcon (tensor_1, ..., tensor_n)
remcon (all)
\end{verbatim}

Removes all the contraction properties from the (\texttt{tensor\_1, ..., tensor\_n}). \texttt{remcon(all)} removes all contraction properties from all indexed objects.

\texttt{contract (expr)} \hspace{1cm} \textbf{[Function]}

Carries out the tensorial contractions in \texttt{expr} which may be any combination of sums and products. This function uses the information given to the \texttt{defcon} function. For best results, \texttt{expr} should be fully expanded. \texttt{ratexpand} is the fastest way to expand products and powers of sums if there are no variables in the denominators of the terms. The \texttt{gcd} switch should be \texttt{false} if GCD cancellations are unnecessary.

\texttt{indexed\_tensor (tensor)} \hspace{1cm} \textbf{[Function]}

Must be executed before assigning components to a \texttt{tensor} for which a built in value already exists as with \texttt{ichr1, ichr2, icurvature}. See the example under \texttt{icurvature}.

\texttt{components (tensor, expr)} \hspace{1cm} \textbf{[Function]}

permits one to assign an indicial value to an expression \texttt{expr} giving the values of the components of \texttt{tensor}. These are automatically substituted for the tensor whenever it occurs with all of its indices. The tensor must be of the form \texttt{t([...,],...)} where either list may be empty. \texttt{expr} can be any indexed expression involving other objects with the same free indices as \texttt{tensor}. When used to assign values to the metric tensor wherein the components contain dummy indices one must be careful to define these indices to avoid the generation of multiple dummy indices. Removal of this assignment is given to the function \texttt{remcomps}.

It is important to keep in mind that \texttt{components} cares only about the valence of a tensor, not about any particular index ordering. Thus assigning components to, say, \texttt{x([i,-j],[]), x([-j,i],[]), or x([[i],[j]]) all produce the same result, namely components being assigned to a tensor named \texttt{x} with valence \texttt{(1,1)}.

Components can be assigned to an indexed expression in four ways, two of which involve the use of the \texttt{components} command:

1) As an indexed expression. For instance:

```
(%i2) components(g([],[i,j]),e([],[i])*p([],[j]))$
```

```
(%i3) ishow(g([],[i,j]))$
```

```
(%i3)
```

```
(\%t3)
```

```
i j
```

```
e p
```

2) As a matrix:

```
(%i5) lg:ident(4)$
[ 1 0 0 0 ]
lg[1,1]:1$lg;
[ 0 -1 0 0 ]
[ ]
[ 0 0 -1 0 ]
[ ]
[ 0 0 0 -1 ]
(%o5)
(%i6) components(g([i,j],[[]]),lg);
(%o6) done
(%i7) ishow(g([i,j],[[]]))$
(%t7) g
i j
(%i8) g([1,1],[[]]);
(%o8) 1
(%i9) g([4,4],[[]]);
(%o9) -1
```

3) As a function. You can use a Maxima function to specify the components of a tensor based on its indices. For instance, the following code assigns \( \delta_{ij} \) to \( h \) if \( h \) has the same number of covariant and contravariant indices and no derivative indices, and \( g \) otherwise:

```
(%i4) h(l1,l2,[l3]):=if length(l1)=length(l2) and length(l3)=0
   then kdelta(l1,l2) else apply(g,append([l1,l2], l3))$
(%i5) ishow(h([i],[j]))$
(%t5) kdelta
j
i
(%i6) ishow(h([i,j],[k],[l]))$
(%t6) g
i j l
```

4) Using Maxima's pattern matching capabilities, specifically the \texttt{de NRF} and \texttt{applyb1} commands:

```
(%i1) load("itensor");
(%i1) /share/tensor/itensor.lisp
(%i2) matchdeclare(l1,listp);
(%o2) done
(%i3) defrule(r1,m(l1,[]),(i1:idummy(),
g([l1[1],l1[2]],[])\ast q([i1],[]\ast e([],[i1])))$)
(%i4) defrule(r2,m([],l1),(i1:idummy(),
w([],l1[1],l1[2]))\ast e([i1],[])\ast q([],[i1])))$
remcomps (tensor)  [Function]
Unbinds all values from tensor which were assigned with the components function.

showcomps (tensor)  [Function]
Shows component assignments of a tensor, as made using the components command.
This function can be particularly useful when a matrix is assigned to an indicial tensor using components, as demonstrated by the following example:

```
(%i1) load("ctensor");
(%o1) /share/tensor/ctensor.mac
(%i2) load("itensor");
(%o2) /share/tensor/itensor.lisp
(%i3) lg:matrix([sqrt(r/(r-2*m)),0,0,0],[0,r,0,0],[0,0,sin(theta)*r,0],[0,0,0,sqrt((r-2*m)/r)]);
    [ r            ]
    [ sqrt(-------) 0 0 0 ]
    [ r - 2 m       ]
    [ 0 r 0 0      ]
(%o3) [ ]
    [ 0 0 r sin(theta) 0 ]
    [ ]
    [ 0 0 0 sqrt(-------) ]
    [ r - 2 m       ]
    [ 0 0 0         ]
    [ r            ]
(%i4) components(g([i,j],[[]]),lg);
(%o4) done
(%i5) showcomps(g([i,j],[[]]));
    [ r            ]
    [ sqrt(-------) 0 0 0 ]
    [ r - 2 m       ]
    [ 0 r 0 0      ]
    [ 0 0 r sin(theta) 0 ]
    [ ]
    [ 0 0 0         ]
    [ r - 2 m       ]
```

```
(\%i5) \texttt{ishow(m([i,n],[[]])*m([],[i,m]))}$
\texttt{m m}$
(\%t5) \texttt{ishow(rename(applyb1(\%r1,r2)))}$
\texttt{m m}$
(\%i6) \texttt{ishow(rename(applyb1(\%r1,r2)))}$
\texttt{m m}$
(\%t6) \texttt{e q w q e g}$
\texttt{m m}$
```
The `showcomps` command can also display components of a tensor of rank higher than 2.

**idummy ()**

Increments `icounter` and returns as its value an index of the form `%n` where `n` is a positive integer. This guarantees that dummy indices which are needed in forming expressions will not conflict with indices already in use (see the example under `indices`).

**idummyx**

Default value: `%`

Is the prefix for dummy indices (see the example under `indices`).

**icounter**

Default value: 1

Determines the numerical suffix to be used in generating the next dummy index in the tensor package. The prefix is determined by the option `idummy` (default: `%`).

**kdelta (L1, L2)**

is the generalized Kronecker delta function defined in the `itensor` package with `L1` the list of covariant indices and `L2` the list of contravariant indices. `kdelta([i],[j])` returns the ordinary Kronecker delta. The command `ev(expr,kdelta)` causes the evaluation of an expression containing `kdelta([],[])` to the dimension of the manifold.

In what amounts to an abuse of this notation, `itensor` also allows `kdelta` to have 2 covariant and no contravariant, or 2 contravariant and no covariant indices, in effect providing a co(ntra)variant "unit matrix" capability. This is strictly considered a programming aid and not meant to imply that `kdelta([i,j],[[]])` is a valid tensorial object.

**kdels (L1, L2)**

Symmetrized Kronecker delta, used in some calculations. For instance:
(\texttt{(%i4)} \texttt{ishow(kdels([a,b],[c,d]))})$
\begin{align*}
\begin{array}{ccc}
c & d & d \\
a & b & a \\
b & a & b \\
\end{array}
\end{align*}
(\texttt{(%t4)})
\begin{align*}
k & \text{delta} & k \text{delta} + k \text{delta} & k \text{delta} \\
a & b & a & b \\
\end{align*}
\textbf{levi\_civita} (L) [Function]

is the permutation (or Levi-Civita) tensor which yields 1 if the list \( L \) consists of an even permutation of integers, -1 if it consists of an odd permutation, and 0 if some indices in \( L \) are repeated.

\textbf{lc2kdt} (expr) [Function]

Simplifies expressions containing the Levi-Civita symbol, converting these to Kronecker-delta expressions when possible. The main difference between this function and simply evaluating the Levi-Civita symbol is that direct evaluation often results in Kronecker expressions containing numerical indices. This is often undesirable as it prevents further simplification. The \texttt{lc2kdt} function avoids this problem, yielding expressions that are more easily simplified with \texttt{rename} or \texttt{contract}.

(\texttt{(%i1)} \texttt{load("itensor");})
(\texttt{(%o1)} /share/tensor/itensor.lisp
(\texttt{(%i2) expr:ishow('levi_civita([],[],i,j))*'levi_civita([k,l],[])*a([j],[k]))}$
\begin{align*}
\begin{array}{ccc}
i & j & k \\
j & k & l \\
\end{array}
\end{align*}
(\texttt{(%t2)})
\begin{align*}
\text{levi\_civita} & \text{a} & \text{levi\_civita} \\
j & k & l \\
\end{align*}
(\texttt{(%i3)} \texttt{ishow(ev(expr,levi_civita))}$
\begin{align*}
\begin{array}{ccc}
i & j & k \\
j & k & l \\
\end{array}
\end{align*}
(\texttt{(%t3)})
\begin{align*}
k \text{delta} & \text{a} & k \text{delta} \\
1 & 2 & j \\
\end{align*}
(\texttt{(%i4)} \texttt{ishow(ev(%,kdelta))}$
\begin{align*}
\begin{array}{ccc}
i & j & i \\
j & i & k \\
\end{array}
\end{align*}
(\texttt{(%t4)} \texttt{(kdelta kdelta - kdelta kdelta) a}$
\begin{align*}
\begin{array}{ccc}
1 & 2 & 2 \\
2 & 1 & j \\
\end{array}
\end{align*}
(\texttt{(%t5)} \texttt{(kdelta kdelta - kdelta kdelta) k}$
\begin{align*}
\begin{array}{ccc}
k & k & l \\
1 & k & l \\
\end{array}
\end{align*}
(\texttt{(%i5)} \texttt{ishow(lc2kdt(expr))}$
\begin{align*}
\begin{array}{ccc}
k & i & j \\
1 & j & i \\
\end{array}
\end{align*}
(\texttt{(%t5)})
\begin{align*}
\text{a} & \text{kdelta} & \text{kdelta} - \text{a} \text{kdelta} \text{kdelta} \\
1 & k & l \\
\end{align*}
(\texttt{(%i6)} \texttt{ishow(contract(expand(%))))}$
\begin{align*}
\begin{array}{ccc}
i & \text{a} \\
\end{array}
\end{align*}
(\texttt{(%t6)})
\begin{align*}
\text{a} - \text{a} \text{kdelta} \\
1 & \text{l} \\
\end{align*}
The `lc2kdt` function sometimes makes use of the metric tensor. If the metric tensor was not defined previously with `imetric`, this results in an error.

```
(%i7) expr:ishow('levi_civita([],[i,j])
      *'levi_civita([],[k,l])*a([],[j,k],[]))$
      i j
(%t7) levi_civita levi_civita a
      j k
(%i8) ishow(lc2kdt(expr))$
Maxima encountered a Lisp error:
Error in $IMETRIC$ [or a callee]:
$IMETRIC$ [or a callee] requires less than two arguments.
Automatically continuing.
To re-enable the Lisp debugger set *debugger-hook* to nil.
(%i9) imetric(g);
     done
(%i10) ishow(lc2kdt(expr))$
     %3 i k %4 j l %3 i l %4 j
(%t10) (g kdelta g kdelta - g kdelta g
     %3 %4 %3
     k
     kdelta ) a
     %4 j k
(%i11) ishow(contract(expand(%)))$
     l i l i j
(%t11) a - g a
     j
```
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(%i5) canform(contract(expand(applyb1(el2,lc_l,lc_u))));
(%t5) 0

lc_u [Function]
Simplification rule used for expressions containing the unevaluated Levi-Civita symbol (levi_civita). Along with lc_u, it can be used to simplify many expressions more efficiently than the evaluation of levi_civita. For details, see lc_l.

canten (expr) [Function]
Simplifies expr by renaming (see rename) and permuting dummy indices. rename is restricted to sums of tensor products in which no derivatives are present. As such it is limited and should only be used if canform is not capable of carrying out the required simplification.

The canten function returns a mathematically correct result only if its argument is an expression that is fully symmetric in its indices. For this reason, canten returns an error if allsym is not set to true.

cconcat (expr) [Function]
Similar to canten but also performs index contraction.

25.2.2 Tensor symmetries

allsym [Option variable]
Default value: false

If true then all indexed objects are assumed symmetric in all of their covariant and contravariant indices. If false then no symmetries of any kind are assumed in these indices. Derivative indices are always taken to be symmetric unless iframe_flag is set to true.

decsym (tensor, m, n, [cov_1, cov_2, ...], [contr_1, contr_2, ...]) [Function]
Declares symmetry properties for tensor of m covariant and n contravariant indices. The cov_i and contr_i are pseudofunctions expressing symmetry relations among the covariant and contravariant indices respectively. These are of the form symoper(index_1, index_2, ...) where symoper is one of sym, anti or cyc and the index_i are integers indicating the position of the index in the tensor. This will declare tensor to be symmetric, antisymmetric or cyclic respectively in the index_i. symoper(all) is also an allowable form which indicates all indices obey the symmetry condition. For example, given an object b with 5 covariant indices, decsym(b,5,3,[sym(1,2),anti(3,4)],[cyc(all)]) declares b symmetric in its first and second and antisymmetric in its third and fourth covariant indices, and cyclic in all of its contravariant indices. Either list of symmetry declarations may be null. The function which performs the simplifications is canform as the example below illustrates.

(%i1) load("itensor");
(%o1) /share/tensor/itensor.lisp
(%i2) expr:contract( expand( a([i1, j1, k1], []))
*kdels([i, j, k], [i1, j1, k1]))$

(%i3) ishow(expr)$

(%o3)

\[
\begin{align*}
  &a + a + a + a + a + a \\
  &\quad k j i \quad i k j \quad i k j \quad i k j \quad i k j \quad i k j
\end{align*}
\]

done

(%o4)

(%i5) ishow(canform(expr))$

(%o5) 6 a

(%i6) remsym(a,3,0);

done

(%i7) decsym(a,3,0,[anti(all)],[]);  

done

(%o7)

(%i8) ishow(canform(expr))$

(%o8) 0

(%i9) remsym(a,3,0);

done

(%i10) decsym(a,3,0,[cyc(all)],[]);  

done

(%o10)

(%i11) ishow(canform(expr))$

(%o11) 3 a + 3 a

\[
\begin{align*}
  &\quad i k j \quad i k j
\end{align*}
\]

(%i12) dispsym(a,3,0);

(%o12) [[cyc, [[1, 2, 3]], []]]
25.2.3 Indicial tensor calculus

`diff (expr, v_1, [n_1, [v_2, n_2] ...])`  [Function]
is the usual Maxima differentiation function which has been expanded in its abilities for itensor. It takes the derivative of expr with respect to v_1 n_1 times, with respect to v_2 n_2 times, etc. For the tensor package, the function has been modified so that the v_i may be integers from 1 up to the value of the variable dim. This will cause the differentiation to be carried out with respect to the v_i-th member of the list vect_coords. If vect_coords is bound to an atomic variable, then that variable subscripted by v_i will be used for the variable of differentiation. This permits an array of coordinate names or subscripted names like x[1], x[2], ... to be used.

A further extension adds the ability to `diff` to compute derivatives with respect to an indexed variable. In particular, the tensor package knows how to differentiate expressions containing combinations of the metric tensor and its derivatives with respect to the metric tensor and its first and second derivatives. This capability is particularly useful when considering Lagrangian formulations of a gravitational theory, allowing one to derive the Einstein tensor and field equations from the action principle.

`idiff (expr, v_1, [n_1, [v_2, n_2] ...])`  [Function]
Indicial differentiation. Unlike `diff`, which differentiates with respect to an independent variable, `idiff` can be used to differentiate with respect to a coordinate. For an indexed object, this amounts to appending the v_i as derivative indices. Subsequently, derivative indices will be sorted, unless iframe_flag is set to true.

`idiff` can also differentiate the determinant of the metric tensor. Thus, if imetric has been bound to G then `idiff(determinant(g),k)` will return `2 * determinant(g) * ichr2([%i,k],[%i])` where the dummy index %i is chosen appropriately.

`liediff (v, ten)`  [Function]
Computes the Lie-derivative of the tensorial expression ten with respect to the vector field v. ten should be any indexed tensor expression; v should be the name (without indices) of a vector field. For example:

```
(%i1) load("itensor");
(%o1) /share/tensor/itensor.lisp
(%i2) ishow(liediff(v,a([i,j],[k])\*b([],[k],l)))$
   k  %2  %2  %2 
(%t2) b (v a + v a + v a )
   ,l  i j,%2  ,j i %2  ,i %2 j
   %1 k  %1 k  %1 k 
   + (v b - b v + v b ) a
   ,%1 l ,l ,%1 ,l ,%1 i j
```

`rediff (ten)`  [Function]
Evaluates all occurrences of the `idiff` command in the tensorial expression ten.
**undiff** (expr)  
Returns an expression equivalent to expr but with all derivatives of indexed objects replaced by the noun form of the idiff function. Its arguments would yield that indexed object if the differentiation were carried out. This is useful when it is desired to replace a differentiated indexed object with some function definition resulting in expr and then carry out the differentiation by saying ev(expr, idiff).

**evundiff** (expr)  
Equivalent to the execution of undiff, followed by ev and rediff. The point of this operation is to easily evaluate expressions that cannot be directly evaluated in derivative form. For instance, the following causes an error:

```
(%i1) load("itensor");
(%o1) /share/tensor/itensor.lisp
(%i2) icurvature([i,j,k],[l],[m]);
Maxima encountered a Lisp error:
Error in $ICURVATURE [or a callee]:
$ICURVATURE [or a callee] requires less than three arguments.
```

Automatically continuing.  
To re-enable the Lisp debugger set *debugger-hook* to nil.

However, if icurvature is entered in noun form, it can be evaluated using evundiff:

```
(%i3) ishow('icurvature([i,j,k],[l],[m]))$
   l
   icurvature
   i j k, m

(%i4) ishow(evundiff(%))$
   l
   icurvature
   i j k, m

(%t4) - ichr2 - ichr2 ichr2 - ichr2 ichr2
   i k, j, m %1 j i, k, m %1 j, m i, k

   l
   i

   l
   %1

   %1

   %1

   + ichr2 + ichr2 ichr2 + ichr2 ichr2
   i j, k, m %1 k i, j, m %1 k, m i, j

Note: In earlier versions of Maxima, derivative forms of the Christoffel-symbols also could not be evaluated. This has been fixed now, so evundiff is no longer necessary for expressions like this:

```
(%i5) imetric(g);
(%o5) done
(%i6) ishow(ichr2([i,j],[k],[l]))$
   k %3
   g
   (g
   j %3, i l i, j, %3 l i %3, j l

(%t6) -----------------------------------------
   2`
\[
\begin{align*}
&k \int \\
g \left( \left( g_{,l} - g_{,j} \right)_{i,j} + g_{,l} \right)_{i,j,3}^3 + \frac{\text{---------}}{2}
\end{align*}
\]

**flush (expr, tensor_1, tensor_2, ...)**

Set to zero, in expr, all occurrences of the tensor\(_i\) that have no derivative indices.

**flushd (expr, tensor_1, tensor_2, ...)**

Set to zero, in expr, all occurrences of the tensor\(_i\) that have derivative indices.

**flushnd (expr, tensor, n)**

Set to zero, in expr, all occurrences of the differentiated object tensor that have \(n\) or more derivative indices as the following example demonstrates.

```lisp
(%i1) load("itensor");
(%o1) /share/tensor/itensor.lisp
(%i2) ishow(a([i],[J,r],k,r)+a([i],[j,r,s],k,r,s))$
    J r  
    j r s
    a + a
    i, k r  i, k r s
(%t2) a + a
    i, k r  i, k r s
(%i3) ishow(flushnd(%,[a,3]))$
    J r  
    a
    i, k r
(%t3) a
    i, k r
```

**coord (tensor_1, tensor_2, ...)**

Gives tensor\(_i\) the coordinate differentiation property that the derivative of contravariant vector whose name is one of the tensor\(_i\) yields a Kronecker delta. For example, if coord(x) has been done then idiff(x([], [i]), j) gives kdelta([i], [j]). coord is a list of all indexed objects having this property.

**remcoord**

remcoord (tensor_1, tensor_2, ...)

remcoord (all)

Removes the coordinate differentiation property from the tensor\(_i\) that was established by the function coord. remcoord(all) removes this property from all indexed objects.

**makebox (expr,g)**

Display expr using the metric \(g\) such that any tensor d'Alembertian occurring in expr will be indicated using the symbol \([\cdot]\). For example, \([\cdot]p([m],[n])\) represents \(g([\cdot],[i,j])p([m],[n],i,j)\).

**conmetderiv (expr, tensor)**

Simplifies expressions containing ordinary derivatives of both covariant and contravariant forms of the metric tensor (the current restriction). For example, conmetderiv
can relate the derivative of the contravariant metric tensor with the Christoffel symbols as seen from the following:

\[
\begin{align*}
(\%i1) \text{load("itensor");} \\
(\%o1) & /share/tensor/itensor.lisp \\
(\%i2) \text{ishow(g(\[],[a,b],c))} \\
& a \ b \\
& g \\
& ,c \\
(\%i3) \text{ishow(conmetderiv(\%,g))} \\
& \%1 \ b \ a \ %1 \ a \ b \\
& - g \ ichr2 \ - g \ ichr2 \\
& %1 \ c \ %1 \ c
\end{align*}
\]

\textit{simpmetderiv}\hspace{1cm}[\text{Function}]

\texttt{simpmetderiv(expr)}

\texttt{simpmetderiv(expr,[stop])}

Simplifies expressions containing products of the derivatives of the metric tensor. Specifically, \texttt{simpmetderiv} recognizes two identities:

\[
\begin{align*}
ab ab a \\
\ g \ g + \ g \ g & = (g \ g ) = (k\delta ) = 0 \\
, d bc & bc ,d & bc ,d & c ,d
\end{align*}
\]

hence

\[
\begin{align*}
ab ab \\
\ g \ g & = - \ g \ g \\
, d bc & bc ,d
\end{align*}
\]

and

\[
\begin{align*}
ab ab \\
\ g \ g & = g \ g \\
, j ab ,i & ,i ab ,j
\end{align*}
\]

which follows from the symmetries of the Christoffel symbols.

The \texttt{simpmetderiv} function takes one optional parameter which, when present, causes the function to stop after the first successful substitution in a product expression. The \texttt{simpmetderiv} function also makes use of the global variable \texttt{flipflag} which determines how to apply a “canonical” ordering to the product indices.

Put together, these capabilities can be used to achieve powerful simplifications that are difficult or impossible to accomplish otherwise. This is demonstrated through the following example that explicitly uses the partial simplification features of \texttt{simpmetderiv} to obtain a contractible expression:
(%i1) load("itensor");
(%o1) /share/tensor/itensor.lisp
(%i2) imetric(g);
(%o2) done
(%i3) ishow(g([],[a,b])*g([],[b,c])*g([a,b],[],d)*g([b,c],[],e))$
   a b b c
   g g g g
   a b,d b c,e
(%i4) ishow(canform(%))$
erreurl has improper indices
-- an error. Quitting. To debug this try debugmode(true);
(%i5) ishow(simpmetderiv(%))$
   a b b c
   g g g g
   a b,d b c,e
(%i6) flipflag:not flipflag;
(%o6) true
(%i7) ishow(simpmetderiv(%th(2)))$
   a b b c
   g g g g
   ,d ,e a b b c
(%i8) flipflag:not flipflag;
(%o8) false
(%i9) ishow(simpmetderiv(%th(2),stop))$
   a b b c
   - g g g g
   ,e a b,d b c
(%i10) ishow(contract(%))$
   b c
   - g g
   ,e c b,d

See also weyl.dem for an example that uses simpmetderiv and conmetderiv together to simplify contractions of the Weyl tensor.

flush1deriv (expr, tensor) [Function]
Set to zero, in expr, all occurrences of tensor that have exactly one derivative index.

25.2.4 Tensors in curved spaces

imetric (g) [Function]
imetric [System variable]
Specifies the metric by assigning the variable imetric:g in addition, the contraction properties of the metric g are set up by executing the commands defcon(g), defcon(g, g, kdelta). The variable imetric (unbound by default), is bound to the metric, assigned by the imetric(g) command.
idim (n) [Function]
Sets the dimensions of the metric. Also initializes the antisymmetry properties of the
Levi-Civita symbols for the given dimension.

ichr1 ([i, j, k]) [Function]
Yields the Christoffel symbol of the first kind via the definition
\[
\frac{(g_{ik,j} + g_{jk,i} - g_{ij,k})}{2}.
\]
To evaluate the Christoffel symbols for a particular metric, the variable imetric must
be assigned a name as in the example under chr2.

ichr2 ([i, j], [k]) [Function]
Yields the Christoffel symbol of the second kind defined by the relation
\[
ichr2([i,j],[k]) = \frac{(g_{is,j} + g_{js,i} - g_{ij,s})}{2}.
\]

icurvature ([i, j, k], [h]) [Function]
Yields the Riemann curvature tensor in terms of the Christoffel symbols of the second
kind (ichr2). The following notation is used:
\[
\text{icurvature} = - ichr2 - ichr2 + ichr2
\]
\[
- ichr2 + ichr2
\]
\[
ichr2
\]
\[
ichr2
\]
\[
ichr2
\]

 covdiff (expr, v_1, v_2, ... ) [Function]
Yields the covariant derivative of expr with respect to the variables v_i in terms of
the Christoffel symbols of the second kind (ichr2). In order to evaluate these, one
should use ev(expr, ichr2).

(%i1) load("itensor");
(%o1) /share/tensor/itensor.lisp
(%i2) enterntensor();$
Enter tensor name: a;
Enter a list of the covariant indices: [i,j];
Enter a list of the contravariant indices: [k];
Enter a list of the derivative indices: [];
    k
(%t2)          a
            i j
(%i3) ishow(covdiff(%s));$
            k  %1  k  %1  k
(%t3) - a  ichr2 - a  ichr2 + a
            i %1  j s  %1 j  i s  i j,s
\[ \text{lorentz\_gauge (expr)} \quad \text{[Function]} \]

Imposes the Lorentz condition by substituting 0 for all indexed objects in expr that have a derivative index identical to a contravariant index.

\[ \text{igeodesic\_coords (expr, name)} \quad \text{[Function]} \]

Causes undifferentiated Christoffel symbols and first derivatives of the metric tensor vanish in expr. The name in the igeodesic\_coords function refers to the metric name (if it appears in expr) while the connection coefficients must be called with the names ichr1 and/or ichr2. The following example demonstrates the verification of the cyclic identity satisfied by the Riemann curvature tensor using the igeodesic\_coords function.

\[
\begin{align*}
\text{(%i1) } & \text{load("itensor");} \\
\text{(%o1) } & \text{/share/tensor/itensor.lisp} \\
\text{(%i2) } & \text{ishow(icurvature([r,s,t],[u]))$} \\
\text{(%t2) } & \text{\texttt{ichr2} \texttt{- ichr2} \texttt{ichr2} \texttt{+ ichr2} \texttt{r t,s} \texttt{\%1 s} \texttt{r t} \texttt{r s,t}} \\
\text{\%1 t} & \text{\%1 r s} \\
\text{(%i3) } & \text{ishow(igeodesic\_coords(\%,ichr2))$} \\
\text{\%1 t} & \text{\%1 r s} \\
\end{align*}
\]
25.2.5 Moving frames

Maxima now has the ability to perform calculations using moving frames. These can be orthonormal frames (tetrads, vielbeins) or an arbitrary frame.

To use frames, you must first set `iframe_flag` to `true`. This causes the Christoffel symbols, `ichr1` and `ichr2`, to be replaced by the more general frame connection coefficients `icc1` and `icc2` in calculations. Specifically, the behavior of `covdiff` and `icurvature` is changed.

The frame is defined by two tensors: the inverse frame field (`ifri`, the dual basis tetrad), and the frame metric `ifg`. The frame metric is the identity matrix for orthonormal frames, or the Lorentz metric for orthonormal frames in Minkowski spacetime. The inverse frame field defines the frame base (unit vectors). Contraction properties are defined for the frame field and the frame metric.

When `iframe_flag` is true, many `itensor` expressions use the frame metric `ifg` instead of the metric defined by `imetric` for raising and lowering indices.

IMPORTANT: Setting the variable `iframe_flag` to `true` does NOT undefine the contraction properties of a metric defined by a call to `defcon` or `imetric`. If a frame field is used, it is best to define the metric by assigning its name to the variable `imetric` and NOT invoke the `imetric` function.

Maxima uses these two tensors to define the frame coefficients (`ifc1` and `ifc2`) which form part of the connection coefficients (`icc1` and `icc2`), as the following example demonstrates:

```
(%i1) load("itensor");
(%o1) /share/tensor/itensor.lisp
(%i2) iframe_flag:true;
(%o2) true
(%i3) ishow(covdiff(v([],i),j))$
   %1
   v + icc2 v
```
(%i4) ishow(ev(%,icc2))$
\%1 j \quad %1 j
(%t4) j \quad j
\%1 i i v ifc2 + v
(%i5) ishow(ev(%,ifc2))$
\%1 i %2 i
(%t5) v ifg ifc1 + v
\%1 j %2 ,j
(%i6) ishow(ev(%,ifc1))$
\%1 i %2
v ifg (ifb - ifb + ifb )
\%1 j %2 %1 %2 %1 j %2 ,j
(%t6) -------------------------------------------------- + v
2
(%i7) ishow(ifb([a,b,c]))$
\%3 %4
(%t7) (ifri - ifri ) ifr ifr
a %3,%4 a %4,%3 b c

An alternate method is used to compute the frame bracket (ifb) if the iframe_bracket_form flag is set to false:

(%i8) block([iframe_bracket_form:false],ishow(ifb([a,b,c])))$
\%6 %5 %5 %6
(%t8) ifri (ifr ifr - ifr ifr )
a %5 b c %6 b %6 c

iframes ()
[Function]

Since in this version of Maxima, contraction identities for ifr and ifri are always defined, as is the frame bracket (ifb), this function does nothing.

ifb
[Variable]

The frame bracket. The contribution of the frame metric to the connection coefficients is expressed using the frame bracket:

- ifb + ifb + ifb
  c a b b c a a b c
ifc1 = ------------------------
abc
2

The frame bracket itself is defined in terms of the frame field and frame metric. Two alternate methods of computation are used depending on the value of frame_bracket_form. If true (the default) or if the itorsion_flag is true:
\[
\text{ifb} = \text{ifr} \quad \text{ifr} \quad (\text{ifr} - \text{ifr} - \text{ifr} \quad \text{itr}) \\
\text{abc} \quad \text{b} \quad \text{c} \quad \text{a} \quad \text{d,e} \quad \text{a,e,d} \quad \text{a,f} \quad \text{d,e}
\]

Otherwise:

\[
\text{ifb} = (\text{ifr} \quad \text{ifr} - \text{ifr} \quad \text{ifr}) \quad \text{ifr} \\
\text{abc} \quad \text{b} \quad \text{c,e} \quad \text{b,e} \quad \text{c} \quad \text{a} \quad \text{d}
\]

**icc1**

Connection coefficients of the first kind. In *itensor*, defined as

\[
icc1 = \text{ichr1} - \text{ikt1} - \text{inmc1} \\
\text{abc} \quad \text{abc} \quad \text{abc} \quad \text{abc}
\]

In this expression, if `iframe_flag` is true, the Christoffel-symbol `ichr1` is replaced with the frame connection coefficient `ifc1`. If `itorsion_flag` is false, `ikt1` will be omitted. It is also omitted if a frame base is used, as the torsion is already calculated as part of the frame bracket. Lastly, if `inonmet_flag` is false, `inmc1` will not be present.

**icc2**

Connection coefficients of the second kind. In *itensor*, defined as

\[
icc2 = \text{ichr2} - \text{ikt2} - \text{inmc2} \\
\text{ab} \quad \text{ab} \quad \text{ab} \quad \text{ab}
\]

In this expression, if `iframe_flag` is true, the Christoffel-symbol `ichr2` is replaced with the frame connection coefficient `ifc2`. If `itorsion_flag` is false, `ikt2` will be omitted. It is also omitted if a frame base is used, as the torsion is already calculated as part of the frame bracket. Lastly, if `inonmet_flag` is false, `inmc2` will not be present.

**ifc1**

Frame coefficient of the first kind (also known as Ricci-rotation coefficients.) This tensor represents the contribution of the frame metric to the connection coefficient of the first kind. Defined as:

\[
- \text{ifb} + \text{ifb} + \text{ifb} \\
\text{c} \quad \text{a} \quad \text{b} \quad \text{b} \quad \text{c} \quad \text{a} \quad \text{a} \quad \text{b} \quad \text{c} \\
\text{ifc1} = \frac{\text{ifb}}{2} \\
\text{abc}
\]
Frame coefficient of the second kind. This tensor represents the contribution of the frame metric to the connection coefficient of the second kind. Defined as a permutation of the frame bracket (ifb) with the appropriate indices raised and lowered as necessary:

\[
ifc2 \equiv ifg \left( ifc1 \right)_{ab, abd}
\]

The frame field. Contracts with the inverse frame field (ifri) to form the frame metric (ifg).

The inverse frame field. Specifies the frame base (dual basis vectors). Along with the frame metric, it forms the basis of all calculations based on frames.

The frame metric. Defaults to \( k\delta \), but can be changed using components.

The inverse frame metric. Contracts with the frame metric (ifg) to \( k\delta \).

Specifies how the frame bracket (ifb) is computed.

### 25.2.6 Torsion and nonmetricity

Maxima can now take into account torsion and nonmetricity. When the flag itorsion_flag is set to true, the contribution of torsion is added to the connection coefficients. Similarly, when the flag inonmet_flag is true, nonmetricity components are included.

The nonmetricity vector. Conformal nonmetricity is defined through the covariant derivative of the metric tensor. Normally zero, the metric tensor’s covariant derivative will evaluate to the following when inonmet_flag is set to true:

\[
g_{ij;k} = -g_{ij} \text{inm}_{ijk}
\]

Covariant permutation of the nonmetricity vector components. Defined as
\[
\text{inm} = \frac{g \text{ inm} - \text{ inm} g - g \text{ inm}}{ab \ c \ a \ bc \ ac \ b}
\]
\[
inmc1 = \frac{1}{abc} \quad 2
\]

(Substitute \(ifg\) in place of \(g\) if a frame metric is used.)

\[\text{inmc2}\]
Contravariant permutation of the nonmetricity vector components. Used in the connection coefficients if \text{inonmet\_flag} is \text{true}. Defined as:
\[
\text{inmc2} = \frac{c \ c \ cd}{ab} \quad 2
\]

(Substitute \(ifg\) in place of \(g\) if a frame metric is used.)

\[\text{ikt1}\]
Covariant permutation of the torsion tensor (also known as contorsion). Defined as:
\[
\text{ikt1} = \frac{d \ d \ d}{abc} \quad 2
\]

(Substitute \(ifg\) in place of \(g\) if a frame metric is used.)

\[\text{ikt2}\]
Contravariant permutation of the torsion tensor (also known as contorsion). Defined as:
\[
\text{ikt2} = g \text{ ikt1}
\]
\[
\text{ikt2} = g \text{ ikt1}
\]
\[
\text{ikt2} = g \text{ ikt1}
\]

(Substitute \(ifg\) in place of \(g\) if a frame metric is used.)

\[\text{itr}\]
The torsion tensor. For a metric with torsion, repeated covariant differentiation on a scalar function will not commute, as demonstrated by the following example:

\[
(%i1) \text{load("itensor");}
(%o1) /share/tensor/itensor.lisp
\]
(%i2) imetric: g;
(%o2)  g
(%i3) covdiff( covdiff( f( [], []), i), j)
    - covdiff( covdiff( f( [], []), j), i)
(%i4) ishow(%);
     \f  \f
    \text{ichr}_2 - \text{ichr}_2
    , \frac{\partial}{\partial i} \frac{\partial}{\partial j}, \frac{\partial}{\partial i} \frac{\partial}{\partial j}
(%i5) canform(%);
(%o5)  0
(%i6) itorsion_flag: true;
(%o6)  true
(%i7) covdiff( covdiff( f( [], []), i), j)
    - covdiff( covdiff( f( [], []), j), i)
(%i8) ishow(%);
     \f  \f
    \text{icc}_2 - \text{icc}_2 - \f + \f
    , \frac{\partial}{\partial i} \frac{\partial}{\partial j}, \frac{\partial}{\partial i} \frac{\partial}{\partial j}, \frac{\partial}{\partial i} \frac{\partial}{\partial j}
(%i9) ishow(canform(%))$;
     \f  \f
    \text{ikk}_2 - \text{ikk}_2
    , \frac{\partial}{\partial i} \frac{\partial}{\partial j}, \frac{\partial}{\partial i} \frac{\partial}{\partial j}
(%i10) ishow(canform(ev(%, icc2)))$;
     \f  \f
    \text{ikt}_2 - \text{ikt}_2
    , \frac{\partial}{\partial i} \frac{\partial}{\partial j}, \frac{\partial}{\partial i} \frac{\partial}{\partial j}
(%i11) ishow(canform(ev(%,ikt2)))$;
     \f  \f  \f  \f
    \text{igt}_1 - \text{igt}_1 - \f \f
    , \frac{\partial}{\partial i} \frac{\partial}{\partial j} \frac{\partial}{\partial i} \frac{\partial}{\partial j}, \frac{\partial}{\partial i} \frac{\partial}{\partial j} \frac{\partial}{\partial i} \frac{\partial}{\partial j}
(%i12) ishow(factor(canform(rename(expand(ev(%,ikt1))))))$;
     \f  \f
    \text{igt}_1 - \text{igt}_1
    , \frac{\partial}{\partial i} \frac{\partial}{\partial j}, \frac{\partial}{\partial i} \frac{\partial}{\partial j}
(%i13) decsym(itr, 2, 1, [anti(all), []]);
(%o13)  done
(%i14) defcon(g, g, kdelta);  
(%o14)  done
(%i15) subst(g, nounify(g), %th(3))$;
(%i16) ishow(canform(contract(%))$;
     \f  \f
    - \f \f
    , \frac{\partial}{\partial i} \frac{\partial}{\partial j}
### 25.2.7 Exterior algebra

The *itensor* package can perform operations on totally antisymmetric covariant tensor fields. A totally antisymmetric tensor field of rank \((0, L)\) corresponds with a differential \(L\)-form. On these objects, a multiplication operation known as the exterior product, or wedge product, is defined.

Unfortunately, not all authors agree on the definition of the wedge product. Some authors prefer a definition that corresponds with the notion of antisymmetrization: in these works, the wedge product of two vector fields, for instance, would be defined as

\[
\frac{a_i a_j - a_j a_i}{2}
\]

More generally, the product of a \(p\)-form and a \(q\)-form would be defined as

\[
\frac{1}{(p+q)!} \epsilon^{i_1..ip j_1..jq} A^{i_1..ip} B^{j_1..jq} = \epsilon^{i_1..ip j_1..jq} A^{i_1..ip} B^{j_1..jq} = \epsilon^{i_1..ip j_1..jq} A^{i_1..ip} B^{j_1..jq}
\]

where \(\epsilon\) stands for the Kronecker-delta.

Other authors, however, prefer a “geometric” definition that corresponds with the notion of the volume element:

\[
\frac{1}{(p+q)!} \epsilon^{i_1..ip j_1..jq} A^{i_1..ip} B^{j_1..jq} = \epsilon^{i_1..ip j_1..jq} A^{i_1..ip} B^{j_1..jq} = \epsilon^{i_1..ip j_1..jq} A^{i_1..ip} B^{j_1..jq}
\]

Since *itensor* is a tensor algebra package, the first of these two definitions appears to be the more natural one. Many applications, however, utilize the second definition. To resolve this dilemma, a flag has been implemented that controls the behavior of the wedge product: if \(\text{igeowedge\_flag}\) is \(\text{false}\) (the default), the first, "tensorial" definition is used, otherwise the second, "geometric" definition will be applied.

\[\text{~}\]

The wedge product operator is denoted by the tilde \(\sim\). This is a binary operator. Its arguments should be expressions involving scalars, covariant tensors of rank one, or covariant tensors of rank 1 that have been declared antisymmetric in all covariant indices.

The behavior of the wedge product operator is controlled by the \(\text{igeowedge\_flag}\) flag, as in the following example:

```lisp
(%i1) load("itensor");
(%o1) /share/tensor/itensor.lisp
(%i2) ishow(a([i])~b([j]))$
\[\frac{a_i a_j - a_j a_i}{2}\]

(\%t2) a i j b i j
```

(\%t2) \[\text{---}\]
The vertical bar | denotes the "contraction with a vector" binary operation. When a totally antisymmetric covariant tensor is contracted with a contravariant vector, the result is the same regardless which index was used for the contraction. Thus, it is possible to define the contraction operation in an index-free manner.

In the itensor package, contraction with a vector is always carried out with respect to the first index in the literal sorting order. This ensures better simplification of expressions involving the | operator. For instance:

\[
\begin{align*}
\text{(i1)} & \quad \text{load(\text{"itensor"});} \\
\text{(i2)} & \quad \text{decsym(a,2,0,[anti(all)],[]);} \\
\text{(i3)} & \quad \text{ishow(a([i,j],[[]])|v)}$
\end{align*}
\]

Note that it is essential that the tensors used with the | operator be declared totally antisymmetric in their covariant indices. Otherwise, the results will be incorrect.

\text{extdiff (expr, i)}  \\
\text{Computes the exterior derivative of expr with respect to the index i. The exterior derivative is formally defined as the wedge product of the partial derivative operator and a differential form. As such, this operation is also controlled by the setting of igeowedge_flag. For instance:}
hodge (expr)
Compute the Hodge-dual of expr. For instance:

(%i11) load("itensor");
(%o11) /share/tensor/itensor.lisp
(%i12) imetric(g);
(%o12) done
(%i13) idim(4);
(%o13) done
(%i14) icounter:100;
(%o14) 100
(%i15) decsym(A,3,0,[anti(all)],[]);
(%o15) done
(%i16) ishow(A([i,j,k],[[]]));
(%o16) A
  i j k
(%i17) ishow(canform(hodge(%)));
  %1 %2 %3 %4
levi_civita g A
  %1 %102 %2 %3 %4
(%o17) -----------------------------------------
   6
(%i18) ishow(canform(hodge(%)));
  %1 %2 %3 %8
levi_civita %4 %5 %6 %7
(%o18) levi_civita
   g
Maxima 5.46.893.g3bfbdcd58 Manual

(%i9) lc2kdt(%)$

(%i10) %,kdelta$

(%i11) ishow(canform(contract(expand(%))))$

(%t11)

igeowedge_flag
[Option variable]
Default value: false
Controls the behavior of the wedge product and exterior derivative. When set to false (the default), the notion of differential forms will correspond with that of a totally antisymmetric covariant tensor field. When set to true, differential forms will agree with the notion of the volume element.

25.2.8 Exporting TeX expressions
The itensor package provides limited support for exporting tensor expressions to TeX. Since itensor expressions appear as function calls, the regular Maxima tex command will not produce the expected output. You can try instead the tentex command, which attempts to translate tensor expressions into appropriately indexed TeX objects.

tex (expr) [Function]
To use the tentex function, you must first load tentex, as in the following example:

(%i1) load("itensor");
(%o1) /share/tensor/itensor.lisp
(%i2) load("tentex");
(%o2) /share/tensor/tentex.lisp
(%i3) idummyx:m;
(%o3) m
(%i4) ishow(icurvature([j,k,l],[i]))$

(%t4) ichr2 ichr2 - ichr2 ichr2 - ichr2

+ ichr2

(%i5) tentex(%)$

$$\Gamma_{j \, k}^{m_1} \Gamma_{l \, m_1}^{i} - \Gamma_{j \, l}^{m_1} \Gamma_{k \, m_1}^{i} - \Gamma_{j \, l, k}^{i} + \Gamma_{j \, k, l}^{i}$$

Note the use of the idummyx assignment, to avoid the appearance of the percent sign in the TeX expression, which may lead to compile errors.

NB: This version of the tentex function is somewhat experimental.
25.2.9 Interfacing with ctensor

The \texttt{itensor} package has the ability to generate Maxima code that can then be executed in the context of the \texttt{ctensor} package. The function that performs this task is \texttt{ic_convert}.

\texttt{ic_convert (eqn)}

[Function]

Converts the \texttt{itensor} equation \texttt{eqn} to a \texttt{ctensor} assignment statement. Implied sums over dummy indices are made explicit while indexed objects are transformed into arrays (the array subscripts are in the order of covariant followed by contravariant indices of the indexed objects). The derivative of an indexed object will be replaced by the noun form of \texttt{diff} taken with respect to \texttt{ct_coords} subcribed by the derivative index. The Christoffel symbols \texttt{ichr1} and \texttt{ichr2} will be translated to \texttt{lcs} and \texttt{mcs}, respectively and if \texttt{metricconvert} is \texttt{true} then all occurrences of the metric with two covariant (contravariant) indices will be renamed to \texttt{lg (ug)}. In addition, \texttt{do} loops will be introduced summing over all free indices so that the transformed assignment statement can be evaluated by just doing \texttt{ev}. The following examples demonstrate the features of this function.

\begin{verbatim}
(%i1) load("itensor");
(%o1) /share/tensor/itensor.lisp
(%i2) eqn:ishow(t([i,j],[k])=f([],[])*g([l,m],[])a([],m,j)*b([i],[l,k]))$
   k   m   l   k
   t = f a b g
   i   j   ,j   i   l   m
(%i3) ic_convert(eqn);
(%o3) for i thru dim do (for j thru dim do (for k thru dim do
    t := f sum(sum(diff(a , ct_coords ) b
       m   j   i   l   k
    g , l , 1, dim), m, 1, dim)))
   l , m
(%i4) imetric(g);
(%o4) done
(%i5) metricconvert:true;
(%o5) true
(%i6) ic_convert(eqn);
(%o6) for i thru dim do (for j thru dim do (for k thru dim do
    t := f sum(sum(diff(a , ct_coords ) b
       m   j   i   l   k
    lg , l , 1, dim), m, 1, dim)))
   l , m
\end{verbatim}

25.2.10 Reserved words

The following Maxima words are used by the \texttt{itensor} package internally and should not be redefined:
<table>
<thead>
<tr>
<th>Keyword</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>indices2()</td>
<td>Internal version of indices()</td>
</tr>
<tr>
<td>conti</td>
<td>Lists contravariant indices</td>
</tr>
<tr>
<td>covi</td>
<td>Lists covariant indices of an indexed object</td>
</tr>
<tr>
<td>deri</td>
<td>Lists derivative indices of an indexed object</td>
</tr>
<tr>
<td>name</td>
<td>Returns the name of an indexed object</td>
</tr>
<tr>
<td>concan</td>
<td></td>
</tr>
<tr>
<td>irpmom</td>
<td></td>
</tr>
<tr>
<td>lc0</td>
<td></td>
</tr>
<tr>
<td>_lc2kdt0</td>
<td></td>
</tr>
<tr>
<td>_lcprod</td>
<td></td>
</tr>
<tr>
<td>_extlc</td>
<td></td>
</tr>
</tbody>
</table>
26 ctensor

26.1 Introduction to ctensor

ctensor is a component tensor manipulation package. To use the ctensor package, type `load("ctensor")`. To begin an interactive session with ctensor, type `csetup()`. You are first asked to specify the dimension of the manifold. If the dimension is 2, 3 or 4 then the list of coordinates defaults to \([x,y]\), \([x,y,z]\) or \([x,y,z,t]\) respectively. These names may be changed by assigning a new list of coordinates to the variable \texttt{ct_coords} (described below) and the user is queried about this. Care must be taken to avoid the coordinate names conflicting with other object definitions.

Next, the user enters the metric either directly or from a file by specifying its ordinal position. The metric is stored in the matrix \(lg\). Finally, the metric inverse is computed and stored in the matrix \(ug\). One has the option of carrying out all calculations in a power series.

A sample protocol is begun below for the static, spherically symmetric metric (standard coordinates) which will be applied to the problem of deriving Einstein’s vacuum equations (which lead to the Schwarzschild solution) as an example. Many of the functions in ctensor will be displayed for the standard metric as examples.

```
(%i1) load("ctensor");
(%o1) /share/tensor/ctensor.mac
(%i2) csetup();
Enter the dimension of the coordinate system:
4;
Do you wish to change the coordinate names?
N;
Do you want to
1. Enter a new metric?
2. Enter a metric from a file?
3. Approximate a metric with a Taylor series?
1;
Answer 1, 2, 3 or 4
1;
Row 1 Column 1:
a;
Row 2 Column 2:
x^2;
Row 3 Column 3:
x^2*sin(y)^2;
Row 4 Column 4:
-d;
```
Matrix entered.
Enter functional dependencies with the DEPENDS function or 'N' if none
depends([a,d],x);
Do you wish to see the metric?
y;

(%o2) done
(%i3) christof(mcs);

(%t3) mcs = ---
      1, 1, 1 2 a
      a
      x
      1

(%t4) mcs = -
      1, 2, 2 x

(%t5) mcs = -
      1, 3, 3 x

(%t6) mcs = ---
      1, 4, 4 2 d
      d
      x

(%t7) mcs = --
      2, 2, 1 a

(%t8) mcs = ------
      2, 3, 3 sin(y)
      cos(y)

(%t9) mcs = --
      3, 3, 1 a
Chapter 26: ctensor

26.2 Functions and Variables for ctensor

26.2.1 Initialization and setup

\texttt{csetup ()}\hspace{1cm}[Function]

A function in the \texttt{ctensor} (component tensor) package which initializes the package and allows the user to enter a metric interactively. See \texttt{ctensor} for more details.

\texttt{cmetric} \hspace{1cm}[Function]

\texttt{cmetric (dis)}
\texttt{cmetric ()}

A function in the \texttt{ctensor} (component tensor) package that computes the metric inverse and sets up the package for further calculations.

If \texttt{cfframe\_flag} is \texttt{false}, the function computes the inverse metric \texttt{ug} from the (user-defined) matrix \texttt{lg}. The metric determinant is also computed and stored in the variable \texttt{gdet}. Furthermore, the package determines if the metric is diagonal and sets the value of \texttt{diagmetric} accordingly. If the optional argument \texttt{dis} is present and not equal to \texttt{false}, the user is prompted to see the metric inverse.

If \texttt{cfframe\_flag} is \texttt{true}, the function expects that the values of \texttt{fri} (the inverse frame matrix) and \texttt{lfg} (the frame metric) are defined. From these, the frame matrix \texttt{fr} and the inverse frame metric \texttt{ufg} are computed.

\texttt{ct\_coordsys}\hspace{1cm}[Function]

\texttt{ct\_coordsys (coordinate\_system, extra\_arg)}
\texttt{ct\_coordsys (coordinate\_system)}

Sets up a predefined coordinate system and metric. The argument \texttt{coordinate\_system} can be one of the following symbols:

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>Dim</th>
<th>Coordinates</th>
<th>Description/comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>cartesian2d</td>
<td>2</td>
<td>[x,y]</td>
<td>Cartesian 2D coordinate system</td>
</tr>
<tr>
<td>polar</td>
<td>2</td>
<td>[r,phi]</td>
<td>Polar coordinate system</td>
</tr>
<tr>
<td>elliptic</td>
<td>2</td>
<td>[r,phi]</td>
<td>Elliptic coord. system</td>
</tr>
<tr>
<td>confocalelliptic</td>
<td>2</td>
<td>[u,v]</td>
<td>Confocal elliptic coordinates</td>
</tr>
<tr>
<td>bipolar</td>
<td>2</td>
<td>[u,v]</td>
<td>Bipolar coord. system</td>
</tr>
</tbody>
</table>
coordinate_system can also be a list of transformation functions, followed by a list containing the coordinate variables. For instance, you can specify a spherical metric as follows:

```
(%i1) load("ctensor");
(%o1) /share/tensor/ctensor.mac
(%i2) ct_coordsys([r*cos(theta)*cos(phi),r*cos(theta)*sin(phi),
                r*sin(theta),[r,theta,phi]]);
     done
(%i3) lg:trigsimp(lg);
     \[
     \begin{bmatrix}
     1 & 0 & 0 \\
     0 & r & 0 \\
     2 & 0 & 0 \\
     \end{bmatrix}
     \]
```
Transformation functions can also be used when `cframe_flag` is `true`:

```lisp
(%i11) load("ctensor");
(%o11) /share/tensor/ctensor.mac
(%i12) cframe_flag:true;
(%o12) true
(%i13) ct_coordsys([r*cos(theta)*cos(phi),r*cos(theta)*sin(phi),
                  r*sin(theta),[r,theta,phi]]);
(%o13) done
(%i14) fri;
(%o14)
[cos(phi)cos(theta) -cos(phi) r sin(theta) -sin(phi) r cos(theta)]
[ ]
[ sin(phi)cos(theta) -sin(phi) r sin(theta) cos(phi) r cos(theta)]
[ ]
[ sin(theta) r cos(theta) 0 ]

(%i15) cmetric();
(%o15) false
(%i16) lg:trigsimp(lg);
(%o16)
[ 1 0 0 ]
[ ]
[ 2 ]
[ ]
[ 0 r 0 ]
[ ]
[ 2 2 ]
[ ]
[ 0 0 r cos (theta) ]
```

The optional argument `extra_arg` can be any one of the following:

- **cylindrical** tells `ct_coordsys` to attach an additional cylindrical coordinate.
- **minkowski** tells `ct_coordsys` to attach an additional coordinate with negative metric signature.
- **all** tells `ct_coordsys` to call `cmetric` and `christof(false)` after setting up the metric.

If the global variable `verbose` is set to `true`, `ct_coordsys` displays the values of `dim`, `ct_coords`, and either `lg` or `lfg` and `fri`, depending on the value of `cframe_flag`. 
The `init_ctensor` function reinitializes the `ctensor` package. It removes all arrays and matrices used by `ctensor`, resets all flags, resets `dim` to 4, and resets the frame metric to the Lorentz-frame.

26.2.2 The tensors of curved space

The main purpose of the `ctensor` package is to compute the tensors of curved space(time), most notably the tensors used in general relativity.

When a metric base is used, `ctensor` can compute the following tensors:

\[
\begin{align*}
lg & \quad ug \\
\backslash & \quad \backslash \\
lcs & \quad mcs \quad ric \quad uric \\
\backslash & \quad \backslash \quad \backslash \\
& \quad tracer \quad ein \quad lein \\
& \quad riem \quad lriem \quad weyl \\
& \quad \backslash \\
& \quad uriem
\end{align*}
\]

`ctensor` can also work using moving frames. When `cframe_flag` is set to `true`, the following tensors can be calculated:

\[
\begin{align*}
lfg & \quad ufg \\
\backslash & \\
fri & \quad fr \quad lcs \quad mcs \quad lriem \quad ric \quad uric \\
\backslash & \quad \backslash \quad \backslash \quad \backslash \\
& \quad lg \quad ug \\
& \quad \backslash \\
& \quad weyl \quad tracer \quad ein \quad lein \\
& \quad riem \\
\end{align*}
\]

`christof (dis)`

A function in the `ctensor` (component tensor) package. It computes the Christoffel symbols of both kinds. The argument `dis` determines which results are to be immediately displayed. The Christoffel symbols of the first and second kinds are stored in the arrays `lcs[i,j,k]` and `mcs[i,j,k]` respectively and defined to be symmetric in the first two indices. If the argument to `christof` is `lcs` or `mcs` then the unique non-zero values of `lcs[i,j,k]` or `mcs[i,j,k]`, respectively, will be displayed. If the argument is `all` then the unique non-zero values of `lcs[i,j,k]` and `mcs[i,j,k]` will be displayed. If the argument is `false` then the display of the elements will not occur.
The array elements $mcs[i,j,k]$ are defined in such a manner that the final index is contravariant.

**ricci (dis)**

A function in the **ctensor** (component tensor) package. **ricci** computes the covariant (symmetric) components $ric[i,j]$ of the Ricci tensor. If the argument *dis* is **true**, then the non-zero components are displayed.

**uricci (dis)**

This function first computes the covariant components $ric[i,j]$ of the Ricci tensor. Then the mixed Ricci tensor is computed using the contravariant metric tensor. If the value of the argument *dis* is **true**, then these mixed components, $uric[i,j]$ (the index $i$ is covariant and the index $j$ is contravariant), will be displayed directly. Otherwise, **ricci(false)** will simply compute the entries of the array $uric[i,j]$ without displaying the results.

**scurvature ()**

Returns the scalar curvature (obtained by contracting the Ricci tensor) of the Riemannian manifold with the given metric.

**einstein (dis)**

A function in the **ctensor** (component tensor) package. **einstein** computes the mixed Einstein tensor after the Christoffel symbols and Ricci tensor have been obtained (with the functions **christof** and **ricci**). If the argument *dis* is **true**, then the non-zero values of the mixed Einstein tensor $ein[i,j]$ will be displayed where $j$ is the contravariant index. The variable **rateinstein** will cause the rational simplification on these components. If **ratfac** is **true** then the components will also be factored.

**leinstein (dis)**

Covariant Einstein-tensor. **leinstein** stores the values of the covariant Einstein tensor in the array $lein$. The covariant Einstein-tensor is computed from the mixed Einstein tensor $ein$ by multiplying it with the metric tensor. If the argument *dis* is **true**, then the non-zero values of the covariant Einstein tensor are displayed.

**riemann (dis)**

A function in the **ctensor** (component tensor) package. **riemann** computes the Riemann curvature tensor from the given metric and the corresponding Christoffel symbols. The following index conventions are used:

$$R[i,j,k,l] = \frac{1}{l} | - | - | - | + | - | - | l$$

$$ijk \quad ijk \quad ikj \quad mk \quad ij \quad mj \quad ik$$

This notation is consistent with the notation used by the **itensor** package and its icurvature function. If the optional argument *dis* is **true**, the unique non-zero components $riem[i,j,k,l]$ will be displayed. As with the Einstein tensor, various switches set by the user control the simplification of the components of the Riemann tensor. If **ratriemann** is **true**, then rational simplification will be done. If **ratfac** is **true** then each of the components will also be factored.
If the variable \texttt{cframe\_flag} is \texttt{false}, the Riemann tensor is computed directly from the Christoffel-symbols. If \texttt{cframe\_flag} is \texttt{true}, the covariant Riemann-tensor is computed first from the frame field coefficients.

\textbf{lriemann} (\textit{dis}) \quad \textbf{[Function]}
\begin{verbatim}
Covariant Riemann-tensor (lriem[]).
Computes the covariant Riemann-tensor as the array lriem. If the argument \textit{dis} is \texttt{true}, unique non-zero values are displayed.
If the variable \texttt{cframe\_flag} is \texttt{true}, the covariant Riemann tensor is computed directly from the frame field coefficients. Otherwise, the (3,1) Riemann tensor is computed first.
For information on index ordering, see \texttt{riemann}.
\end{verbatim}

\textbf{uriemann} (\textit{dis}) \quad \textbf{[Function]}
\begin{verbatim}
Computes the contravariant components of the Riemann curvature tensor as array elements uriem[i,j,k,l]. These are displayed if \textit{dis} is \texttt{true}.
\end{verbatim}

\textbf{rinvariant} () \quad \textbf{[Function]}
\begin{verbatim}
Forms the Kretschmann-invariant (kinvariant) obtained by contracting the tensors lriem[i,j,k,l]*uriem[i,j,k,l].
This object is not automatically simplified since it can be very large.
\end{verbatim}

\textbf{weyl} (\textit{dis}) \quad \textbf{[Function]}
\begin{verbatim}
Computes the Weyl conformal tensor. If the argument \textit{dis} is \texttt{true}, the non-zero components weyl[i,j,k,l] will be displayed to the user. Otherwise, these components will simply be computed and stored. If the switch \texttt{ratweyl} is set to \texttt{true}, then the components will be rationally simplified; if \texttt{ratfac} is \texttt{true} then the results will be factored as well.
\end{verbatim}

\textbf{26.2.3 Taylor series expansion}

The \texttt{ctensor} package has the ability to truncate results by assuming that they are Taylor-series approximations. This behavior is controlled by the \texttt{ctayswitch} variable; when set to \texttt{true}, \texttt{ctensor} makes use internally of the function \texttt{ctaylor} when simplifying results.

The \texttt{ctaylor} function is invoked by the following \texttt{ctensor} functions:

\begin{verbatim}
<table>
<thead>
<tr>
<th>Function</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>christof()</td>
<td>For mcs only</td>
</tr>
<tr>
<td>ricci()</td>
<td></td>
</tr>
<tr>
<td>uricci()</td>
<td></td>
</tr>
<tr>
<td>einstein()</td>
<td></td>
</tr>
<tr>
<td>riemann()</td>
<td></td>
</tr>
<tr>
<td>weyl()</td>
<td></td>
</tr>
<tr>
<td>checkdiv()</td>
<td></td>
</tr>
</tbody>
</table>
\end{verbatim}
The `ctaylor` function truncates its argument by converting it to a Taylor-series using `taylor`, and then calling `ratdisrep`. This has the combined effect of dropping terms higher order in the expansion variable `ctayvar`. The order of terms that should be dropped is defined by `ctaypov`; the point around which the series expansion is carried out is specified in `ctaypt`.

As an example, consider a simple metric that is a perturbation of the Minkowski metric. Without further restrictions, even a diagonal metric produces expressions for the Einstein tensor that are far too complex:

```plaintext
(%i1) load("ctensor");
(%o1) /share/tensor/ctensor.mac
(%i2) ratfac:true;
(%o2) true
(%i3) derivabbrev:true;
(%o3) true
(%i4) ct_coords:[t,r,theta,phi];
(%o4) [t, r, theta, phi]
(%i5) lg:matrix([-1,0,0,0],[0,1,0,0],[0,0,r^2,0],[0,0,0,r^2*sin(theta)^2]);
(%o5) [ - 1 0 0 0 ]
     [ ]
     [ 0 1 0 0 ]
     [ ]
     [ 0 0 2 r 0 ]
     [ ]
     [ 0 0 0 r^2*sin(theta)^2 ]
(%i6) h:matrix([h11,0,0,0],[0,h22,0,0],[0,0,h33,0],[0,0,0,h44]);
(%o6) [ h11 0 0 0 ]
     [ ]
     [ 0 h22 0 0 ]
     [ ]
     [ 0 0 h33 0 ]
     [ ]
     [ 0 0 0 h44 ]
(%i7) depends(l,r);
(%o7) [l(r)]
(%i8) lg:lg+l*h;
     [ h11 l - 1 0  0  0  0 ]
     [ ]
     [ 0     h22 l + 1 0  0  0 ]
     [ ]
     [ 2 ]
     [ 0     0  r + h33 l  0  0 ]
```
[ 0 0 0 r sin (theta) + h44 l ]
\( \text{cmetric}(\text{false}); \) done
\( \text{einstein}(\text{false}); \) done
\( \text{ntermst}(\text{ein}); \) done
[[1, 1], 62]
[[1, 2], 0]
[[1, 3], 0]
[[1, 4], 0]
[[2, 1], 0]
[[2, 2], 24]
[[2, 3], 0]
[[2, 4], 0]
[[3, 1], 0]
[[3, 2], 0]
[[3, 3], 46]
[[3, 4], 0]
[[4, 1], 0]
[[4, 2], 0]
[[4, 3], 0]
[[4, 4], 46]
\( \text{done} \)

However, if we recompute this example as an approximation that is linear in the variable \( l \), we get much simpler expressions:

\( \text{ctayswitch:} \text{true}; \) true
\( \text{ctayvar:} l; \) 1
\( \text{ctaypov:} 1; \) 1
\( \text{ctaypt:} 0; \) 0
\( \text{christof}(\text{false}); \) done
\( \text{ricci}(\text{false}); \) done
\( \text{einstein}(\text{false}); \) done
\( \text{ntermst}(\text{ein}); \) done
[[1, 1], 6]
[[1, 2], 0]
This capability can be useful, for instance, when working in the weak field limit far from a gravitational source.

### 26.2.4 Frame fields

When the variable `cframe_flag` is set to true, the `ctensor` package performs its calculations using a moving frame.

```plaintext
frame_bracket (fr, fri, diagframe)  [Function]

The frame bracket (fb[]).

Computes the frame bracket according to the following definition:

\[
\text{ifb} = \left( \frac{\text{ifr} - \text{ifi}}{\text{ifr}} \right) \text{ifr} \text{ifr} \\
\text{ab} \quad \text{d,e} \quad \text{e,d} \quad \text{a,b}
\]
```

### 26.2.5 Algebraic classification

A new feature (as of November, 2004) of `ctensor` is its ability to compute the Petrov classification of a 4-dimensional spacetime metric. For a demonstration of this capability, see the file `share/tensor/petrov.dem`. 
nptetrad ()

Computes a Newman-Penrose null tetrad (np) and its raised-index counterpart (npi). See petrov for an example.

The null tetrad is constructed on the assumption that a four-dimensional orthonormal frame metric with metric signature (-,+,+,+) is being used. The components of the null tetrad are related to the inverse frame matrix as follows:

\[
\begin{align*}
np &= (f_{ri} + f_{ri}) / \sqrt{2} \\
&\quad (f_{ri} + f_{ri}) / \sqrt{2} \\
&\quad (f_{ri} + \%i f_{ri}) / \sqrt{2} \\
&\quad (f_{ri} - \%i f_{ri}) / \sqrt{2}
\end{align*}
\]

psi (dis)

Computes the five Newman-Penrose coefficients psi[0]...psi[4]. If dis is set to true, the coefficients are displayed. See petrov for an example.

These coefficients are computed from the Weyl-tensor in a coordinate base. If a frame base is used, the Weyl-tensor is first converted to a coordinate base, which can be a computationally expensive procedure. For this reason, in some cases it may be more advantageous to use a coordinate base in the first place before the Weyl tensor is computed. Note however, that constructing a Newman-Penrose null tetrad requires a frame base. Therefore, a meaningful computation sequence may begin with a frame base, which is then used to compute lg (computed automatically by cmetric) and then ug. See petrov for an example. At this point, you can switch back to a coordinate base by setting cframe_flag to false before beginning to compute the Christoffel symbols. Changing to a frame base at a later stage could yield inconsistent results, as you may end up with a mixed bag of tensors, some computed in a frame base, some in a coordinate base, with no means to distinguish between the two.

petrov ()

Computes the Petrov classification of the metric characterized by psi[0]...psi[4].

For example, the following demonstrates how to obtain the Petrov-classification of the Kerr metric:

\[
\begin{align*}
(\%i1) \text{load("ctensor");} \\
(\%o1) \quad /\text{share/tensor/ctensor.mac} \\
(\%i2) \quad (\text{cframe_flag:}\text{true,}\gcd:spmod,\text{ctrgsimp:}\text{true,}\text{ratfac:}\text{true}); \\
(\%o2) \quad \text{true} \\
(\%i3) \quad \text{ct_coordsys(exteriorschwarzschild,all);} \\
(\%o3) \quad \text{done}
\end{align*}
\]
\begin{verbatim}
(%i4) ug:invert(lg)$
(%i5) weyl(false);
(%o5) done
(%i6) nptetrad(true);
(%o6) done
(%t6) np =
    [ sqrt(r - 2 m) sqrt(r) ]
    [ --------------------- --------------------- 0 0 ]
    [ sqrt(2) sqrt(r) sqrt(2) sqrt(r - 2 m) ]
    [ ]
    [ sqrt(r - 2 m) sqrt(r) ]
    [ --------------------- - --------------------- 0 0 ]
    [ sqrt(2) sqrt(r) sqrt(2) sqrt(r - 2 m) ]
    [ ]
    [ r %i r sin(theta) ]
    [ 0 0 ------- --------------- ]
    [ sqrt(2) sqrt(2) ]
    [ ]
    [ r %i r sin(theta)]
    [ 0 0 ------- - ---------------]
    [ sqrt(2) sqrt(2) ]

(%t7) npi = matrix([- ---------------------,---------------, 0, 0],
                    [sqrt(2) sqrt(r - 2 m) sqrt(2) sqrt(r)])

(%t8) psi(true);

(%t9) psi = 0
\end{verbatim}
The Petrov classification function is based on the algorithm published in "Classifying geometries in general relativity: III Classification in practice" by Pollney, Skea, and d’Inverno, Class. Quant. Grav. 17 2885-2902 (2000). Except for some simple test cases, the implementation is untested as of December 19, 2004, and is likely to contain errors.

26.2.6 Torsion and nonmetricity

ctensor has the ability to compute and include torsion and nonmetricity coefficients in the connection coefficients.

The torsion coefficients are calculated from a user-supplied tensor tr, which should be a rank (2,1) tensor. From this, the torsion coefficients kt are computed according to the following formulae:

\[
kt = \frac{-g_{im} tr_{kj} - g_{mj} tr_{ki} - tr_{ij} g_{km}}{2}
\]

Note that only the mixed-index tensor is calculated and stored in the array kt.

The nonmetricity coefficients are calculated from the user-supplied nonmetricity vector nm. From this, the nonmetricity coefficients nmc are computed as follows:

\[
nmc = \frac{-nm_{ik} D_{jm} - D_{nm} + g_{nm} g_{ij} m_{ij}}{m_{ij}}
\]
where D stands for the Kronecker-delta.

When `ctorsion_flag` is set to `true`, the values of `kt` are subtracted from the mixed-indexed connection coefficients computed by `christof` and stored in `mcs`. Similarly, if `cnonmet_flag` is set to `true`, the values of `nm` are subtracted from the mixed-indexed connection coefficients.

If necessary, `christof` calls the functions `contortion` and `nonmetricity` in order to compute `kt` and `nm`.

**Function**

Contortion (tr)

Computes the (2,1) contortion coefficients from the torsion tensor `tr`.

**Function**

Nonmetricity (nm)

Computes the (2,1) nonmetricity coefficients from the nonmetricity vector `nm`.

### 26.2.7 Miscellaneous features

**Function**

ctransform (M)

A function in the `ctensor` (component tensor) package which will perform a coordinate transformation upon an arbitrary square symmetric matrix `M`. The user must input the functions which define the transformation. (Formerly called `transform`.) These may also be supplied in the form of a list as an optional second argument.

**Function**

findde (A, n)

returns a list of the unique differential equations (expressions) corresponding to the elements of the `n` dimensional square array `A`. Presently, `n` may be 2 or 3. `deindex` is a global list containing the indices of `A` corresponding to these unique differential equations. For the Einstein tensor (`ein`), which is a two dimensional array, if computed for the metric in the example below, `findde` gives the following independent differential equations:

```
(%i1) load("ctensor");
(%o1) /share/tensor/ctensor.mac
(%i2) derivabbrev:true;
(%o2) true
(%i3) dim:4;
(%o3) 4
(%i4) lg:matrix([a, 0, 0, 0], [0, x^2, 0, 0],
                 [0, 0, x^2*sin(y)^2, 0], [0,0,0,-d]);
(%o4)
   [ a  0  0  0 ]
   [            ]
   [ 0 x^2  0  0 ]
   [            ]
   [ 2        ]
   [            ]
   [ 0 x  0  0 ]
   [            ]
   [ 0 0 x sin(y) 0 ]
   [ 0 0  0   ]
```
[ 0 0 0 - d ]
(\%i5) depends([a,d],x);
(\%o5) [a(x), d(x)]
(\%i6) ct_coords:[x,y,z,t];
(\%o6) [x, y, z, t]
(\%i7) cmetric();
(\%o7) done
(\%i8) einstein(false);
(\%o8) done
(\%i9) findde(ein,2);
(\%o9) 
2
[dx - a dx + dx, 2 ad dx - a(dx) x - ad x dx
x x x x x
2
+ 2 ad x - 2 ad , a + a - a]

x x x
(\%i10) deindex;
(\%o10) [[1, 1], [2, 2], [4, 4]]

cograd ()
[Function]
Computes the covariant gradient of a scalar function allowing the user to choose the corresponding vector name as the example under contragrad illustrates.

contragrad ()
[Function]
Computes the contravariant gradient of a scalar function allowing the user to choose the corresponding vector name as the example below for the Schwarzschild metric illustrates:

(\%i1) load("ctensor");
(\%o1) /share/tensor/ctensor.mac
(\%i2) derivabbrev:true;
(\%o2) true
(\%i3) ct_coordsys(exteriorschwarzschild,all);
(\%o3) done
(\%i4) depends(f,r);
(\%o4) [f(r)]
(\%i5) cograd(f,g1);
(\%o5) done
(\%i6) listarray(g1);
(\%o6) [0, f, 0, 0]
(\%i7) contragrad(f,g2);
(\%o7) done
(\%i8) listarray(g2);
(\%o8) [0, ------------, 0, 0]
**dscalar ()**

computes the tensor d'Alembertian of the scalar function once dependencies have been declared upon the function. For example:

```
(%i1) load("ctensor");
(%o1) /share/tensor/ctensor.mac
(%i2) derivabbrev:true;
(%o2) true
(%i3) ct_coordsys(exteriorschwarzschild,all);
(%o3) done
(%i4) depends(p,r);
(%o4) [p(r)]
(%i5) factor(dscalar(p));
2     2
-----
  r
```

**checkdiv ()**

computes the covariant divergence of the mixed second rank tensor (whose first index must be covariant) by printing the corresponding n components of the vector field (the divergence) where n = dim. If the argument to the function is g then the divergence of the Einstein tensor will be formed and must be zero. In addition, the divergence (vector) is given the array name div.

**cgeodesic (dis)**

A function in the ctensor (component tensor) package. cgeodesic computes the geodesic equations of motion for a given metric. They are stored in the array geod[i]. If the argument dis is true then these equations are displayed.

**bdvac (f)**

generates the covariant components of the vacuum field equations of the Brans-Dicke gravitational theory. The scalar field is specified by the argument f, which should be a (quoted) function name with functional dependencies, e.g., 'p(x).

The components of the second rank covariant field tensor are represented by the array bd.

**invariant1 ()**

generates the mixed Euler-Lagrange tensor (field equations) for the invariant density of R^2. The field equations are the components of an array named inv1.

**invariant2 ()**

*** NOT YET IMPLEMENTED ***

generates the mixed Euler-Lagrange tensor (field equations) for the invariant density of ric[i,j]*uriem[i,j]. The field equations are the components of an array named inv2.
bimetric ()
*** NOT YET IMPLEMENTED ***
generates the field equations of Rosen's bimetric theory. The field equations are the components of an array named rosen.

26.2.8 Utility functions
diagmatrixp (M,n)
Returns true if the first n rows and n columns of M form a diagonal matrix or (2D) array.
symmetricp (M,n)
Returns true if M is a n by n symmetric matrix or two-dimensional array, otherwise false.
If n is less than the size of M, symmetricp considers only the n by n submatrix (respectively, subarray) comprising rows 1 through n and columns 1 through n.

ntermst (f)
gives the user a quick picture of the "size" of the doubly subscripted tensor (array) f. It prints two element lists where the second element corresponds to NTERMS of the components specified by the first elements. In this way, it is possible to quickly find the non-zero expressions and attempt simplification.
cdisplay (ten)
displays all the elements of the tensor ten, as represented by a multidimensional array. Tensors of rank 0 and 1, as well as other types of variables, are displayed as with ldisplay. Tensors of rank 2 are displayed as 2-dimensional matrices, while tensors of higher rank are displayed as a list of 2-dimensional matrices. For instance, the Riemann-tensor of the Schwarzschild metric can be viewed as:

```maxima
%i1) load("ctensor");
%i1) /share/tensor/ctensor.mac
%i2) ratfac:true;
%i2) true
%i3) ct_coordsys(exteriorschwarzschild,all);
%i3) done
%i4) riemann(false);
%i4) done
%i5) cdisplay(riem);
```

```maxima
riem = [
 0                   0                   0                   0
 0                   0                   0                   0
 0                   0                   0                   0
 3 m (r - 2 m) m 2 m
 0 - --------------- + -- + ---- 0                   0
 4                   3                   4
 4                   3                   4
 r                   r                   r
 r                   r                   r
 m (r - 2 m)
 1, 1 [ 0                   0                   -- ---  0 ]
```
\[
\text{riem} = \begin{bmatrix}
1, 2 & [0 & 0 & 0 & 0] \\
1, 3 & [0 & 0 & 0 & 0] \\
1, 4 & [0 & 0 & 0 & 0] \\
2, 1 & [r (r - 2m)]
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
2m (r - 2m) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
2m & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
m (r - 2m) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
m (r - 2m) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
2, 2 & m \\
0, 0, 0, 0 & 2 \\
r (r - 2m) \\
0, 0, 0, 0 & 0
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
2, 3 & m \\
0, 0, 0, 0 & 2 \\
r (r - 2m) \\
0, 0, 0, 0 & 0
\end{bmatrix}
\]

\[
\text{riem} = \begin{bmatrix}
2, 4 & m \\
0, 0, 0, 0 & 2 \\
r (r - 2m) \\
0, 0, 0, 0 & 0
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
\begin{array}{cccc}
\end{array}
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
\begin{array}{cccc}
-0&0&0
\end{array}
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
\begin{array}{cccc}
0&-0&0
\end{array}
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
\begin{array}{cccc}
0&0&0
\end{array}
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
\begin{array}{cccc}
0&0&0
\end{array}
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
\begin{array}{cccc}
0&0&0
\end{array}
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
\begin{array}{cccc}
2&-m&-r
\end{array}
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
\begin{array}{cccc}
0&0&0&-1
\end{array}
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
\begin{array}{cccc}
0&0&0&0
\end{array}
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
\begin{array}{cccc}
0&0&0&0
\end{array}
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
\begin{array}{cccc}
0&0&0
\end{array}
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
\begin{array}{cccc}
0&0&0
\end{array}
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
\begin{array}{cccc}
0&0&0
\end{array}
\end{bmatrix}
\]
\[
\text{riem} = \begin{bmatrix}
\begin{array}{cccc}
0&0&0
\end{array}
\end{bmatrix}
\]
riem = [ 0 0 0 0 ]
4, 2
[ 2 ]
[ m sin (theta) ]
[ 0 0 0 0 ]
[ r ]

riem = [ 0 0 0 0 ]
4, 3
[ 2 ]
[ 2 m sin (theta) ]
[ 0 0 - -------------- 0 ]
[ r ]

riem = [ 0 0 0 0 ]
4, 4
[ 2 ]
[ m sin (theta) ]
[ - -------------- 0 0 ]
[ r ]
[ 0 0 0 0 ]

(%o5) done
deleten \((L, n)\)  
\[\text{[Function]}\]  
Returns a new list consisting of \(L\) with the \(n\)'th element deleted.

### 26.2.9 Variables used by ctensor

**dim**  
[Option variable]  
Default value: 4  
An option in the ctensor (component tensor) package. \(\text{dim}\) is the dimension of the manifold with the default 4. The command \(\text{dim: n}\) will reset the dimension to any other value \(n\).

**diagmetric**  
[Option variable]  
Default value: false  
An option in the ctensor (component tensor) package. If \(\text{diagmetric}\) is true special routines compute all geometrical objects (which contain the metric tensor explicitly) by taking into consideration the diagonality of the metric. Reduced run times will, of course, result. Note: this option is set automatically by csetup if a diagonal metric is specified.

**ctrgsimp**  
[Option variable]  
Causes trigonometric simplifications to be used when tensors are computed. Presently, ctrgsimp affects only computations involving a moving frame.

**cframe_flag**  
[Option variable]  
Causes computations to be performed relative to a moving frame as opposed to a holonomic metric. The frame is defined by the inverse frame array \(\text{fri}\) and the frame metric \(\text{lfg}\). For computations using a Cartesian frame, \(\text{lfg}\) should be the unit matrix of the appropriate dimension; for computations in a Lorentz frame, \(\text{lfg}\) should have the appropriate signature.

**ctorsion_flag**  
[Option variable]  
Causes the contortion tensor to be included in the computation of the connection coefficients. The contortion tensor itself is computed by contortion from the user-supplied tensor \(\text{tr}\).

**cnonmet_flag**  
[Option variable]  
Causes the nonmetricity coefficients to be included in the computation of the connection coefficients. The nonmetricity coefficients are computed from the user-supplied nonmetricity vector \(\text{nm}\) by the function nonmetricity.

**ctayswitch**  
[Option variable]  
If set to true, causes some ctensor computations to be carried out using Taylor-series expansions. Presently, christof, ricci, uricci, einstein, and weyl take into account this setting.

**ctayvar**  
[Option variable]  
Variable used for Taylor-series expansion if ctayswitch is set to true.

**ctaypov**  
[Option variable]  
Maximum power used in Taylor-series expansion when ctayswitch is set to true.
ctaypt \hspace{1em} \text{[Option variable]} \\
Point around which Taylor-series expansion is carried out when \texttt{ctayswitch} is set to \texttt{true}.

gdet \hspace{1em} \text{[System variable]} \\
The determinant of the metric tensor \texttt{lg}. Computed by \texttt{cmetric} when \texttt{cframe\_flag} is set to \texttt{false}.

ratchristof \hspace{1em} \text{[Option variable]} \\
Causes rational simplification to be applied by \texttt{christof}.

rateinstein \hspace{1em} \text{[Option variable]} \\
Default value: \texttt{true} \\
If \texttt{true} rational simplification will be performed on the non-zero components of Ein-stein tensors; if \texttt{ratfac} is \texttt{true} then the components will also be factored.

ratriemann \hspace{1em} \text{[Option variable]} \\
Default value: \texttt{true} \\
One of the switches which controls simplification of Riemann tensors; if \texttt{true}, then rational simplification will be done; if \texttt{ratfac} is \texttt{true} then each of the components will also be factored.

ratweyl \hspace{1em} \text{[Option variable]} \\
Default value: \texttt{true} \\
If \texttt{true}, this switch causes the \texttt{weyl} function to apply rational simplification to the values of the Weyl tensor. If \texttt{ratfac} is \texttt{true}, then the components will also be factored.

lfg \hspace{1em} \text{[Variable]} \\
The covariant frame metric. By default, it is initialized to the 4-dimensional Lorentz frame with signature (+,+,+,-). Used when \texttt{cframe\_flag} is \texttt{true}.

ufg \hspace{1em} \text{[Variable]} \\
The inverse frame metric. Computed from \texttt{lfg} when \texttt{cmetric} is called while \texttt{cframe\_flag} is set to \texttt{true}.

riem \hspace{1em} \text{[Variable]} \\
The (3,1) Riemann tensor. Computed when the function \texttt{riemann} is invoked. For information about index ordering, see the description of \texttt{riemann}.

\text{If \texttt{cframe\_flag} is \texttt{true}, \texttt{riem} is computed from the covariant Riemann-tensor \texttt{lriem}.}

lriem \hspace{1em} \text{[Variable]} \\
The covariant Riemann tensor. Computed by \texttt{lriemann}.

uriem \hspace{1em} \text{[Variable]} \\
The contravariant Riemann tensor. Computed by \texttt{uriemann}.

ric \hspace{1em} \text{[Variable]} \\
The covariant Ricci-tensor. Computed by \texttt{ricci}.
uric
The mixed-index Ricci-tensor. Computed by uricci.

lg
The metric tensor. This tensor must be specified (as a dim by dim matrix) before other computations can be performed.

ug
The inverse of the metric tensor. Computed by cmetric.

weyl
The Weyl tensor. Computed by weyl.

fb
Frame bracket coefficients, as computed by frame_bracket.

kinvariant
The Kretschmann invariant. Computed by rinvariant.

np
A Newman-Penrose null tetrad. Computed by nptetrad.

npi
The raised-index Newman-Penrose null tetrad. Computed by nptetrad. Defined as ug.np. The product np.transpose(npi) is constant:

\[
\begin{bmatrix}
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

tr
User-supplied rank-3 tensor representing torsion. Used by contortion.

kt
The contortion tensor, computed from tr by contortion.

nm
User-supplied nonmetricity vector. Used by nonmetricity.

nmc
The nonmetricity coefficients, computed from nm by nonmetricity.

tensorkill
Variable indicating if the tensor package has been initialized. Set and used by csetup, reset by init_ctensor.
ct_coords

Default value: []

An option in the ctensor (component tensor) package. ct_coords contains a list of coordinates. While normally defined when the function csetup is called, one may redefine the coordinates with the assignment ct_coords: [j1, j2, ..., jn] where the j's are the new coordinate names. See also csetup.

### 26.2.10 Reserved names

The following names are used internally by the ctensor package and should not be redefined:

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>_lg()</td>
<td>Evaluates to lfg if frame metric used, lg otherwise</td>
</tr>
<tr>
<td>_ug()</td>
<td>Evaluates to ufg if frame metric used, ug otherwise</td>
</tr>
<tr>
<td>cleanup()</td>
<td>Removes items from the deindex list</td>
</tr>
<tr>
<td>contract4()</td>
<td>Used by psi()</td>
</tr>
<tr>
<td>filemet()</td>
<td>Used by csetup() when reading the metric from a file</td>
</tr>
<tr>
<td>findde1()</td>
<td>Used by findde()</td>
</tr>
<tr>
<td>findde2()</td>
<td>Used by findde()</td>
</tr>
<tr>
<td>findde3()</td>
<td>Used by findde()</td>
</tr>
<tr>
<td>kdelta()</td>
<td>Kronecker-delta (not generalized)</td>
</tr>
<tr>
<td>newmet()</td>
<td>Used by csetup() for setting up a metric interactively</td>
</tr>
<tr>
<td>setflags()</td>
<td>Used by init_ctensor()</td>
</tr>
<tr>
<td>readvalue()</td>
<td></td>
</tr>
<tr>
<td>resimp()</td>
<td></td>
</tr>
<tr>
<td>sermet()</td>
<td>Used by csetup() for entering a metric as Taylor-series</td>
</tr>
<tr>
<td>txyzsum()</td>
<td></td>
</tr>
<tr>
<td>tmetric()</td>
<td>Frame metric, used by cmetric() when cframe_flag:true</td>
</tr>
<tr>
<td>triemann()</td>
<td>Riemann-tensor in frame base, used when cframe_flag:true</td>
</tr>
<tr>
<td>tricci()</td>
<td>Ricci-tensor in frame base, used when cframe_flag:true</td>
</tr>
<tr>
<td>trrc()</td>
<td>Ricci rotation coefficients, used by christof()</td>
</tr>
<tr>
<td>yesp()</td>
<td></td>
</tr>
</tbody>
</table>

### 26.2.11 Changes

In November, 2004, the ctensor package was extensively rewritten. Many functions and variables have been renamed in order to make the package compatible with the commercial version of Macsyma.

<table>
<thead>
<tr>
<th>New Name</th>
<th>Old Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>ctaylor()</td>
<td>DLGTAYLOR()</td>
<td>Taylor-series expansion of an expression</td>
</tr>
<tr>
<td>lggeod[]</td>
<td>EM</td>
<td>Geodesic equations</td>
</tr>
<tr>
<td>ein[]</td>
<td>G[]</td>
<td>Mixed Einstein-tensor</td>
</tr>
<tr>
<td>ric[]</td>
<td>LR[]</td>
<td>Mixed Ricci-tensor</td>
</tr>
<tr>
<td>ricci()</td>
<td>LRICCICOM()</td>
<td>Compute the mixed Ricci-tensor</td>
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<tr>
<td>ctaypov</td>
<td>MINP</td>
<td>Maximum power in Taylor-series expansion</td>
</tr>
<tr>
<td>cgeodesic()</td>
<td>MOTION</td>
<td>Compute geodesic equations</td>
</tr>
<tr>
<td>ct_coords</td>
<td>OMEGA</td>
<td>Metric coordinates</td>
</tr>
</tbody>
</table>
ctayvar  PARAM    
|lriem[]  R[]       | Covariant Riemann-tensor |
|uriemann() RAISERIEMANN() | Compute the contravariant Riemann-tensor |
|ratriemann RATRIEMAN | Rational simplif. of the Riemann-tensor |
|uric[]  RICCI[]   | Contravariant Ricci-tensor |
|uricci() RICCICOM() | Compute the contravariant Ricci-tensor |
|cmetric() SETMETRIC() | Set up the metric |
|ctaypt TAYPT | Point for Taylor-series expansion |
|ctayswitch TAYSWITCH | Taylor-series setting switch |
|csetup() TSETUP() | Start interactive setup session |
|ctransform() TTRANSFORM() | Interactive coordinate transformation |
|uriem[] UR[] | Contravariant Riemann-tensor |
|weyl[]  W[] | (3,1) Weyl-tensor |
27 atensor

27.1 Introduction to atensor

atensor is an algebraic tensor manipulation package. To use atensor, type load("atensor"), followed by a call to the init_atensor function.

The essence of atensor is a set of simplification rules for the noncommutative (dot) product operator ("."). atensor recognizes several algebra types; the corresponding simplification rules are put into effect when the init_atensor function is called.

The capabilities of atensor can be demonstrated by defining the algebra of quaternions as a Clifford-algebra Cl(0,2) with two basis vectors. The three quaternionic imaginary units are then the two basis vectors and their product, i.e.:

\[
i = v_1 \\
\hat{j} = v_2 \\
\hat{k} = v_1 \cdot v_2
\]

Although the atensor package has a built-in definition for the quaternion algebra, it is not used in this example, in which we endeavour to build the quaternion multiplication table as a matrix:

```
(%i1) load("atensor");
(%o1) /share/tensor/atensor.mac
(%i2) init_atensor(clifford,0,0,2);
(%o2) done
(%i3) atensimp(v[1].v[1]);
(%o3) - 1
(%i4) atensimp((v[1].v[2]).(v[1].v[2]));
(%o4) - 1
(%i5) q:zeromatrix(4,4);
   [ 0 0 0 0 ]
   [        ]
   [ 0 0 0 0 ]
   [        ]
   [ 0 0 0 0 ]
   [        ]
(%o5) [ 0 0 0 0 ]
(%i6) q[1,1]:1;
(%o6) 1
(%i7) for i thru adim do q[1,i+1]:q[i+1,1]:v[i];
(%o7) done
(%i8) q[1,4]:q[4,1]:v[1].v[2];
(%o8) v . v
   1 2
(%i9) for i from 2 thru 4 do for j from 2 thru 4 do
   q[i,j]:atensimp(q[i,1].q[1,j]);
(%o9) done
(%i10) q;
```
atensor recognizes as base vectors indexed symbols, where the symbol is that stored in asymbol and the index runs between 1 and adim. For indexed symbols, and indexed symbols only, the bilinear forms sf, af, and av are evaluated. The evaluation substitutes the value of aform[i,j] in place of fun(v[i],v[j]) where v represents the value of asymbol and fun is either af or sf; or, it substitutes v[aform[i,j]] in place of av(v[i],v[j]).

Needless to say, the functions sf, af and av can be redefined.

When the atensor package is loaded, the following flags are set:

```
dotscrules:true;
dotdistrib:true;
dotexptsimp:false;
```

If you wish to experiment with a nonassociative algebra, you may also consider setting dotassoc to false. In this case, however, atensimp will not always be able to obtain the desired simplifications.

### 27.2 Functions and Variables for atensor

**init_atensor**

```
init_atensor (alg_type, opt_dims)
init_atensor (alg_type)
```

Initializes the atensor package with the specified algebra type. alg_type can be one of the following:

- **universal**: The universal algebra has no commutation rules.
- **grassmann**: The Grassman algebra is defined by the commutation relation \( u \cdot v + v \cdot u = 0 \).
- **clifford**: The Clifford algebra is defined by the commutation relation \( u \cdot v + v \cdot u = -2sf(u,v) \) where sf is a symmetric scalar-valued function. For this algebra, opt_dims can be up to three nonnegative integers, representing the number of positive, degenerate, and negative dimensions of the algebra, respectively. If any opt_dims values are supplied, atensor will configure the values of adim and aform appropriately. Otherwise, adim will default to 0 and aform will not be defined.
- **symmetric**: The symmetric algebra is defined by the commutation relation \( u \cdot v - v \cdot u = 0 \).
- **symplectic**: The symplectic algebra is defined by the commutation relation \( u \cdot v - v \cdot u = 2af(u,v) \) where af is an antisymmetric scalar-valued function. For the
symplectic algebra, opt_dims can be up to two nonnegative integers, representing the nondegenerate and degenerate dimensions, respectively. If any opt_dims values are supplied, atensor will configure the values of adim and aform appropriately. Otherwise, adim will default to 0 and aform will not be defined.

lie_envelop: The algebra of the Lie envelope is defined by the commutation relation \( u.v - v.u = 2*av(u,v) \) where \( av \) is an antisymmetric function.

The init_atensor function also recognizes several predefined algebra types:

- complex implements the algebra of complex numbers as the Clifford algebra Cl(0,1).
  The call init_atensor(complex) is equivalent to init_atensor(clifford,0,0,1).

- quaternion implements the algebra of quaternions. The call init_atensor(quaternion) is equivalent to init_atensor(clifford,0,0,2).

- pauli implements the algebra of Pauli-spinors as the Clifford-algebra Cl(3,0). A call to init_atensor(pauli) is equivalent to init_atensor(clifford,3).

- dirac implements the algebra of Dirac-spinors as the Clifford-algebra Cl(3,1). A call to init_atensor(dirac) is equivalent to init_atensor(clifford,3,0,1).

\( \text{atensimp} \ (\text{expr}) \) [Function]
Simplifies an algebraic tensor expression expr according to the rules configured by a call to init_atensor. Simplification includes recursive application of commutation relations and resolving calls to sf, af, and av where applicable. A safeguard is used to ensure that the function always terminates, even for complex expressions.

\( \text{alg_type} \) [Function]
The algebra type. Valid values are universal, grassmann, clifford, symmetric, symplectic and lie_envelop.

\( \text{adim} \) [Variable]
Default value: 0
The dimensionality of the algebra. atensor uses the value of adim to determine if an indexed object is a valid base vector. See abasep.

\( \text{aform} \) [Variable]
Default value: \( \text{ident}(3) \)
Default values for the bilinear forms sf, af, and av. The default is the identity matrix \( \text{ident}(3) \).

\( \text{asymbol} \) [Variable]
Default value: v
The symbol for base vectors.

\( \text{sf} \ (u, v) \) [Function]
A symmetric scalar function that is used in commutation relations. The default implementation checks if both arguments are base vectors using abasep and if that is the case, substitutes the corresponding value from the matrix aform.

\( \text{af} \ (u, v) \) [Function]
An antisymmetric scalar function that is used in commutation relations. The default implementation checks if both arguments are base vectors using abasep and if that is the case, substitutes the corresponding value from the matrix aform.
av (u, v)  [Function]
An antisymmetric function that is used in commutation relations. The default implementation checks if both arguments are base vectors using abasep and if that is the case, substitutes the corresponding value from the matrix aform.

For instance:

```
(%i1) load("atensor");
(%o1) /share/tensor/atensor.mac
(%i2) adim:3;
(%o2) 3
(%i3) aform:matrix([0,3,-2],[-3,0,1],[2,-1,0]);
     [ 0  3  -2 ]
   [        ]
(%o3) [ -3  0  1 ]
     [        ]
     [  2  -1  0 ]
(%i4) asymbol:x;
(%o4) x
(%i5) av(x[1],x[2]);
(%o5) x
3
```

abasep (v)  [Function]
Checks if its argument is an atensor base vector. That is, if it is an indexed symbol, with the symbol being the same as the value of asymbol, and the index having a numeric value between 1 and adim.
28 Sums, Products, and Series

28.1 Functions and Variables for Sums and Products

bashindices (expr) [Function]
Transforms the expression expr by giving each summation and product a unique index. This gives changevar greater precision when it is working with summations or products. The form of the unique index is jnumber. The quantity number is determined by referring to gensumnum, which can be changed by the user. For example, gensumnum:0$ resets it.

lsum (expr, x, L) [Function]
Represents the sum of expr for each element x in L. A noun form 'lsum is returned if the argument L does not evaluate to a list.

Examples:
(%i1) lsum (x^i, i, [1, 2, 7]);
  7 2
(%o1) x + x + x
(%i2) lsum (i^2, i, rootsof (x^3 - 1, x));
  \ 2
(%o2) > i
  /
  ===
  3
  i in rootsof(x - 1, x)

intosum (expr) [Function]
Moves multiplicative factors outside a summation to inside. If the index is used in the outside expression, then the function tries to find a reasonable index, the same as it does for sumcontract. This is essentially the reverse idea of the outative property of summations, but note that it does not remove this property, it only bypasses it.

In some cases, a scanmap (multthru, expr) may be necessary before the intosum.

simpproduct [Option variable]
Default value: false

When simpproduct is true, the result of a product is simplified. This simplification may sometimes be able to produce a closed form. If simpproduct is false or if the quoted form 'product is used, the value is a product noun form which is a representation of the pi notation used in mathematics.

product (expr, i, i_0, i_1) [Function]
Represents a product of the values of expr as the index i varies from i_0 to i_1. The noun form 'product is displayed as an uppercase letter pi.

product evaluates expr and lower and upper limits i_0 and i_1, product quotes (does not evaluate) the index i.
If the upper and lower limits differ by an integer, \( expr \) is evaluated for each value of the index \( i \), and the result is an explicit product.

Otherwise, the range of the index is indefinite. Some rules are applied to simplify the product. When the global variable \( \text{simpproduct} \) is \( \text{true} \), additional rules are applied.

In some cases, simplification yields a result which is not a product; otherwise, the result is a noun form \( \text{'}product\text{'} \).

See also \text{nouns} and \text{evflag}.

Examples:

\[
\begin{align*}
(\%i1) & \quad \text{product} \ (x + i*(i+1)/2, i, 1, 4); \\
(\%o1) & \quad (x + 1) (x + 3) (x + 6) (x + 10) \\
(\%i2) & \quad \text{product} \ (i^2, i, 1, 7); \\
(\%o2) & \quad 25401600 \\
(\%i3) & \quad \text{product} \ (a[i], i, 1, 7); \\
(\%o3) & \quad a a a a a a a \\
& \quad 1 2 3 4 5 6 7 \\
(\%i4) & \quad \text{product} \ (a(i), i, 1, 7); \\
(\%o4) & \quad a(1) a(2) a(3) a(4) a(5) a(6) a(7) \\
(\%i5) & \quad \text{product} \ (a(i), i, 1, n); \\
& \quad n \\
& \quad /===\ \\
& \quad ! ! \\
(\%o5) & \quad ! ! a(i) \\
& \quad ! ! \\
& \quad i = 1 \\
(\%i6) & \quad \text{product} \ (k, k, 1, n); \\
& \quad n \\
& \quad /===\ \\
& \quad ! ! \\
(\%o6) & \quad ! ! k \\
& \quad ! ! \\
& \quad k = 1 \\
(\%i7) & \quad \text{product} \ (k, k, 1, n), \text{simpproduct}; \\
(\%o7) & \quad n! \\
(\%i8) & \quad \text{product} \ (\text{integrate} \ (x^k, x, 0, 1), k, 1, n); \\
& \quad n \\
& \quad /===\ \\
& \quad ! ! 1 \\
(\%o8) & \quad ! ! ---- \\
& \quad ! ! k + 1 \\
& \quad k = 1 \\
(\%i9) & \quad \text{product} \ (\text{if} \ k \leq 5 \ \text{then} \ a^k \ \text{else} \ b^k, k, 1, 10); \\
& \quad 15 \ 40 \\
(\%o9) & \quad a \ b
\end{align*}
\]

\text{\texttt{simpsum}}

\hspace{1cm} [Option variable]

Default value: \text{false}
When \texttt{simpsum} is \texttt{true}, the result of a \texttt{sum} is simplified. This simplification may sometimes be able to produce a closed form. If \texttt{simpsum} is \texttt{false} or if the quoted form \texttt{'sum} is used, the value is a sum noun form which is a representation of the sigma notation used in mathematics.

\begin{verbatim}
sum (expr, i, i_0, i_1) [Function]
Represents a summation of the values of \texttt{expr} as the index \texttt{i} varies from \texttt{i_0} to \texttt{i_1}. The noun form \texttt{'sum} is displayed as an uppercase letter sigma.

\texttt{sum} evaluates its summand \texttt{expr} and lower and upper limits \texttt{i_0} and \texttt{i_1}, \texttt{sum} quotes (does not evaluate) the index \texttt{i}.

If the upper and lower limits differ by an integer, the summand \texttt{expr} is evaluated for each value of the summation index \texttt{i}, and the result is an explicit sum.

Otherwise, the range of the index is indefinite. Some rules are applied to simplify the summation. When the global variable \texttt{simpsum} is \texttt{true}, additional rules are applied. In some cases, simplification yields a result which is not a summation; otherwise, the result is a noun form \texttt{'sum}.

When the \texttt{evflag} (evaluation flag) \texttt{cauchysum} is \texttt{true}, a product of summations is expressed as a Cauchy product, in which the index of the inner summation is a function of the index of the outer one, rather than varying independently.

The global variable \texttt{genindex} is the alphabetic prefix used to generate the next index of summation, when an automatically generated index is needed.

\texttt{gensumnum} is the numeric suffix used to generate the next index of summation, when an automatically generated index is needed. When \texttt{gensumnum} is \texttt{false}, an automatically-generated index is only \texttt{genindex} with no numeric suffix.

See also \texttt{lsum}, \texttt{sumcontract}, \texttt{intosum}, \texttt{bashindices}, \texttt{niceindices}, \texttt{nouns}, \texttt{evflag}, and Chapter 95 [zeilberger-pkg], page 1247.

Examples:

\begin{verbatim}
(%i1) sum (i^2, i, 1, 7);
(%o1) 140
(%i2) sum (a[i], i, 1, 7);
    a + a + a + a + a + a + a
    7 6 5 4 3 2 1
(%o2) a(i)
(%i3) sum (a(i), i, 1, 7);
(%o3) a(7) + a(6) + a(5) + a(4) + a(3) + a(2) + a(1)
(%i4) sum (a(i), i, 1, n);
          n
        \====
      /    \ a(i)
     / 
    / 
   /  i = 1
(%o4) > a(i)
(%i5) sum (2^i + i^2, i, 0, n);
       n
      ===
\end{verbatim}
\end{verbatim}
(%o5) \> (2 + i^2) \\
====
\i = 0
(%i6) sum (2^i + i^2, i, 0, n), simpsum;
\ 3
\n + 1
\ 2n + 3n + n
\ 2 + ------------------- - 1
6
(%o6)

(%i7) sum (1/3^i, i, 1, inf);
inf
====
\ 1
\ > --
\i
==== 3
\i = 1
(%i8) sum (1/3^i, i, 1, inf), simpsum;
1
\ -
2
(%i9) sum (i^2, i, 1, 4) * sum (1/i^2, i, 1, inf);
inf
====
\ 1
\ > --
\i
==== 2
\i = 1
(%i10) sum (i^2, i, 1, 4) * sum (1/i^2, i, 1, inf), simpsum;
2
(%o10)
\ 5 \%pi
5
(%i11) sum (integrate (x^k, x, 0, 1), k, 1, n);
n
====
\ 1
\ > -----
\k + 1
====
\k = 1
(%i12) sum (if k <= 5 then a^k else b^k, k, 1, 10);
\ 10 9 8 7 6 5 4 3 2
\b + b + b + b + b + a + a + a + a + a
(%o12)
sumcontract (expr)  [Function]
Combines all sums of an addition that have upper and lower bounds that differ by constants. The result is an expression containing one summation for each set of such summations added to all appropriate extra terms that had to be extracted to form this sum. sumcontract combines all compatible sums and uses one of the indices from one of the sums if it can, and then try to form a reasonable index if it cannot use any supplied.

It may be necessary to do an intosum (expr) before the sumcontract.

sumexpand  [Option variable]
Default value: false
When sumexpand is true, products of sums and exponentiated sums simplify to nested sums.
See also cauchysum.

Examples:
(%i1) sumexpand: true$
(%i2) sum (f (i), i, 0, m) * sum (g (j), j, 0, n);
    m    n
   ---- ----
  \    \   \f(i1) g(i2)
    /    /   /
   ---- ----
  i1 = 0 i2 = 0
(%o2)
(%i3) sum (f (i), i, 0, m)^2;
    m    m
   ---- ----
  \    \   f(i3) f(i4)
    /    /   /
   ---- ----
  i3 = 0 i4 = 0
(%o3)

28.2 Introduction to Series
Maxima contains functions taylor and powerseries for finding the series of differentiable functions. It also has tools such as nusum capable of finding the closed form of some series. Operations such as addition and multiplication work as usual on series. This section presents the global variables which control the expansion.

28.3 Functions and Variables for Series
cauchysum  [Option variable]
Default value: false
When multiplying together sums with inf as their upper limit, if sumexpand is true and cauchysum is true then the Cauchy product will be used rather than the usual
product. In the Cauchy product the index of the inner summation is a function of the index of the outer one rather than varying independently.

Example:

(%i1) sumexpand: false$
(%i2) cauchysum: false$
(%i3) s: sum (f(i), i, 0, inf) * sum (g(j), j, 0, inf);

inf inf
==== ====
\ / \\
(%o3) ( > f(i)) > g(j)
/ /
==== ====

i = 0 j = 0

(%i4) sumexpand: true$
(%i5) cauchysum: true$
(%i6) expand(s,0,0);

inf i1
==== ====
\ / \\
(%o6) > > g(i1 - i2) f(i2)
/ /
==== ====

i1 = 0 i2 = 0

\texttt{deftaylor} \ (f_1(x_1), \ expr_1, \ldots, \ f_n(x_n), \ expr_n) \quad \text{[Function]}

For each function \(f_i\) of one variable \(x_i\), \texttt{deftaylor} defines \(expr_i\) as the Taylor series about zero. \(expr_i\) is typically a polynomial in \(x_i\) or a summation; more general expressions are accepted by \texttt{deftaylor} without complaint.

\texttt{powerseries} \ (f_i(x_i), \ x_i, \ 0) \text{ returns the series defined by } \texttt{deftaylor}.

\texttt{deftaylor} returns a list of the functions \(f_1, \ldots, f_n\). \texttt{deftaylor} evaluates its arguments.

Example:

(%i1) deftaylor (f(x), x^2 + sum(x^i/(2^i*i!^2), i, 4, inf));
(%o1) \ [f]$
(%i2) powerseries (f(x), x, 0);

inf
==== i1
\ / \
(%o2) > -------- + x
/ i1 2
==== 2 i1!

i1 = 4

(%i3) taylor (exp (sqrt (f(x))), x, 0, 4);

2 3 4
x 3073 x 12817 x

(%o3)/T/ 1 + x + -- + ------- + ------- + . . .
maxtayorder

Default value: true

When maxtayorder is true, then during algebraic manipulation of (truncated) Taylor series, taylor tries to retain as many terms as are known to be correct.

niceindices (expr)

Renames the indices of sums and products in expr. niceindices attempts to rename each index to the value of niceindicespref[1], unless that name appears in the summand or multiplicand, in which case niceindices tries the succeeding elements of niceindicespref in turn, until an unused variable is found. If the entire list is exhausted, additional indices are constructed by appending integers to the value of niceindicespref[1], e.g., i0, i1, i2, ...

niceindices returns an expression. niceindices evaluates its argument.

Example:

(%i1) niceindicespref;
(%o1) [i, j, k, l, m, n]
(%i2) product (sum (f (foo + i*j*bar), foo, 1, inf), bar, 1, inf);
       inf   inf
       /===\\  =====
     !!!  \
   !!! >  f(bar i j + foo)
     !!! /
bar = 1 =====
foo = 1
(%i3) niceindices (%);
inf   inf
/===\\  =====
!!!  \
!!! >  f(i j l + k)
!!! /
l = 1 =====
k = 1

niceindicespref

Default value: [i, j, k, l, m, n]

niceindicespref is the list from which niceindices takes the names of indices for sums and products.

The elements of niceindicespref are must be names of simple variables.

Example:

(%i1) niceindicespref: [p, q, r, s, t, u]$%i2) product (sum (f (foo + i*j*bar), foo, 1, inf), bar, 1, inf);
inf   inf
/===\\  =====
!!!  \

(%o2) ! > f(bar i j + foo)
    ! /  
    bar = 1 ====  
           foo = 1
(%i3) niceindices (%);
    inf  inf
/====\  ====
   ! !  
(%o3) ! > f(i j q + p)
    ! /  
    q = 1 ====  
           p = 1

nusum (expr, x, i_0, i_1) [Function]
Carries out indefinite hypergeometric summation of expr with respect to x using a
decision procedure due to R.W. Gosper. expr and the result must be expressible as
products of integer powers, factorials, binomials, and rational functions.
The terms "definite" and "indefinite summation" are used analogously to "definite"
and "indefinite integration". To sum indefinitely means to give a symbolic result for
the sum over intervals of variable length, not just e.g. 0 to inf. Thus, since there is
no formula for the general partial sum of the binomial series, nusum can't do it.
nusum and unsum know a little about sums and differences of finite products. See also
unsum.
Examples:
(%i1) nusum (n*n!, n, 0, n);
Dependent equations eliminated: (1)
   (n + 1)! - 1
(%o1) 4 3 2 n
   693 binomial(2 n, n) 3 11 7
(%i2) nusum (n^4*4^n/binomial(2*n,n), n, 0, n);
    2 (n + 1) (63 n + 112 n + 18 n - 22 n + 3) 4 2
   ---------------------------------- - ------
    693 binomial(2 n, n) 3 11 7
(%o2) 4 n
   n 4
(%i3) unsum (%), n;
    4 n
   binomial(2 n, n)
(%o3) unsum (%), n;
    -------------------
   binomial(2 n, n)
(%i4) unsum (prod (i^2, i, 1, n), n);
    n - 1
   /====\  2
   ! !  
   ( ! ! i ) (n - 1) (n + 1)
   ! !  
   i = 1
(%o4) unsum (%), n, 1, n);
Dependent equations eliminated: (2 3)

\[
\begin{array}{c}
\sum_{i=1}^{n} \\
\end{array}
\]

(%o5)

\[
\begin{array}{c}
\sum_{i=1}^{n} \\
\end{array}
\]

pade (taylor_series, numer_deg_bound, denom_deg_bound)

[Function]

Returns a list of all rational functions which have the given Taylor series expansion where the sum of the degrees of the numerator and the denominator is less than or equal to the truncation level of the power series, i.e. are "best" approximants, and which additionally satisfy the specified degree bounds.

taylor_series is an univariate Taylor series. numer_deg_bound and denom_deg_bound are positive integers specifying degree bounds on the numerator and denominator.
taylor_series can also be a Laurent series, and the degree bounds can be inf which causes all rational functions whose total degree is less than or equal to the length of the power series to be returned. Total degree is defined as numer_deg_bound + denom_deg_bound. Length of a power series is defined as "truncation level" + 1 - min(0, "order of series").

(%i1) taylor (1 + x + x^2 + x^3, x, 0, 3);

2 3
(%o1)/T/ 1 + x + x + x + . . .

(%i2) pade (%i1, 1, 1);

1
(%o2) \[- \frac{1}{x - 1} \]

(%i3) t: taylor(-(83787*x^10 - 45552*x^9 - 187296*x^8
+ 387072*x^7 + 86016*x^6 - 1507328*x^5
+ 1966080*x^4 + 4194304*x^3 - 25165824*x^2
+ 67108864*x - 134217728)
/134217728, x, 0, 10);

2 3 4 5 6 7
x 3 x x 15 x 23 x 21 x 189 x
(%o3)/T/ 1 - - + ----- - --- - ------- + ------ - ------- - -------

2 16 32 1024 2048 32768 65536

8 9 10
5853 x 2847 x 83787 x
+ ------- + ------- - ---------- + . . .

4194304 8388608 134217728

(%i4) pade (t, 4, 4);

(%o4) []

There is no rational function of degree 4 numerator/denominator, with this power series expansion. You must in general have degree of the numerator and degree of
the denominator adding up to at least the degree of the power series, in order to have enough unknown coefficients to solve.

(%i5) pade (t, 5, 5);

5 4 3
(%o5) \[- (520256329 x - 96719020632 x - 489651410240 x
2
- 1619100813312 x - 2176885157888 x - 2386516803584)\]
5 4 3
/ (47041365435 x + 381702613848 x + 1360678489152 x
2
+ 2856700692480 x + 3370143559680 x + 2386516803584)]

\textbf{powerseries} (expr, x, a)

Returns the general form of the power series expansion for expr in the variable x about the point a (which may be \texttt{inf} for infinity):

\[
\inf \int \frac{b (x - a)}{n} dx
\]

If \texttt{powerseries} is unable to expand expr, \texttt{taylor} may give the first several terms of the series.

When \texttt{verbose} is true, \texttt{powerseries} prints progress messages.

(%i1) verbose: true$
(%i2) powerseries (log(sin(x)/x), x, 0);

\texttt{can't expand log(sin(x)) so we'll try again after applying the rule:}

\[
d \log(sin(x))
\]

\[
[ dx \frac{i \cot(x) dx - \log(x)}{sin(x)} ]
\]

\[
in the first simplification we have returned:
\]

\[
/\]

\[
/\]

\[
inf
\]
\[ \frac{(-1)^{2i} \text{bern}(2i) x}{2i (2i)!} \]

\[ i_1 = 1 \]

\[ \text{psexpand} \]

Default value: \text{false}

When \text{psexpand} is \text{true}, an extended rational function expression is displayed fully expanded. The switch \text{ratexpand} has the same effect.

When \text{psexpand} is \text{false}, a multivariate expression is displayed just as in the rational function package.

When \text{psexpand} is \text{multi}, then terms with the same total degree in the variables are grouped together.

\text{revert} (\text{expr}, x)

\text{revert2} (\text{expr}, x, n)

These functions return the reversion of \text{expr}, a Taylor series about zero in the variable \text{x}. \text{revert} returns a polynomial of degree equal to the highest power in \text{expr}. \text{revert2} returns a polynomial of degree \text{n}, which may be greater than, equal to, or less than the degree of \text{expr}.

\text{load} ("\text{revert}") loads these functions.

Examples:

\[
(\%i1) \quad \text{load} ("\text{revert}");
\]

\[
(\%i2) \quad t: \text{taylor} (\exp(x) - 1, x, 0, 6); \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad x \quad x \quad x \quad x \quad x \quad \frac{x + --- + --- + --- + --- + --- + . . .}{2 \quad 6 \quad 24 \quad 120 \quad 720} \quad (\%o2)/T/ \quad \frac{x + --- + --- + --- + --- + --- + . . .}{2 \quad 6 \quad 24 \quad 120 \quad 720} \quad (\%i3) \quad \text{revert} (t, x); \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 10 \quad x \quad -12 \quad x \quad +15 \quad x \quad -20 \quad x \quad +30 \quad x \quad -60 \quad x \quad (\%o3)/R/ \quad - \frac{- - + - - - + - - - + x}{60} \quad (\%i4) \quad \text{ratexpand} (\%); \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad x \quad x \quad x \quad x \quad x \quad (\%o4) \quad - - + - - - + - - - + x \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad (\%i5) \quad \text{taylor} (\log(x+1), x, 0, 6); \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad x \quad x \quad x \quad x \quad x \quad (\%o5)/T/ \quad x \quad - - + - - - + - - - + - - - + . . . \]
\[
\frac{2}{a+1} + \frac{2}{(a + 2 + a + 1) \cdot x}
\]

\[
\frac{1}{2} \left( x + \frac{x^2}{2} \right)
\]

taylor

taylor(expr, x, a, n)
taylor(expr, [x_1, x_2, ...], a, n)
taylor(expr, [x_1, x_2, ...], [a_1, a_2, ...], [n_1, n_2, ...])
taylor(expr, [x_1, a_1, n_1], [x_2, a_2, n_2], ...)
taylor(expr, [x, a, n], 'asymp)

\[\text{taylor} \quad \text{(Function)}\]

\[
\text{taylor} \quad \text{expands the expression } expr \quad \text{in a truncated Taylor or Laurent series in the variable } x \quad \text{around the point } a, \quad \text{containing terms through } (x - a)^n.
\]

If \( expr \) is of the form \( f(x)/g(x) \) and \( g(x) \) has no terms up to degree \( n \) then \( \text{taylor} \) attempts to expand \( g(x) \) up to degree \( 2n \). If there are still no nonzero terms, \( \text{taylor} \) doubles the degree of the expansion of \( g(x) \) so long as the degree of the expansion is less than or equal to \( 2^{\text{taylordepth}} \).

\[
\text{taylor} \quad \text{returns a truncated power series of degree } n \quad \text{in all variables } x_1, x_2, ... \quad \text{about the point } (a, a, ...).
\]

\[
\text{taylor} \quad \text{returns a truncated power series in the variables } x_1, x_2, ... \quad \text{about the point } (a_1, a_2, ...), \quad \text{truncated at } n_1, n_2, ...
\]

\[
\text{taylor} \quad \text{returns a truncated power series in the variables } x_1, x_2, ... \quad \text{about the point } (a_1, a_2, ...), \quad \text{truncated at } n_1, n_2, ...
\]

\[
\text{taylor} \quad \text{returns an expansion of } expr \quad \text{in negative powers of } x - a. \quad \text{The highest order term is } (x - a)^{-n}.
\]

When \( \text{maxtayorder} \) is true, then during algebraic manipulation of (truncated) Taylor series, \( \text{taylor} \) tries to retain as many terms as are known to be correct.

When \( \text{psexpand} \) is true, an extended rational function expression is displayed fully expanded. The switch \( \text{ratexpand} \) has the same effect. When \( \text{psexpand} \) is false, a multivariate expression is displayed just as in the rational function package. When \( \text{psexpand} \) is multi, then terms with the same total degree in the variables are grouped together.

See also the \( \text{taylor_logexpand} \) switch for controlling expansion.

Examples:

\[
(\%i1) \text{taylor} \quad (\sqrt{\sin(x) + a \cdot x + 1}), \quad x, \quad 0, \quad 3);
\]

\[
\frac{1}{2} \left( x + \frac{x^2}{2} \right)
\]

\[
\frac{2}{a+1} + \frac{2}{(a + 2 + a + 1) \cdot x}
\]

\[
\frac{1}{2} \left( x + \frac{x^2}{2} \right)
\]
\[
\frac{3^2}{48}
\]

\[\frac{3}{48}(3a + 9a + 9a - 1)x + \ldots\]

\[
1 + (a + 1)x - \ldots
\]

\[
taylor(\sqrt{x + 1}, x, 0, 5);
\]

\[
1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{x^4}{128} + \frac{x^5}{256} + \ldots
\]

\[
1 + x + \ldots
\]

\[
product((1 + x^i)^{2.5}, i, 1, \infty)/(1 + x^2);
\]

\[
\frac{1}{2x + 1}
\]

\[
ev(taylor(%, x, 0, 3), \text{keepfloat});
\]

\[
1 + 2.5x + 3.375x + 6.5625x + \ldots
\]

\[
taylor(1/\log(x + 1), x, 0, 3);
\]

\[
-1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{24}x^3 + \frac{1}{720}x^4 + \ldots
\]

\[
taylor((\cos(x) - \sec(x))^3, x, 0, 5);
\]

\[
0 + \ldots
\]

\[
taylor(1/(\cos(x) - \sec(x))^3, x, 0, 5);
\]

\[
\frac{2}{6} - \frac{4}{15120} + \ldots
\]
\[
x \quad 2 x \quad 120 x
\]

(%i11) taylor (sqrt (1 - k^2*sin(x)^2), x, 0, 6);
\[
\frac{2 k x}{2} \quad \frac{4 k x}{24} \quad \frac{1}{1 - \frac{\left(3 k - 4 k^2\right) x}{2}}
\]

(%o11)/T/ 1 - \frac{\left(45 k - 60 k^2 + 16 k^3\right) x}{720}

(%i12) taylor ((x + 1)^n, x, 0, 4);
\[
\frac{(n - n) x}{2} \quad \frac{\left(n - 3 n + 2 n\right) x}{6}
\]

(%o12)/T/ 1 + n x + \frac{\left(n - 6 n + 11 n - 6 n\right) x}{24}

(%i13) taylor (sin (y + x), x, 0, 3, y, 0, 3);
\[
\frac{y}{6} \quad \frac{2 y}{3}
\]

(%o13)/T/ y - \frac{y}{6} + \ldots + \frac{1 - \frac{y}{6} + \ldots}{2}

(%i14) taylor (sin (y + x), [x, y], 0, 3);
\[
\frac{3 y}{2} \quad \frac{2 y}{12} \quad \frac{1 y}{2}
\]

(%o14)/T/ y + x - \frac{x + 3 y x + 3 y x + y}{6}

(%i15) taylor (1/sin (y + x), x, 0, 3, y, 0, 3);
\[
\frac{1 y}{6} \quad \frac{1}{2 y} \quad \frac{1}{2}
\]

(%o15)/T/ - + - + \ldots + \frac{\left(- - + + \ldots\right) x + \left(- -- + + \ldots\right) x}{2}

(%i15)/T/ - + - + \ldots + \frac{\left(- -- + + \ldots\right) x + \left(- -- + + \ldots\right) x}{4}

\[
\frac{1 y}{6} \quad \frac{2 y}{6} \quad \frac{3 y}{3}
\]

(%o15)/T/ - + - + \ldots + \frac{\left(- -- + + \ldots\right) x + \left(- -- + + \ldots\right) x}{4}
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\[
(\%i16) \text{taylor}\left(\frac{1}{\sin(y + x)}, [x, y], 0, 3\right);
\]
\[
\frac{\sin(y + x) - y - y^3}{\left(y - y^3\right)} + \frac{\sin^2(y + x) - y^2}{\left(y - y^3\right)^2} + \frac{\cos(y + x) - y}{\left(y - y^3\right)^3} + \ldots
\]

\[
(\%o16)/T/ \quad \frac{1}{x + y} + \frac{3}{6} + \frac{21 x y}{360} + \ldots
\]

`taylordepth` [Option variable]

Default value: 3

If there are still no nonzero terms, `taylor` doubles the degree of the expansion of \( g(x) \) so long as the degree of the expansion is less than or equal to \( n \cdot 2^{\text{taylordepth}} \).

`taylorinfo (expr)` [Function]

Returns information about the Taylor series `expr`. The return value is a list of lists. Each list comprises the name of a variable, the point of expansion, and the degree of the expansion.

`taylorinfo` returns `false` if `expr` is not a Taylor series.

Example:

\[
(\%i1) \text{taylor}\left(\frac{(1 - y^2)/(1 - x)}{x}, x, 0, 3, [y, a, \text{inf}]\right);
\]
\[
\frac{2}{2} (1 - a - 2 a (y - a) - (y - a) x
\]
\[
+ (1 - a - 2 a (y - a) - (y - a) x)
\]
\[
+ (1 - a - 2 a (y - a) - (y - a) x)
\]
\[
+ \ldots
\]
\[
(\%i2) \text{taylorinfo(\%)};
\]
\[
(\%o2) \quad [[y, a, \text{inf}], [x, 0, 3]]
\]

`taylorp (expr)` [Function]

Returns `true` if `expr` is a Taylor series, and `false` otherwise.

`taylor_logexpand` [Option variable]

Default value: `true`

`taylor_logexpand` controls expansions of logarithms in `taylor` series.

When `taylor_logexpand` is `true`, all logarithms are expanded fully so that zero-recognition problems involving logarithmic identities do not disturb the expansion process. However, this scheme is not always mathematically correct since it ignores branch information.

When `taylor_logexpand` is set to `false`, then the only expansion of logarithms that occur is that necessary to obtain a formal power series.
taylor_order_coefficients

[Option variable]

Default value: true

taylor_order_coefficients controls the ordering of coefficients in a Taylor series.

When taylor_order_coefficients is true, coefficients of taylor series are ordered canonically.

taylor_simplifier (expr)

[Function]

Simplifies coefficients of the power series expr. taylor calls this function.

taylor_truncate_polynomials

[Option variable]

Default value: true

When taylor_truncate_polynomials is true, polynomials are truncated based upon the input truncation levels.

Otherwise, polynomials input to taylor are considered to have infinite precision.

taytorat (expr)

[Function]

Converts expr from taylor form to canonical rational expression (CRE) form. The effect is the same as rat (ratdisrep (expr)), but faster.

trunc (expr)

[Function]

Annotates the internal representation of the general expression expr so that it is displayed as if its sums were truncated Taylor series. expr is not otherwise modified.

Example:

(%i1) expr: x^2 + x + 1;
2
(%o1) x + x + 1
(%i2) trunc (expr);
2
(%o2) 1 + x + x + . . .
(%i3) is (expr = trunc (expr));
true

unsum (f, n)

[Function]

Returns the first backward difference \( f(n) - f(n - 1) \). Thus unsum in a sense is the inverse of sum.

See also nusum.

Examples:

(%i1) g(p) := p*4^n/binomial(2*n,n);

(%o1) g(p) := -----------------
    binomial(2 n, n)
(%i2) g(n^4);

(%o2) ----------------
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\[
\text{binomial}(2\ n, \ n)
\]

\[
\begin{align*}
(\%i3) \ \text{nusum} & \ (\%, \ n, \ 0, \ n) ; \\
& 4 \quad 3 \quad 2 \\
& 2 \ (n + 1) \ (63 \ n \ + \ 112 \ n \ + \ 18 \ n \ - \ 22 \ n \ - \ 3) \ 4 \quad 2 \\
(\%o3) & \frac{4 \ n \ (n + 1) \ (63 \ n \ + \ 112 \ n \ + \ 18 \ n \ - \ 22 \ n \ + \ 3) \ 4 \ 2}{693 \ \text{binomial}(2 \ n, \ n)} \\
(\%i4) \ \text{unsum} & \ (\%, \ n) ; \\
& 4 \ n \\
& n \ 4 \\
(\%o4) & \frac{4 \ n \ (n + 1) \ (63 \ n \ + \ 112 \ n \ + \ 18 \ n \ - \ 22 \ n \ + \ 3) \ 4 \ 2}{693 \ \text{binomial}(2 \ n, \ n)}
\]

[Option variable]

\text{verbose}  \\
Default value: false

When \text{verbose} is true, \text{powerseries} prints progress messages.

28.4 Introduction to Fourier series

The \text{fourie} package comprises functions for the symbolic computation of Fourier series. There are functions in the \text{fourie} package to calculate Fourier integral coefficients and some functions for manipulation of expressions.

28.5 Functions and Variables for Fourier series

\text{equalp} \ (x, \ y)  \\
\text{equalp} \ (x, \ y) \text{ returns true if equal} \ (x, \ y) \text{ otherwise false (doesn’t give an error message like equal} \ (x, \ y) \text{ would do in this case).}

\text{remfun}  \\
\text{remfun} \ (f, \ expr) \\
\text{remfun} \ (f, \ expr, \ x) \\
\text{remfun} \ (f, \ expr) \text{ replaces all occurrences of} \ f \ (\text{arg}) \text{ by arg in expr.}
\text{remfun} \ (f, \ expr, \ x) \text{ replaces all occurrences of} \ f \ (\text{arg}) \text{ by arg in expr only if arg contains the variable x.}

\text{funp}  \\
\text{funp} \ (f, \ expr) \\
\text{funp} \ (f, \ expr, \ x) \\
\text{funp} \ (f, \ expr) \text{ returns true if expr contains the function} \ f. 
\text{funp} \ (f, \ expr, \ x) \text{ returns true if expr contains the function} \ f \text{ and the variable} \ x \text{ is somewhere in the argument of one of the instances of} \ f.

\text{absint}  \\
\text{absint} \ (f, \ x, \ \text{halfplane}) \\
\text{absint} \ (f, \ x) \\
\text{absint} \ (f, \ x, \ a, \ b) \\
\text{absint} \ (f, \ x, \ \text{halfplane}) \text{ returns the indefinite integral of} \ f \text{ with respect to} \ x \text{ in the given halfplane (pos, neg, or both).} \ f \text{ may contain expressions of the form abs} \ (x), \text{ abs} \ (\sin (x)), \text{ abs} \ (a) \ * \ \text{exp} \ (-\text{abs} \ (b) \ * \ \text{abs} \ (x)).
Absint \( f, x \) is equivalent to \( \text{absint} (f, x, \text{pos}) \).

\( \text{absint} (f, x, a, b) \) returns the definite integral of \( f \) with respect to \( x \) from \( a \) to \( b \). \( f \) may include absolute values.

\texttt{fourier} \( (f, x, p) \) \hspace{2cm} \textbf{[Function]}

Returns a list of the Fourier coefficients of \( f(x) \) defined on the interval \([-p, p]\).

\texttt{foursimp} \( (l) \) \hspace{2cm} \textbf{[Function]}

Simplifies \( \sin (n \%pi) \) to 0 if \( \text{sinnpiflag} \) is \text{true} and \( \cos (n \%pi) \) to \((-1)^n \) if \( \text{cosnpiflag} \) is \text{true}.

\texttt{sinnpiflag} \hspace{2cm} \textbf{[Option variable]}

Default value: \text{true}

See \texttt{foursimp}.

\texttt{cosnpiflag} \hspace{2cm} \textbf{[Option variable]}

Default value: \text{true}

See \texttt{foursimp}.

\texttt{fourexpand} \( (l, x, p, \text{limit}) \) \hspace{2cm} \textbf{[Function]}

Constructs and returns the Fourier series from the list of Fourier coefficients \( l \) up through \( \text{limit} \) terms (\( \text{limit} \) may be \text{inf}). \( x \) and \( p \) have same meaning as in \texttt{fourier}.

\texttt{fourcos} \( (f, x, p) \) \hspace{2cm} \textbf{[Function]}

Returns the Fourier cosine coefficients for \( f(x) \) defined on \([0, p]\).

\texttt{foursin} \( (f, x, p) \) \hspace{2cm} \textbf{[Function]}

Returns the Fourier sine coefficients for \( f(x) \) defined on \([0, p]\).

\texttt{totalfourier} \( (f, x, p) \) \hspace{2cm} \textbf{[Function]}

Returns \texttt{fourexpand} (\texttt{foursimp} (\texttt{fourier} \( (f, x, p) \))), \( x \), \( p \), \text{'}inf\).

\texttt{fourint} \( (f, x) \) \hspace{2cm} \textbf{[Function]}

Constructs and returns a list of the Fourier integral coefficients of \( f(x) \) defined on \([\text{minf}, \text{inf}]\).

\texttt{fourintcos} \( (f, x) \) \hspace{2cm} \textbf{[Function]}

Returns the Fourier cosine integral coefficients for \( f(x) \) on \([0, \text{inf}]\).

\texttt{fourintsin} \( (f, x) \) \hspace{2cm} \textbf{[Function]}

Returns the Fourier sine integral coefficients for \( f(x) \) on \([0, \text{inf}]\).

\textbf{28.6 Functions and Variables for Poisson series}

\texttt{intopois} \( (a) \) \hspace{2cm} \textbf{[Function]}

Converts \( a \) into a Poisson encoding.

\texttt{outofpois} \( (a) \) \hspace{2cm} \textbf{[Function]}

Converts \( a \) from Poisson encoding to general representation. If \( a \) is not in Poisson form, \texttt{outofpois} carries out the conversion, i.e., the return value is \texttt{outofpois} \( (\text{intopois} \ (a)) \). This function is thus a canonical simplifier for sums of powers of sine and cosine terms of a particular type.
poisdif \(a, b\) \[Function\]
Differentiates \(a\) with respect to \(b\). \(b\) must occur only in the trig arguments or only in the coefficients.

poisexpr \(a, b\) \[Function\]
Functionally identical to \(\text{intopois}(a^*b)\). \(b\) must be a positive integer.

poisint \(a, b\) \[Function\]
Integrates in a similarly restricted sense (to \(\text{poisdif}\)). Non-periodic terms in \(b\) are dropped if \(b\) is in the trig arguments.

poislim \[Option variable\]
Default value: 5
\(\text{poislim}\) determines the domain of the coefficients in the arguments of the trig functions. The initial value of 5 corresponds to the interval \([-2^{(5-1)}+1, 2^{(5-1)}]\), or \([-15, 16]\), but it can be set to \([-2^{(n-1)}+1, 2^{(n-1)}]\).

poismap \(\text{series}, \text{sinfn}, \text{cosfn}\) \[Function\]
will map the functions \(\text{sinfn}\) on the sine terms and \(\text{cosfn}\) on the cosine terms of the Poisson series given. \(\text{sinfn}\) and \(\text{cosfn}\) are functions of two arguments which are a coefficient and a trigonometric part of a term in series respectively.

poisplus \(a, b\) \[Function\]
Is functionally identical to \(\text{intopois}(a + b)\).

poisimp \(a\) \[Function\]
Converts \(a\) into a Poisson series for \(a\) in general representation.

poisson \[Special symbol\]
The symbol \(/\text{P}/\) follows the line label of Poisson series expressions.

poissubst \(a, b, c\) \[Function\]
Substitutes \(a\) for \(b\) in \(c\). \(c\) is a Poisson series.
(1) Where \(B\) is a variable \(u, v, w, x, y,\) or \(z\), then \(a\) must be an expression linear in those variables (e.g., \(6*u + 4*v\)).
(2) Where \(b\) is other than those variables, then \(a\) must also be free of those variables, and furthermore, free of sines or cosines.
\(\text{poissubst}(a, b, c, d, n)\) is a special type of substitution which operates on \(a\) and \(b\) as in type (1) above, but where \(d\) is a Poisson series, expands \(\cos(d)\) and \(\sin(d)\) to order \(n\) so as to provide the result of substituting \(a + d\) for \(b\) in \(c\). The idea is that \(d\) is an expansion in terms of a small parameter. For example, \(\text{poissubst}(u, v, \cos(v), %e, 3)\) yields \(\cos(u)*(1 - %e^2/2) - \sin(u)*(%e - %e^3/6)\).

poistimes \(a, b\) \[Function\]
Is functionally identical to \(\text{intopois}(a*b)\).

poistrim \[Function\]
is a reserved function name which (if the user has defined it) gets applied during Poisson multiplication. It is a predicate function of 6 arguments which are the coefficients of the \(u, v, ..., z\) in a term. Terms for which \(\text{poistrim}\) is \(\text{true}\) (for the coefficients of that term) are eliminated during multiplication.
printpois (a)

Prints a Poisson series in a readable format. In common with outofpois, it will convert a into a Poisson encoding first, if necessary.
29 Number Theory

29.1 Functions and Variables for Number Theory

bern (n)  [Function]
Returns the \( n \)'th Bernoulli number for integer \( n \). Bernoulli numbers equal to zero are suppressed if \( \text{zerobern} \) is \text{false}.

See also \text{burn}.

\[
\text{(%i1) zerobern: true}\$
\text{(%i2) map (bern, [0, 1, 2, 3, 4, 5, 6, 7, 8]);}
\text{[1, - -\, -\, 0, - --\, 0, - --\, 0, - --]}\n\text{(%o2) [1, - -, - , 0, - --, 0, - --, 0, - --]}
\text{(%i3) zerobern: false}\$
\text{(%i4) map (bern, [0, 1, 2, 3, 4, 5, 6, 7, 8]);}
\text{[1, - -\, -\, -\, --\, -\, --\, -\, ----\, -]}\n\text{(%o4) [1, - -, - , - --, - --, - --, - --, - ----, -]}$bernpoly (x, n)  [Function]
Returns the \( n \)'th Bernoulli polynomial in the variable \( x \).

bfzeta (s, n)  [Function]
Returns the Riemann zeta function for the argument \( s \). The return value is a big float (bfloat); \( n \) is the number of digits in the return value.

bfhzeta (s, h, n)  [Function]
Returns the Hurwitz zeta function for the arguments \( s \) and \( h \). The return value is a big float (bfloat); \( n \) is the number of digits in the return value.

The Hurwitz zeta function is defined as

\[
\zeta(s, h) = \sum_{k=0}^{\infty} \frac{1}{(k + h)^s}
\]

\text{load ("bffac") loads this function.}

burn (n)  [Function]
Returns a rational number, which is an approximation of the \( n \)'th Bernoulli number for integer \( n \). \text{burn} exploits the observation that (rational) Bernoulli numbers can be approximated by (transcendental) zetas with tolerable efficiency:

\[
B(2^n) = \frac{n - 1}{2} \frac{1 - 2^n}{zeta(2n)} \frac{(2n)!}{2^n \pi^n}
\]
**burn** may be more efficient than **bern** for large, isolated $n$ as **bern** computes all the Bernoulli numbers up to index $n$ before returning. **burn** invokes the approximation for even integers $n > 255$. For odd integers and $n \leq 255$ the function **bern** is called. `load ("bffac")` loads this function. See also **bern**.

**chinese** ([$r_1, \ldots, r_n$], [$m_1, \ldots, m_n$])

Solves the system of congruences $x = r_1 \mod m_1, \ldots, x = r_n \mod m_n$. The remainders $r_n$ may be arbitrary integers while the moduli $m_n$ have to be positive and pairwise coprime integers.

```maxima
(%i1) mods : [1000, 1001, 1003, 1007];
(%o1) [1000, 1001, 1003, 1007]
(%i2) lreduce('gcd, mods);
(%o2) 1
(%i3) x : random(apply(*, mods));
(%o3) 685124877004
(%i4) rem : map(lambda([z], mod(x, z)), mods);
(%o4) [4, 568, 54, 624]
(%i5) chinese(rem, mods);
(%o5) 685124877004
(%i6) chinese([1, 2], [3, n]);
(%o6) chinese([1, 2], [3, n])
(%i7) n = 4;
(%o7) 10
```

**cf** (expr)

Computes a continued fraction approximation. expr is an expression comprising continued fractions, square roots of integers, and literal real numbers (integers, rational numbers, ordinary floats, and bigfloats). **cf** computes exact expansions for rational numbers, but expansions are truncated at `ratepsilon` for ordinary floats and $10^{-fpprec}$ for bigfloats.

Operands in the expression may be combined with arithmetic operators. Maxima does not know about operations on continued fractions outside of **cf**. **cf** evaluates its arguments after binding `listarith` to **false**. **cf** returns a continued fraction, represented as a list.

A continued fraction $a + 1/(b + 1/(c + \ldots))$ is represented by the list [a, b, c, \ldots]. The list elements a, b, c, \ldots must evaluate to integers. expr may also contain `sqrt` (n) where n is an integer. In this case **cf** will give as many terms of the continued fraction as the value of the variable `cflength` times the period.

A continued fraction can be evaluated to a number by evaluating the arithmetic representation returned by **cfdisrep**. See also **cfexpand** for another way to evaluate a continued fraction.

See also **cfdisrep**, **cfexpand**, and `cflength`.

Examples:

- expr is an expression comprising continued fractions and square roots of integers.

  ```maxima
  (%i11) cf ([[5, 3, 1]*[11, 9, 7] + [3, 7]/[4, 3, 2]]);
  ```
(%%o1) [59, 17, 2, 1, 1, 1, 27]
(%%i2) cf ((3/17)*[1, -2, 5]/sqrt(11) + (8/13));
(%%o2) [0, 1, 1, 1, 3, 2, 1, 4, 1, 9, 1, 9, 2]

• **cflength** controls how many periods of the continued fraction are computed for algebraic, irrational numbers.

(%%i1) cflength: 1$
(%%i2) cf ((1 + sqrt(5))/2);
(%%o2) [1, 1, 1, 1, 2]
(%%i3) cflength: 2$
(%%i4) cf ((1 + sqrt(5))/2);
(%%o4) [1, 1, 1, 1, 1, 1, 1, 2]
(%%i5) cflength: 3$
(%%i6) cf ((1 + sqrt(5))/2);
(%%o6) [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2]

• A continued fraction can be evaluated by evaluating the arithmetic representation returned by **cfdisrep**.

(%%i1) cflength: 3$
(%%i2) cfdisrep (cf (sqrt (3)))$
(%%i3) ev (% , numer);
(%%o3) 1.731707317073171

• Maxima does not know about operations on continued fractions outside of **cf**.

(%%i1) cf ([1,1,1,1,1,2] * 3);
(%%o1) [4, 1, 5, 2]
(%%i2) cf ([1,1,1,1,1,2]) * 3;
(%%o2) [3, 3, 3, 3, 3, 6]

**cfdisrep (list)** [Function]

Constructs and returns an ordinary arithmetic expression of the form $a + 1/(b + 1/(c + \ldots))$ from the list representation of a continued fraction $[a, b, c, \ldots]$.

(%%i1) cf ([1, 2, -3] + [1, -2, 1]);
(%%o1) [1, 1, 1, 2]
(%%i2) cfdisrep (%);

\[
\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}}
\]

**cfexpand (x)** [Function]

Returns a matrix of the numerators and denominators of the last (column 1) and next-to-last (column 2) convergents of the continued fraction $x$.

(%%i1) cf (rat (ev (%pi, numer)));

'rat' replaced 3.141592653589793 by 103993/33102 =3.141592653011902
(%o1) [3, 7, 15, 1, 292]
(%i2) cfexpand (%);
     [ 103993   355   ]
     [             ]
     [ 33102   113 ]
(%o2) [%[1,1]/%[2,1], numer; 3.141592653011902]

**cflength**

[Option variable]

Default value: 1

cflength controls the number of terms of the continued fraction the function cf will give, as the value cflength times the period. Thus the default is to give one period.

(%i1) cflength: 1$
(%i2) cf ((1 + sqrt(5))/2);
     [1, 1, 1, 1, 2]
(%o2) [1, 1, 1, 1, 1, 1, 1, 2]
(%i3) cflength: 2$
(%i4) cf ((1 + sqrt(5))/2);
     [1, 1, 1, 1, 1, 1, 1, 1, 2]
(%o4) [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2]

**divsum**

[Function]

divsum (n, k)
divsum (n)

divsum (n, k) returns the sum of the divisors of n raised to the k'th power.
divsum (n) returns the sum of the divisors of n.

(%i1) divsum (12);
(%o1) 28
(%i2) 1 + 2 + 3 + 4 + 6 + 12;
(%o2) 28
(%i3) divsum (12, 2);
(%o3) 210
(%i4) 1^2 + 2^2 + 3^2 + 4^2 + 6^2 + 12^2;
(%o4) 210

**euler**

[Function]

Returns the n'th Euler number for nonnegative integer n. Euler numbers equal to zero are suppressed if zerobern is false.

For the Euler-Mascheroni constant, see %gamma.

(%i1) zerobern: true$
(%i2) map (euler, [0, 1, 2, 3, 4, 5, 6]);
(%o2) [1, 0, - 1, 0, 5, 0, - 61]
(%i3) zerobern: false$
(%i4) map (euler, [0, 1, 2, 3, 4, 5, 6]);
(%o4) [1, - 1, 5, - 61, 1385, - 50521, 2702765]
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**factors_only**

[Option variable]

Default value: `false`

Controls the value returned by `ifactors`. The default `false` causes `ifactors` to provide information about multiplicities of the computed prime factors. If `factors_only` is set to `true`, `ifactors` returns nothing more than a list of prime factors.

Example: See `ifactors`.

**fib (n)**

[Function]

Returns the n’th Fibonacci number. `fib(0)` is equal to 0 and `fib(1)` equal to 1, and `fib (-n)` equal to \((-1)^{n+1} * fib(n)\).

After calling `fib`, `prevfib` is equal to `fib(n - 1)`, the Fibonacci number preceding the last one computed.

```lisp
(%i1) map (fib, [-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8]);
(%o1) [- 3, 2, - 1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21]
```

**fibtophi (expr)**

[Function]

Expresses Fibonacci numbers in `expr` in terms of the constant `%phi`, which is \((1 + \sqrt{5})/2\), approximately 1.61803399.

Examples:

```lisp
(%i1) fibtophi (fib (n));
\[ \frac{n \cdot \%phi - (1 - \%phi)}{2 \cdot \%phi - 1} \]
(%o1)
```

```lisp
(%i2) fib (n-1) + fib (n) - fib (n+1);
(%o2) - fib(n + 1) + fib(n) + fib(n - 1)
(%i3) fibtophi (%);
\[ \frac{n + 1 \cdot \%phi - (1 - \%phi)}{2 \cdot \%phi - 1} + \frac{n \cdot \%phi - (1 - \%phi)}{2 \cdot \%phi - 1} \]
(%o3)
```

```lisp
(%i4) ratsimp (%);
(%o4) 0
```

**ifactors (n)**

[Function]

For a positive integer `n` returns the factorization of `n`. If `n=p1^e1..pk^nk` is the decomposition of `n` into prime factors, `ifactors` returns `[[p1, e1], ... , [pk, ek]]`.

Factorization methods used are trial divisions by primes up to 9973, Pollard’s rho and p-1 method and elliptic curves.

If the variable `ifactor_verbose` is set to `true` ifactor produces detailed output about what it is doing including immediate feedback as soon as a factor has been found.
The value returned by `ifactors` is controlled by the option variable `factors_only`. The default `false` causes `ifactors` to provide information about the multiplicities of the computed prime factors. If `factors_only` is set to `true`, `ifactors` simply returns the list of prime factors.

```lisp
(%i1) ifactors(51575319651600);
(%o1) [[2, 4], [3, 2], [5, 2], [1583, 1], [9050207, 1]]

(%i2) apply("*", map(lambda([u], u[1]^u[2]), %));
(%o2) 51575319651600

(%i3) ifactors(51575319651600), factors_only : true;
(%o3) [2, 3, 5, 1583, 9050207]
```

**igcdex** \((n, k)\)  
Returns a list \([a, b, u]\) where \(u\) is the greatest common divisor of \(n\) and \(k\), and \(u\) is equal to \(a n + b k\). The arguments \(n\) and \(k\) must be integers. `igcdex` implements the Euclidean algorithm. See also `gcdex`.

The command `load("gcdex")` loads the function.

Examples:

```lisp
(%i1) load("gcdex")$

(%i2) igcdex(30,18);  
(%o2) [- 1, 2, 6]

(%i3) igcdex(1526757668, 783562735736);  
(%o3) [845922341123, - 164826435, 4]

(%i4) igcdex(fib(20), fib(21));  
(%o4) [4181, - 2584, 1]
```

**inrt** \((x, n)\)  
Returns the integer \(n\)'th root of the absolute value of \(x\).

```lisp
(%i1) l: [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]$

(%i2) map (lambda ([a], inrt (10^a, 3)), l);  
(%o2) [2, 4, 10, 21, 46, 100, 215, 464, 1000, 2154, 4641, 10000]
```

**inv_mod** \((n, m)\)  
Computes the inverse of \(n\) modulo \(m\). `inv_mod (n,m)` returns `false`, if \(n\) is a zero divisor modulo \(m\).

```lisp
(%i1) inv_mod(3, 41);  
(%o1) 14

(%i2) ratsimp(3^-1), modulus = 41;  
(%o2) 14

(%i3) inv_mod(3, 42);  
(%o3) false
```

**isqrt** \((x)\)  
Returns the "integer square root" of the absolute value of \(x\), which is an integer.

**jacobi** \((p, q)\)  
Returns the Jacobi symbol of \(p\) and \(q\).
(%i2) map (lambda ([a], jacobi (a, 9)), 1); 
(%o2) [1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0] 

\textbf{lcm (expr\_1, \ldots, expr\_n)} \quad \textbf{[Function]}

Returns the least common multiple of its arguments. The arguments may be general expressions as well as integers.

load ("functs") loads this function.

\textbf{lucas (n)} \quad \textbf{[Function]}

Returns the \(n\)'th Lucas number. \(\text{lucas}(0)\) is equal to 2 and \(\text{lucas}(1)\) equal to 1, and in general, \(\text{lucas}(n) = \text{lucas}(n-1) + \text{lucas}(n-2)\). Also \(\text{lucas}(-n)\) is equal to \((-1)^{-n} \cdot \text{lucas}(n)\).

(%i1) map (lucas, [-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8]); 
(%o1) [7, -4, 3, -1, 2, 1, 3, 4, 7, 11, 18, 29, 47] 

After calling \(\text{lucas}\), the global variable \text{next_lucas} is equal to \(\text{lucas}(n+1)\), the Lucas number following the last returned. The example shows how Fibonacci numbers can be computed via \text{lucas} and \text{next_lucas}.

(%i1) fib_via_lucas(n) := 
block([lucas : lucas(n)], 
    signum(n) * (2*next_lucas - lucas)/5 )$ 
(%i2) map (fib_via_lucas, [-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8]); 
(%o2) [-3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21] 

\textbf{mod (x, y)} \quad \textbf{[Function]}

If \(x\) and \(y\) are real numbers and \(y\) is nonzero, return \(x - y \times \text{floor}(x / y)\). Further for all real \(x\), we have \(\text{mod} (x, 0) = x\). For a discussion of the definition \(\text{mod} (x, 0) = x\), see Section 3.4, of "Concrete Mathematics," by Graham, Knuth, and Patashnik.

The function \(\text{mod} (x, 1)\) is a sawtooth function with period 1 with \(\text{mod} (1, 1) = 0\) and \(\text{mod} (0, 1) = 0\).

To find the principal argument (a number in the interval (-\%pi, %pi]) of a complex number, use the function \(x \mapsto \%pi - \text{mod} (\%pi - x, 2*%pi)\), where \(x\) is an argument.

When \(x\) and \(y\) are constant expressions (10 * %pi, for example), \text{mod} uses the same big float evaluation scheme that \text{floor} and \text{ceiling} uses. Again, it's possible, although unlikely, that \text{mod} could return an erroneous value in such cases.

For nonnumerical arguments \(x\) or \(y\), \text{mod} knows several simplification rules:

(%i1) \text{mod} (x, 0); 
(%o1) x 
(%i2) \text{mod} (a*x, a*y); 
(%o2) a \text{mod}(x, y) 
(%i3) \text{mod} (0, x); 
(%o3) 0 

\textbf{next_prime (n)} \quad \textbf{[Function]}

Returns the smallest prime bigger than \(n\).

(%i1) \text{next_prime}(27); 
(%o1) 29
partfrac (expr, var)  
Expands the expression expr in partial fractions with respect to the main variable var. 
partfrac does a complete partial fraction decomposition. The algorithm employed is 
based on the fact that the denominators of the partial fraction expansion (the factors 
of the original denominator) are relatively prime. The numerators can be written as 
linear combinations of denominators, and the expansion falls out.

partfrac ignores the value true of the option variable keepfloat.

\[
\frac{1}{(1+x)^2} - \frac{2}{1+x} + \frac{2}{2+x} \\
\frac{2}{x+2} - \frac{2}{x+1} + \frac{1}{(x+1)}
\]

\[
\text{ratsimp (\%)}; \\
\frac{x}{3} - \frac{3}{2} \frac{x + 4}{x + 5} + \frac{1}{x + 2}
\]

\[
\text{partfrac (\%, x)}; \\
\frac{2}{x+2} - \frac{2}{x+1} + \frac{1}{(x+1)}
\]

power_mod (a, n, m)  
Uses a modular algorithm to compute \(a^n \mod m\) where \(a\) and \(n\) are integers and \(m\) is 
a positive integer. If \(n\) is negative, inv_mod is used to find the modular inverse.

\[
\text{power_mod(3, 15, 5);} \\
\text{mod(3^15,5);} \\
\text{power_mod(2, -1, 5);} \\
\text{inv_mod(2,5);} \\
\]

primep (n)  
Primality test. If primep (n) returns false, \(n\) is a composite number and if it returns 
true, \(n\) is a prime number with very high probability.

For \(n\) less than 341550071728321 a deterministic version of Miller-Rabin’s test is used. 
If primep (n) returns true, then \(n\) is a prime number.

For \(n\) bigger than 341550071728321 primep uses primep_number_of_tests Miller- 
Rabin’s pseudo-primality tests and one Lucas pseudo-primality test. The probability 
that a non-prime \(n\) will pass one Miller-Rabin test is less than 1/4. Using the default 
value 25 for primep_number_of_tests, the probability of \(n\) being composite is much 
smaller that \(10^{-15}\).
primep_number_of_tests  
Default value: 25  
Number of Miller-Rabin’s tests used in primep.

primes (start, end)  
Returns the list of all primes from start to end.

(%i1) primes(3, 7);  
(%o1) [3, 5, 7]

prev_prime (n)  
Returns the greatest prime smaller than n.

(%i1) prev_prime(27);  
(%o1) 23

qunit (n)  
Returns the principal unit of the real quadratic number field sqrt (n) where n is an integer, i.e., the element whose norm is unity. This amounts to solving Pell’s equation a^2 - n b^2 = 1.

(%i1) qunit (17);  
(%o1) sqrt(17) + 4  
(%i2) expand (% * (sqrt(17) - 4));  
(%o2) 1

totient (n)  
Returns the number of integers less than or equal to n which are relatively prime to n.

zerobern  
Default value: true  
When zerobern is false, bern excludes the Bernoulli numbers and euler excludes the Euler numbers which are equal to zero. See bern and euler.

zeta (n)  
Returns the Riemann zeta function. If n is a negative integer, 0, or a positive even integer, the Riemann zeta function simplifies to an exact value. For a positive even integer the option variable zeta%pi has to be true in addition (See zeta%pi). For a floating point or bigfloat number the Riemann zeta function is evaluated numerically. Maxima returns a noun form zeta (n) for all other arguments, including rational noninteger, and complex arguments, or for even integers, if zeta%pi has the value false.

zeta(1) is undefined, but Maxima knows the limit limit(zeta(x), x, 1) from above and below.

The Riemann zeta function distributes over lists, matrices, and equations.

See also bfzeta and zeta%pi.

Examples:

(%i1) zeta([-2, -1, 0, 0.5, 2, 3, 1+%i]);
\[
\begin{align*}
\text{(%o1)} & \quad 0, -\frac{\pi}{12}, \frac{\pi}{2}, -\frac{\pi}{6}, -1.460354508809586, -\frac{\pi}{2}, \zeta(3), \frac{\pi}{2}, \zeta(\pi + 1) \\
\text{(%i2)} & \quad \text{limit}(\zeta(x), x, 1, \text{plus}); \\
\text{(%o2)} & \quad \text{inf} \\
\text{(%i3)} & \quad \text{limit}(\zeta(x), x, 1, \text{minus}); \\
\text{(%o3)} & \quad \text{minf}
\end{align*}
\]

\text{zeta}\%\pi

[Option variable]

Default value: true

When \texttt{zeta}\%\pi is true, \texttt{zeta} returns an expression proportional to \(\pi^n\) for even integer \(n\). Otherwise, \texttt{zeta} returns a noun form \texttt{zeta}(n) for even integer \(n\).

Examples:

\[
\begin{align*}
\text{(%i1)} & \quad \text{zeta}\%\pi: \text{true}\$ \\
\text{(%i2)} & \quad \zeta(4); \\
\text{(%o2)} & \quad \frac{\pi}{90} \\
\text{(%i3)} & \quad \text{zeta}\%\pi: \text{false}\$ \\
\text{(%i4)} & \quad \zeta(4); \\
\text{(%o4)} & \quad \text{zeta}(4)
\end{align*}
\]

\text{zn_add_table} (n)

[Function]

Shows an addition table of all elements in \((Z/nZ)\).

See also \texttt{zn_mult_table}, \texttt{zn_power_table}.

\text{zn_characteristic_factors} (n)

[Function]

Returns a list containing the characteristic factors of the totient of \(n\).

Using the characteristic factors a multiplication group modulo \(n\) can be expressed as a group direct product of cyclic subgroups.

In case the group itself is cyclic the list only contains the totient and using \texttt{zn_primroot} a generator can be computed. If the totient splits into more than one characteristic factors \texttt{zn_factor_generators} finds generators of the corresponding subgroups.

Each of the \(r\) factors in the list divides the right following factors. For the last factor \(f_r\) therefore holds \(a^{f_r} = 1 \pmod{n}\) for all \(a\) coprime to \(n\). This factor is also known as Carmichael function or Carmichael lambda.

If \(n > 2\), then \(\text{totient}(n)/2^r\) is the number of quadratic residues, and each of these has \(2^r\) square roots.

See also \texttt{totient}, \texttt{zn_primroot}, \texttt{zn_factor_generators}.

Examples:

The multiplication group modulo 14 is cyclic and its 6 elements can be generated by a primitive root.

\[
\begin{align*}
\text{(%i1)} & \quad \text{zn_characteristic_factors}(14), \phi: \text{totient}(14); \\
\end{align*}
\]
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(%o1)    [[6], 6]
(%i2) [zn_factor_generators(14), g: zn_primroot(14)];
(%o2)    [[3], 3]
(%i3) M14: makelist(power_mod(g, i, 14), i, 0, phi-1);
(%o3)    [1, 3, 9, 13, 11, 5]

The multiplication group modulo 15 is not cyclic and its 8 elements can be generated by two factor generators.

(%i1) [[f1,f2]: zn_characteristic_factors(15), totient(15)];
(%o1)    [[2, 4], 8]
(%i2) [[g1,g2]: zn_factor_generators(15), zn_primroot(15)];
(%o2)    [[11, 7], false]
(%i3) UG1: makelist(power_mod(g1, i, 15), i, 0, f1-1);
(%o3)    [1, 11]
(%i4) UG2: makelist(power_mod(g2, i, 15), i, 0, f2-1);
(%o4)    [1, 7, 4, 13]
(%i5) M15: create_list(mod(i*j, 15), i, UG1, j, UG2);
(%o5)    [1, 7, 4, 13, 11, 2, 14, 8]

For the last characteristic factor 4 it holds that \(a^4 = 1 \pmod{15}\) for all \(a\) in M15. M15 has two characteristic factors and therefore \(8/2^2\) quadratic residues, and each of these has \(2^2\) square roots.

(%i6) zn_power_table(15);

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 4 & 8 & 1 \\
4 & 1 & 4 & 1 \\
7 & 4 & 13 & 1 \\
8 & 4 & 2 & 1 \\
11 & 1 & 11 & 1 \\
13 & 4 & 7 & 1 \\
14 & 1 & 14 & 1 \\
\end{array}
\]

(%o6)

(%i7) map(lambda([i], zn_nth_root(i, 2, 15)), [1, 4]);
(%o7)    [[1, 4, 11, 14], [2, 7, 8, 13]]

\textbf{zn_carmichael_lambda (n)} \quad [\text{Function}]

Returns 1 if \(n\) is 1 and otherwise the greatest characteristic factor of the totient of \(n\).

For remarks and examples see \textbf{zn_characteristic_factors}.

\textbf{zn_determinant (matrix, p)} \quad [\text{Function}]

Uses the technique of LU-decomposition to compute the determinant of \(matrix\) over \((\mathbb{Z}/p\mathbb{Z})\). \(p\) must be a prime.
However if the determinant is equal to zero the LU-decomposition might fail. In that case `zn_determinant` computes the determinant non-modular and reduces thereafter.

See also `zn_invert_by_lu`.

Examples:

```
(%i1) m : matrix([1,3],[2,4]);
     [ 1 3 ]
     [ ]
     [ 2 4 ]
(%o1)                  [ ]

(%i2) zn_determinant(m, 5);
(%o2) 3

(%i3) m : matrix([2,4,1],[3,1,4],[4,3,2]);
     [ 2 4 1 ]
     [ ]
     [ 3 1 4 ]
     [ ]
     [ 4 3 2 ]
(%o3)                  [ ]

(%i4) zn_determinant(m, 5);
(%o4) 0
```

`zn_factor_generators (n)`

Returns a list containing factor generators corresponding to the characteristic factors of the totient of `n`.

For remarks and examples see `zn_characteristic_factors`.

`zn_invert_by_lu (matrix, p)`

Uses the technique of LU-decomposition to compute the modular inverse of `matrix` over `(Z/pZ)`. `p` must be a prime and `matrix` invertible. `zn_invert_by_lu` returns `false` if `matrix` is not invertible.

See also `zn_determinant`.

Example:

```
(%i1) m : matrix([1,3],[2,4]);
     [ 1 3 ]
     [ ]
     [ 2 4 ]
(%o1)                  [ ]

(%i2) zn_determinant(m, 5);
(%o2) 3

(%i3) mi : zn_invert_by_lu(m, 5);
     [ 3 4 ]
     [ ]
     [ 1 2 ]
(%o3)                  [ ]

(%i4) matrixmap(lambda([a], mod(a, 5)), m . mi);
     [ 1 0 ]
     [ ]
     [ 0 1 ]
(%o4)                  [ ]
```
zn_log

zn_log (a, g, n)
zn_log (a, g, n, [[p1, e1], ..., [pk, ek]])

 Computes the discrete logarithm. Let \((Z/nZ)^*\) be a cyclic group, \(g\) a primitive root modulo \(n\) or a generator of a subgroup of \((Z/nZ)^*\) and let \(a\) be a member of this group. \(zn_log (a, g, n)\) then solves the congruence \(g^x = a \mod n\). Please note that if \(a\) is not a power of \(g\) modulo \(n\), \(zn_log\) will not terminate.

The applied algorithm needs a prime factorization of \(zn_order(g)\) resp. \(totient(n)\) in case \(g\) is a primitive root modulo \(n\). A precomputed list of factors of \(zn_order(g)\) might be used as the optional fourth argument. This list must be of the same form as the list returned by \(ifactors(zn_order(g))\) using the default option \(factors_only : false\). However, compared to the running time of the logarithm algorithm providing the list of factors has only a quite small effect.

The algorithm uses a Pohlig-Hellman-reduction and Pollard’s Rho-method for discrete logarithms. The running time of \(zn_log\) primarily depends on the bitlength of the greatest prime factor of \(zn_order(g)\).

See also \(zn_primroot, zn_order, ifactors, totient\).

Examples:

\(zn_log (a, g, n)\) solves the congruence \(g^x = a \mod n\).

\[
\begin{align*}
(\%i1) & \ n : 22 \\
(\%i2) & \ g : zn_primroot(n); \\
(\%o2) & \ 7 \\
(\%i3) & \ ord_7 : zn_order(7, n); \\
(\%o3) & \ 10 \\
(\%i4) & \ powers_7 : makelist(power_mod(g, x, n), x, 0, ord_7 - 1); \\
(\%o4) & \ [1, 7, 5, 13, 3, 21, 15, 17, 9, 19] \\
(\%i5) & \ zn_log(9, g, n); \\
(\%o5) & \ 8 \\
(\%i6) & \ map(lambda([x], zn_log(x, g, n)), powers_7); \\
(\%o6) & \ [0, 1, 2, 3, 4, 5, 6, 7, 8, 9] \\
(\%i7) & \ ord_5 : zn_order(5, n); \\
(\%o7) & \ 5 \\
(\%i8) & \ powers_5 : makelist(power_mod(5,x,n), x, 0, ord_5 - 1); \\
(\%o8) & \ [1, 5, 3, 15, 9] \\
(\%i9) & \ zn_log(9, 5, n); \\
(\%o9) & \ 4
\end{align*}
\]

The optional fourth argument must be of the same form as the list returned by \(ifactors(zn_order(g))\). The running time primarily depends on the bitlength of the totient’s greatest prime factor.

\[
\begin{align*}
(\%i1) & \ (p : 2^127-1, primep(p)); \\
(\%o1) & \ true \\
(\%i2) & \ ifs : ifactors(p - 1)$ \\
(\%i3) & \ g : zn_primroot(p, ifs); \\
(\%o3) & \ 43 \\
(\%i4) & \ a : power_mod(g, 4711, p)$
\end{align*}
\]
(%i5) zn_log(a, g, p, ifs);
(%o5) 4711

(%i6) f_max : last(ifs);
(%o6) [77158673929, 1]

(%i7) ord_5 : zn_order(5, p, ifs)$
(%i8) (p - 1)/ord_5;
(%o8) 73

(%i9) ifs_5 : ifactors(ord_5)$
(%i10) a : power_mod(5, 4711, p)$
(%i11) zn_log(a, 5, p, ifs_5);
(%o11) 4711

\begin{verbatim}
zn_mult_table
zn_mult_table (n)
zn_mult_table (n, gcd)
\end{verbatim}

Without the optional argument \textit{gcd}, \texttt{zn_mult_table}(n) shows a multiplication table of all elements in \((\mathbb{Z}/n\mathbb{Z})^\ast\) which are all elements coprime to \(n\).

The optional second argument \textit{gcd} allows to select a specific subset of \((\mathbb{Z}/n\mathbb{Z})\). If \textit{gcd} is an integer, a multiplication table of all residues \(x\) with \(gcd(x, n) = \textit{gcd}\) are returned. Additionally row and column headings are added for better readability. If necessary, these can be easily removed by \texttt{submatrix(1, table, 1)}.

If \textit{gcd} is set to \texttt{all}, the table is printed for all non-zero elements in \((\mathbb{Z}/n\mathbb{Z})\).

The second example shows an alternative way to create a multiplication table for subgroups.

See also \texttt{zn_add_table}, \texttt{zn_power_table}.

Examples:

The default table shows all elements in \((\mathbb{Z}/n\mathbb{Z})^\ast\) and allows to demonstrate and study basic properties of modular multiplication groups. E.g. the principal diagonal contains all quadratic residues, each row and column contains every element, the tables are symmetric, etc.

If \textit{gcd} is set to \texttt{all}, the table is printed for all non-zero elements in \((\mathbb{Z}/n\mathbb{Z})\).

\begin{verbatim}
(%i1) zn_mult_table(8);
[ 1 3 5 7 ]
[ ]
[ 3 1 7 5 ]
[ ]
[ 5 7 1 3 ]
[ ]
[ 7 5 3 1 ]
[ ]

(%o1)

(%i2) zn_mult_table(8, all);
[ 1 2 3 4 5 6 7 ]
[ ]
[ 2 4 6 0 2 4 6 ]
[ ]
[ 3 6 1 4 7 2 5 ]
[ ]
\end{verbatim}
If \( \gcd \) is an integer, row and column headings are added for better readability.

If the subset chosen by \( \gcd \) is a group there is another way to create a multiplication table. An isomorphic mapping from a group with 1 as identity builds a table which is easy to read. The mapping is accomplished via CRT.

In the second version of T36_4 the identity, here 28, is placed in the top left corner, just like in table T9.

\[
\begin{bmatrix}
4 & 0 & 4 & 0 & 4 & 0 & 4 \\
5 & 2 & 7 & 4 & 1 & 6 & 3 \\
6 & 4 & 2 & 0 & 6 & 4 & 2 \\
7 & 6 & 5 & 4 & 3 & 2 & 1
\end{bmatrix}
\]
zn_nth_root

zn_nth_root (x, n, m)

zn_nth_root (x, n, m, [[p1, e1], ..., [pk, ek]])

Returns a list with all n-th roots of x from the multiplication subgroup of \((\mathbb{Z}/m\mathbb{Z})\) which contains x, or false, if x is no n-th power modulo m or not contained in any multiplication subgroup of \((\mathbb{Z}/m\mathbb{Z})\).

x is an element of a multiplication subgroup modulo m, if the greatest common divisor \(g = \gcd(x, m)\) is coprime to \(m/g\).

zn_nth_root is based on an algorithm by Adleman, Manders and Miller and on theorems about modulo multiplication groups by Daniel Shanks.

The algorithm needs a prime factorization of the modulus m. So in case the factorization of m is known, the list of factors can be passed as the fourth argument. This optional argument must be of the same form as the list returned by \(\text{ifactors}(m)\) using the default option factors_only: false.

Examples:

A power table of the multiplication group modulo 14 followed by a list of lists containing all n-th roots of 1 with n from 1 to 6.

(%i1) zn_power_table(14);

(%o1) [[1, 1, 1, 1, 1, 1], [3, 9, 13, 11, 5, 1], [5, 11, 13, 9, 3, 1], [9, 11, 1, 9, 11, 1], [11, 9, 1, 11, 9, 1], [13, 1, 13, 1, 13, 1]]

(%i2) makelist(zn_nth_root(1,n,14), n,1,6);

(%o2) [[1], [1, 13], [1, 9, 11], [1, 13], [1, 13], [1, 3, 5, 9, 11, 13]]

In the following example x is not coprime to m, but is a member of a multiplication subgroup of \((\mathbb{Z}/m\mathbb{Z})\) and any n-th root is a member of the same subgroup.

The residue class 3 is no member of any multiplication subgroup of \((\mathbb{Z}/63\mathbb{Z})\) and is therefore not returned as a third root of 27.

Here zn_power_table shows all residues x in \((\mathbb{Z}/63\mathbb{Z})\) with \(\gcd(x, 63) = 9\). This subgroup is isomorphic to \((\mathbb{Z}/7\mathbb{Z})^*\) and its identity 36 is computed via CRT.

(%i11) m: 7*9$
Chapter 29: Number Theory

In the following RSA-like example, where the modulus \( N \) is squarefree, i.e. it splits into exclusively first power factors, every \( x \) from 0 to \( N-1 \) is contained in a multiplication subgroup.

The process of decryption needs the \( e \)-th root. \( e \) is coprime to \( \text{totient}(N) \) and therefore the \( e \)-th root is unique. In this case \text{zn_nth_root} effectively performs CRT-RSA. (Please note that \text{flatten} removes braces but no solutions.)

In the following example the factorization of the modulus is known and passed as the fourth argument.

In the following example the factorization of the modulus is known and passed as the fourth argument.
\textbf{zn_order} \hfill [Function]
\begin{verbatim}
zn_order (x, n)
zn_order (x, n, [[p1, e1], ..., [pk, ek]])
\end{verbatim}

Returns the order of \(x\) if it is a unit of the finite group \((\mathbb{Z}/n\mathbb{Z})^*\) or returns \texttt{false}. \(x\) is a unit modulo \(n\) if it is coprime to \(n\).

The applied algorithm needs a prime factorization of \texttt{totient(n)}. This factorization might be time consuming in some cases and it can be useful to factor first and then to pass the list of factors to \texttt{zn_log} as the third argument. The list must be of the same form as the list returned by \texttt{ifactors(totient(n))} using the default option \texttt{factors_only : false}.

See also \texttt{zn_primroot}, \texttt{ifactors}, \texttt{totient}.

Examples:
\begin{verbatim}
zn_order computes the order of the unit \(x\) in \((\mathbb{Z}/n\mathbb{Z})^*\).

(%i1) n: 22$
(%i2) g: zn_primroot(n); 7
(%i3) units_22: sublist(makelist(i,i,1,21), lambda([x], gcd(x,n)=1));
([1, 3, 5, 7, 9, 13, 15, 17, 19, 21])
(%i4) (ord_7: zn_order(7, n)) = totient(n); 10 = 10
(%i5) powers_7: makelist(power_mod(g,i,n), i,0,ord_7 - 1);
([1, 7, 5, 13, 3, 21, 15, 17, 9, 19])
(%i6) map(lambda([x], zn_order(x, n)), powers_7); 10
(%i7) map(lambda([x], ord_7/gcd(x,ord_7)), makelist(i,i,0,ord_7-1));
([1, 10, 5, 10, 5, 2, 5, 10, 5, 10])
(%i8) totient(totient(n)); 4
\end{verbatim}

The optional third argument must be of the same form as the list returned by \texttt{ifactors(totient(n))}.

\begin{verbatim}
(%i1) (p : 2^142 + 217, primep(p));
(%o1) true
(%i2) ifs: ifactors( totient(p) )$
(%i3) g: zn_primroot(p, ifs);
3
(%i4) is( (ord_3 : zn_order(g, p, ifs)) = totient(p) );
(%o4) true
(%i5) map(lambda([x], ord_3/zn_order(x,p,ifs)), makelist(i,i,2,15));
([22, 1, 44, 10, 5, 2, 22, 2, 8, 2, 1, 1, 20, 1])
\end{verbatim}

\textbf{zn_power_table} \hfill [Function]
\begin{verbatim}
zn_power_table (n)
zn_power_table (n, gcd)
zn_power_table (n, gcd, max_exp)
\end{verbatim}

Without any optional argument \texttt{zn_power_table(n)} shows a power table of all elements in \((\mathbb{Z}/n\mathbb{Z})^*\) which are all residue classes coprime to \(n\). The exponent loops
from 1 to the greatest characteristic factor of \( \text{totient}(n) \) (also known as Carmichael function or Carmichael lambda) and the table ends with a column of ones on the right side.

The optional second argument \( \gcd \) allows to select powers of a specific subset of \((\mathbb{Z}/n\mathbb{Z})\). If \( \gcd \) is an integer, powers of all residue classes \( x \) with \( \gcd(x,n) = \gcd \) are returned, i.e. the default value for \( \gcd \) is 1. If \( \gcd \) is set to all, the table contains powers of all elements in \((\mathbb{Z}/n\mathbb{Z})\).

If the optional third argument \( \text{max}\_\text{exp} \) is given, the exponent loops from 1 to \( \text{max}\_\text{exp} \).

See also \texttt{zn_add_table}, \texttt{zn_mult_table}.

Examples:

The default which is \( \gcd = 1 \) allows to demonstrate and study basic theorems of e.g. Fermat and Euler.

The argument \( \gcd \) allows to select subsets of \((\mathbb{Z}/n\mathbb{Z})\) and to study multiplication subgroups and isomorphisms. E.g. the groups \( G_{10} \) and \( G_{10\_2} \) are under multiplication both isomorphic to \( G_{5} \). 1 is the identity in \( G_{5} \). So are 1 resp. 6 the identities in \( G_{10} \) resp. \( G_{10\_2} \). There are corresponding mappings for primitive roots, n-th roots, etc..

\[
\begin{align*}
(\%i1) \ \text{zn_power_table}(10); & \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 9 & 7 & 1 \\ 7 & 9 & 3 & 1 \\ 9 & 1 & 9 & 1 \end{bmatrix} \\
(\%o1) & \quad \begin{bmatrix} 2 & 4 & 8 & 6 \\ 4 & 6 & 4 & 6 \\ 6 & 6 & 6 & 6 \end{bmatrix} \\
(\%i2) \ \text{zn_power_table}(10,2); & \quad \begin{bmatrix} 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \end{bmatrix} \\
(\%o2) & \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
(\%i3) \ \text{zn_power_table}(10,5); & \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 2 & 8 & 4 \\ 1 & 3 & 7 & 9 \end{bmatrix} \\
(\%o3) & \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 2 & 8 & 4 \end{bmatrix} \\
(\%i4) \ \text{zn_power_table}(10,10); & \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
(\%o4) & \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
(\%i5) \ G_{5}: [1,2,3,4]; & \quad \begin{bmatrix} 1, 2, 3, 4 \end{bmatrix} \\
(\%o5) & \quad \begin{bmatrix} 1, 2, 3, 4 \end{bmatrix} \\
(\%i6) \ G_{10\_2}: \ \text{map}(\lambda[x], \text{chinese}([0,x],[2,5])), G_{5}); & \quad \begin{bmatrix} 6, 2, 8, 4 \end{bmatrix} \\
(\%o6) & \quad \begin{bmatrix} 6, 2, 8, 4 \end{bmatrix} \\
(\%i7) \ G_{10}: \ \text{map}(\lambda[x], \text{power_mod}(3, \text{zn_log}(x,2,5), 10)), G_{5}); & \quad \begin{bmatrix} 1, 3, 7, 9 \end{bmatrix} \\
(\%o7) & \quad \begin{bmatrix} 1, 3, 7, 9 \end{bmatrix}
\end{align*}
\]

If \( \gcd \) is set to all, the table contains powers of all elements in \((\mathbb{Z}/n\mathbb{Z})\).
The third argument \textit{max\_exp} allows to set the highest exponent. The following table shows a very small example of RSA.

\begin{verbatim}
(%i1) N:2*5$ phi:totient(N)$ e:7$ d:inv_mod(e,phi)$

(%i5) zn_power_table(N, all, e*d);
\end{verbatim}

\begin{verbatim}
[ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 ]
[ 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 ]
[ 2 4 8 6 2 4 8 6 2 4 8 6 2 4 8 6 2 4 8 6 2 ]
[ 3 9 7 1 3 9 7 1 3 9 7 1 3 9 7 1 3 9 7 1 3 ]
[ 4 6 4 6 4 6 4 6 4 6 4 6 4 6 4 6 4 6 4 6 4 ]

(%o5) [ ]

[ 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 ]
[ 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 ]
[ 7 9 3 1 7 9 3 1 7 9 3 1 7 9 3 1 7 9 3 1 7 ]
[ 8 4 2 6 8 4 2 6 8 4 2 6 8 4 2 6 8 4 2 6 8 ]
[ 9 1 9 1 9 1 9 1 9 1 9 1 9 1 9 1 9 1 9 1 9 ]
\end{verbatim}

\textbf{zn\_primroot} \textbf{[Function]}

\textbf{zn\_primroot (n)}

\textbf{zn\_primroot (n, [[p1, e1], ..., [pk, ek]])}

If the multiplicative group \((\mathbb{Z}/n\mathbb{Z})^*\) is cyclic, \textbf{zn\_primroot} computes the smallest primitive root modulo \(n\). \((\mathbb{Z}/n\mathbb{Z})^*\) is cyclic if \(n\) is equal to 2, 4, \(p^k\) or \(2*p^k\), where \(p\) is prime and greater than 2 and \(k\) is a natural number. \textbf{zn\_primroot} performs an according pretest if the option variable \textbf{zn\_primroot\_pretest} (default: false) is set to \textbf{true}. In any case the computation is limited by the upper bound \textbf{zn\_primroot\_limit}.

If \((\mathbb{Z}/n\mathbb{Z})^*\) is not cyclic or if there is no primitive root up to \textbf{zn\_primroot\_limit}, \textbf{zn\_primroot} returns \textbf{false}.

The applied algorithm needs a prime factorization of \textbf{totient(n)}. This factorization might be time consuming in some cases and it can be useful to factor first and then to pass the list of factors to \textbf{zn\_log} as an additional argument. The list must be of the same form as the list returned by \textbf{ifactors(totient(n))} using the default option \textbf{factors\_only : false}.

See also \textbf{zn\_primroot\_p}, \textbf{zn\_order}, \textbf{ifactors}, \textbf{totient}.

Examples:

\textbf{zn\_primroot} computes the smallest primitive root modulo \(n\) or returns \textbf{false}.

\begin{verbatim}
(%i11) n : 14$
(%i12) g : zn\_primroot(n);
\end{verbatim}
(%o2) 3
(%i3) zn_order(g, n) = totient(n);
(%o3) 6 = 6
(%i4) n : 15$
(%i5) zn_primroot(n);
(%o5) false

The optional second argument must be of the same form as the list returned by
ifactors(totient(n)).

(%i1) (p : 2^142 + 217, primep(p));
(%o1) true
(%i2) ifs : ifactors( totient(p) )$
(%i3) g : zn_primroot(p, ifs);
(%o3) 3
(%i4) [time(%o2), time(%o3)];
(%o4) [[15.556972], [0.004]]
(%i5) is(zn_order(g, p, ifs) = p - 1);
(%o5) true
(%i6) n : 2^142 + 216$
(%i7) ifs : ifactors(totient(n))$
(%i8) zn_primroot(n, ifs),
zn_primroot_limit : 200, zn_primroot_verbose : true;
'zn_primroot' stopped at zn_primroot_limit = 200
(%o8) false

zn_primroot_limit [Option variable]
Default value: 1000
If zn_primroot cannot find a primitive root, it stops at this upper bound. If the
option variable zn_primroot_verbose (default: false) is set to true, a message will
be printed when zn_primroot_limit is reached.

zn_primroot_p [Function]
zn_primroot_p (x, n)
zn_primroot_p (x, n, [[p1, e1], ..., [pk, ek]])
Checks whether x is a primitive root in the multiplicative group \((Z/nZ)^*\).
The applied algorithm needs a prime factorization of totient(n). This factorization
might be time consuming and in case zn_primroot_p will be consecutively applied to
a list of candidates it can be useful to factor first and then to pass the list of factors
to zn_log as a third argument. The list must be of the same form as the list returned
by ifactors(totient(n)) using the default option factors_only : false.
See also zn_primroot, zn_order, ifactors, totient.
Examples:
zn_primroot_p as a predicate function.

(%i1) n : 14$
(%i2) units_14 : sublist(makelist(i,i,1,13), lambda([i], gcd(i, n) = 1));
(%o2) [1, 3, 5, 9, 11, 13]
(%i3) zn_primroot_p(13, n);
(%i3) sublist(units_14, lambda([x], zn_primroot_p(x, n)));  
(%o4) [3, 5]

(%i5) map(lambda([x], zn_order(x, n)), units_14);  
(%o5) [1, 6, 6, 3, 3, 2]

The optional third argument must be of the same form as the list returned by
ifactors(totient(n)).

(%i1) (p: 2^142 + 217, primep(p));  
(%o1) true

(%i2) ifs: ifactors( totient(p) )$

(%i3) sublist(makelist(i,i,1,50), lambda([x], zn_primroot_p(x,p,ifs)));  
(%o3) [3, 12, 13, 15, 21, 24, 26, 27, 29, 33, 38, 42, 48]

(%i4) [time(%o2), time(%o3)];  
(%o4) [[7.748484], [0.036002]]

zn_primroot_pretest  [Option variable]
Default value: false
The multiplicative group $\mathbb{Z}/n\mathbb{Z}^*$ is cyclic if $n$ is equal to 2, 4, $p^k$ or $2*p^k$, where
$p$ is prime and greater than 2 and $k$ is a natural number.

zn_primroot_pretest controls whether zn_primroot will check if one of these cases
occur before it computes the smallest primitive root. Only if zn_primroot_pretest
is set to true this pretest will be performed.

zn_primroot_verbose  [Option variable]
Default value: false
Controls whether zn_primroot prints a message when reaching zn_primroot_limit.
30 Symmetries

30.1 Introduction to Symmetries

sym is a package for working with symmetric groups of polynomials.

It was written for Macsyma-Symbolics by Annick Valibouze\(^1\). The algorithms are described in the following papers:


30.2 Functions and Variables for Symmetries

30.2.1 Changing bases

\texttt{comp2pui (n, L)} \hspace{1em} \textbf{[Function]}

implements passing from the complete symmetric functions given in the list \(L\) to the elementary symmetric functions from 0 to \(n\). If the list \(L\) contains fewer than \(n+1\) elements, it will be completed with formal values of the type \(h1, h2,\) etc. If the first element of the list \(L\) exists, it specifies the size of the alphabet, otherwise the size is set to \(n\).

\begin{verbatim}
(%i1) comp2pui (3, [4, g]);
(%o1) [4, g, 2 h2 - g, 3 h3 - g h2 + g (g - 2 h2)]
\end{verbatim}

\texttt{ele2pui (m, L)} \hspace{1em} \textbf{[Function]}

goes from the elementary symmetric functions to the complete functions. Similar to \texttt{comp2ele} and \texttt{comp2pui}.

Other functions for changing bases: \texttt{comp2ele}.

\texttt{ele2comp (m, L)} \hspace{1em} \textbf{[Function]}

goes from the elementary symmetric functions to the compete functions. Similar to \texttt{comp2ele} and \texttt{comp2pui}.

Other functions for changing bases: \texttt{comp2ele}.

\(^1\) \url{www-calfor.lip6.fr/~avb}
\(^2\) \url{www.stix.polytechnique.fr/publications/1984-1994.html}
\(^3\) \url{www-calfor.lip6.fr/~avb/DonneesTelechargeables/MesArticles/issac89ACMValibouze.pdf}
\(^4\) \url{www.stix.polytechnique.fr/publications/1984-1994.html}
**elem (ele, sym, lvar)**

[Function]

decomposes the symmetric polynomial sym, in the variables contained in the list lvar, in terms of the elementary symmetric functions given in the list ele. If the first element of ele is given, it will be the size of the alphabet, otherwise the size will be the degree of the polynomial sym. If values are missing in the list ele, formal values of the type e1, e2, etc. will be added. The polynomial sym may be given in three different forms: contracted (ele should then be 1, its default value), partitioned (ele should be 3), or extended (i.e. the entire polynomial, and ele should then be 2). The function pui is used in the same way.

On an alphabet of size 3 with e1, the first elementary symmetric function, with value 7, the symmetric polynomial in 3 variables whose contracted form (which here depends on only two of its variables) is $x^4 - 2xy$ decomposes as follows in elementary symmetric functions:

```
(%i1) elem ([3, 7], x^4 - 2*x*y, [x, y]);
(%o1) 7 (e3 - 7 e2 + 7 (49 - e2)) + 21 e3
     + (- 2 (49 - e2) - 2) e2
```

```
(%i2) ratsimp (%);
2
(%o2) 28 e3 + 2 e2 - 198 e2 + 2401
```

Other functions for changing bases: comp2ele.

**mon2schur (L)**

[Function]

The list L represents the Schur function $S_L$: we have $L = [i_1, i_2, \ldots, i_q]$, with $i_1 \leq i_2 \leq \ldots \leq i_q$. The Schur function $S_{i_1, i_2, \ldots, i_q}$ is the minor of the infinite matrix $h_{i-j}$, $i \geq 1, j \geq 1$, consisting of the q first rows and the columns $i_1 + 1, i_2 + 2, \ldots, i_q + q$.

This Schur function can be written in terms of monomials by using treinat and kostka. The form returned is a symmetric polynomial in a contracted representation in the variables $x_1, x_2, \ldots$

```
(%i1) mon2schur ([1, 1, 1]);
(%o1) x1 x2 x3

(%i2) mon2schur ([3]);
    2     3
(%o2) x1 x2 x3 + x1 x2 + x1

(%i3) mon2schur ([1, 2]);
    2
(%o3) 2 x1 x2 x3 + x1 x2
```

which means that for 3 variables this gives:

$$2 x_1 x_2 x_3 + x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + x_3^2 x_1 + x_2^2 x_3 + x_3^2 x_2$$

Other functions for changing bases: comp2ele.

**multi_elem (l_elem, multi_pc, l_var)**

[Function]

decomposes a multi-symmetric polynomial in the multi-contracted form multi_pc in the groups of variables contained in the list of lists l_var in terms of the elementary symmetric functions contained in l_elem.
multi_elem ([[2, e1, e2], [2, f1, f2]], a*x + a^2 + x^3, [[x, y], [a, b]]);
(%o1) - 2 f2 + f1 (f1 + e1) - 3 e1 e2 + e1
(%o2) ratsimp (%);
2 3
- 2 f2 + f1 + e1 f1 - 3 e1 e2 + e1

Other functions for changing bases: comp2ele.

multi_pui

is to the function pui what the function multi_elem is to the function elem.

multi_pui ([[2, p1, p2], [2, t1, t2]], a*x + a^2 + x^3, [[x, y], [a, b]]);
(%o1) t2 + p1 t1 + ------- - ---
    2    2

pui (L, sym, lvar)

decomposes the symmetric polynomial sym, in the variables in the list lvar, in terms of the power functions in the list L. If the first element of L is given, it will be the size of the alphabet, otherwise the size will be the degree of the polynomial sym. If values are missing in the list L, formal values of the type p1, p2, etc. will be added. The polynomial sym may be given in three different forms: contracted (elem should then be 1, its default value), partitioned (elem should be 3), or extended (i.e. the entire polynomial, and elem should then be 2). The function pui is used in the same way.

pui2comp (n, lpui)

renders the list of the first n complete functions (with the length first) in terms of the power functions given in the list lpui. If the list lpui is empty, the cardinal is n, otherwise it is its first element (as in comp2ele and comp2pui).
(%i1) pui2comp (2, []);
    2
  p2 + p1
(%o1) [2, p1, -----------]

(%i2) pui2comp (3, [2, a1]);
2
  a1 (p2 + a1 )
  2
p3 + ------------------- + a1 p2
    2
p2 + a1
(%o2) [2, a1, -----------, --------------------------]
     3

(%i3) ratsimp (%);
    2
3 p1 p2 - p1
(%o3) [2, a1, -----------, --------------------------]
     2

Other functions for changing bases: comp2ele.

pui2ele (n, lpui) [Function]
effects the passage from power functions to the elementary symmetric functions. If
the flag pui2ele is girard, it will return the list of elementary symmetric functions
from 1 to n, and if the flag is close, it will return the n-th elementary symmetric
function.

Other functions for changing bases: comp2ele.

puireduc (n, lpui) [Function]
lpui is a list whose first element is an integer m. pui2ele gives the first n power
functions in terms of the first m.

(%i1) pui2ele (3, [2]);
    2
  p1 (p1 - p2)
(%o1) [2, p1, p2, p1 p2 - -----------]

(%i2) ratsimp (%);
    3
3 p1 p2 - p1
(%o2) [2, p1, p2, -----------]

schur2comp (P, l_var) [Function]
P is a polynomial in the variables of the list l_var. Each of these variables represents
a complete symmetric function. In l_var the i-th complete symmetric function is
represented by the concatenation of the letter h and the integer i: hi. This function
expresses P in terms of Schur functions.

(%i1) schur2comp (h1*h2 - h3, [h1, h2, h3]);
  s
(%o1) 1, 2
30.2.2 Changing representations

\texttt{cont2part (pc, lvar)}

returns the partitioned polynomial associated to the contracted form \( pc \) whose variables are in \( lvar \).

\begin{verbatim}
(%i1) pc: 2*a^3*b*x^4*y + x^5;
    3 4 5
(%o1) 2 a b x y + x
(%i2) cont2part (pc, [x, y]);
    3
(%o2) [[1, 5, 0], [2 a b, 4, 1]]
\end{verbatim}

\texttt{contract (psym, lvar)}

returns a contracted form (i.e. a monomial orbit under the action of the symmetric group) of the polynomial \( psym \) in the variables contained in the list \( lvar \). The function \texttt{explose} performs the inverse operation. The function \texttt{tcontract} tests the symmetry of the polynomial.

\begin{verbatim}
(%i1) psym: explode (2*a^3*b*x^4*y, [x, y, z]);
    3 4 3 4 3 4
(%o1) 2 a b y z + 2 a b x z + 2 a b y z + 2 a b x z
    + 2 a b x y + 2 a b x y
(%i2) contract (psym, [x, y, z]);
    3 4
(%o2) 2 a b x y
\end{verbatim}

\texttt{explose (pc, lvar)}

returns the symmetric polynomial associated with the contracted form \( pc \). The list \( lvar \) contains the variables.

\begin{verbatim}
(%i1) explode (a*x + 1, [x, y, z]);
    (%o1) a z + a y + a x + 1
\end{verbatim}

\texttt{part2cont (ppart, lvar)}

goes from the partitioned form to the contracted form of a symmetric polynomial. The contracted form is rendered with the variables in \( lvar \).

\begin{verbatim}
(%i1) part2cont ([[2*a^3*b, 4, 1]], [x, y]);
    3 4
(%o1) 2 a b x y
\end{verbatim}

\texttt{partpol (psym, lvar)}

\( psym \) is a symmetric polynomial in the variables of the list \( lvar \). This function returns its partitioned representation.

\begin{verbatim}
(%i1) partpol (-a*(x + y) + 3*x*y, [x, y]);
    (%o1) [[3, 1, 1], [- a, 1, 0]]
\end{verbatim}
tcontract (pol, lvar) [Function]
tests if the polynomial pol is symmetric in the variables of the list lvar. If so, it returns a contracted representation like the function contract.

tpartpol (pol, lvar) [Function]
tests if the polynomial pol is symmetric in the variables of the list lvar. If so, it returns its partitioned representation like the function partpol.

30.2.3 Groups and orbits

direct ([p_1, ..., p_n], y, f, [lvar_1, ..., lvar_n]) [Function]
calculates the direct image (see M. Giusti, D. Lazard et A. Valibouze, ISSAC 1988, Rome) associated to the function f, in the lists of variables lvar_1, ..., lvar_n, and in the polynomials p_1, ..., p_n in a variable y. The arity of the function f is important for the calculation. Thus, if the expression for f does not depend on some variable, it is useless to include this variable, and not including it will also considerably reduce the amount of computation.

(%i1) direct ([z^2 - e1* z + e2, z^2 - f1* z + f2], z, b*v + a*u, [[u, v], [a, b]]);
2 2 2 2
(%o1) y - e1 f1 y - 4 e2 f2 - (e1 - 2 e2) (f1 - 2 f2) + e1 f1 + -----------------------------------------------
2
(%i2) ratsimp (%);
2 2 2
(%o2) y - e1 f1 y + (e1 - 4 e2) f2 + e2 f1
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(%i3) ratsimp (direct ([z^3-e1*z^2+e2*z-e3,z^2 - f1* z + f2],
          z, b*v + a*u, [[u, v], [a, b]]));

   6 5 2 2 2 4
(%o3) y - 2 e1 f1 y + ((2 e1 - 6 e2) f2 + (2 e2 + e1 ) f1 ) y

   3 3 3
+ ((9 e3 + 5 e1 e2 - 2 e1 ) f1 f2 + (- 2 e3 - 2 e1 e2) f1 ) y

   2 2 4 2
+ ((9 e2 - 6 e1 e2 + e1 ) f2

   2 2 2 2 4
+ (- 9 e1 e3 - 6 e2 + 3 e1 e2) f1 f2 + (2 e1 e3 + e2 ) f1 )

2 2 2 3 2
y + (((9 e1 - 27 e2) e3 + 3 e1 e2 - e1 e2) f1 f2

2 2 2 3 5
+ ((15 e2 - 2 e1 ) e3 - e1 e2 ) f1 f2 - 2 e2 e3 f1 ) y

2 3 3 2 2 3
+ (- 27 e3 + (18 e1 e2 - 4 e1 ) e3 - 4 e2 + e1 e2 ) f2

2 3 3 2 2
+ (27 e3 + (e1 - 9 e1 e2) e3 + e2 ) f1 f2

2 4 2 6
+ (e1 e2 e3 - 9 e3 ) f1 f2 + e3 f1

Finding the polynomial whose roots are the sums $a+u$ where $a$ is a root of $z^2-e_1 z+e_2$ and $u$ is a root of $z^2 - f_1 z + f_2$.

(%i1) ratsimp (direct ([z^2-e1* z + e2, z^2 - f1* z + f2],
          z, a + u, [[u], [a]]));

   4
(%o1) y + (- 2 f1 - 2 e1) y + (2 f2 + f1 + 3 e1 f1 + 2 e2

   2 2 2
+ e1 ) y + (2 f1 - 2 e1) f2 - e1 f1 + (- 2 e2 - e1 ) f1

   2 2 2
- 2 e1 e2) y + f2 + (e1 f1 - 2 e2 + e1 ) f2 + e2 f1 + e1 e2 f1

   2
+ e2
direct accepts two flags: elementaires and puissances (default) which allow decomposing the symmetric polynomials appearing in the calculation into elementary symmetric functions, or power functions, respectively.
Functions of \texttt{sym} used in this function:
\texttt{multi_orbit} (so \texttt{orbit}), \texttt{pui_direct}, \texttt{multi_elem} (so \texttt{elem}), \texttt{multi_pui} (so \texttt{pui}), \texttt{pui2ele}, \texttt{ele2pui} (if the flag \texttt{direct} is in puissances).

\textbf{multi_orbit} \((P, \{lvar_1, lvar_2, \ldots, lvar_p\})\) \quad [\text{Function}]
P is a polynomial in the set of variables contained in the lists \texttt{lvar_1, lvar_2, \ldots, lvar_p}. This function returns the orbit of the polynomial \(P\) under the action of the product of the symmetric groups of the sets of variables represented in these \(p\) lists.

\begin{verbatim}
(%i1) multi_orbit (a*x + b*y, [[x, y], [a, b]]);
(%o1) [b*y + a*x, a*y + b*x]
(%i2) multi_orbit (x + y + 2*a, [[x, y], [a, b, c]]);
(%o2) [y + x + 2*c, y + x + 2*b, y + x + 2*a]
\end{verbatim}

Also see: \texttt{orbit} for the action of a single symmetric group.

\textbf{multsym} \((ppart_1, ppart_2, n)\) \quad [\text{Function}]
returns the product of the two symmetric polynomials in \(n\) variables by working only modulo the action of the symmetric group of order \(n\). The polynomials are in their partitioned form.

Given the 2 symmetric polynomials in \(x, y\): \(3*(x + y) + 2*x*y\) and \(5*(x^2 + y^2)\) whose partitioned forms are \([[3, 1], [2, 1, 1]]\) and \([[5, 2]]\), their product will be

\begin{verbatim}
(%i1) multsym ([[3, 1], [2, 1, 1]], [[5, 2]], 2);
(%o1) [[10, 3, 1], [15, 3, 0], [15, 2, 1]]
\end{verbatim}

that is \(10*(x^3*y + y^3*x) + 15*(x^2*y + y^2*x) + 15*(x^3 + y^3)\).

Functions for changing the representations of a symmetric polynomial:
\texttt{contract, cont2part, explode, part2cont, partpol, tcontract, tpartpol}.

\textbf{orbit} \((P, lvar)\) \quad [\text{Function}]
computes the orbit of the polynomial \(P\) in the variables in the list \texttt{lvar} under the action of the symmetric group of the set of variables in the list \texttt{lvar}.

\begin{verbatim}
(%i1) orbit (a*x + b*y, [x, y]);
(%o1) [a*y + b*x, b*y + a*x]
(%i2) orbit (2*x + x^2, [x, y]);
2 2
(%o2) [y + 2*y, x + 2*x]
\end{verbatim}

See also \texttt{multi_orbit} for the action of a product of symmetric groups on a polynomial.

\textbf{pui_direct} \((orbite, [lvar_1, \ldots, lvar_n], [d_1, d_2, \ldots, d_n])\) \quad [\text{Function}]
Let \(f\) be a polynomial in \(n\) blocks of variables \texttt{lvar_1, \ldots, lvar_n}. Let \(c.i\) be the number of variables in \texttt{lvar_i}, and \(SC\) be the product of \(n\) symmetric groups of degree \(c.1, \ldots, c.n\). This group acts naturally on \(f\). The list \texttt{orbite} is the orbit, denoted \(SC(f)\), of the function \(f\) under the action of \(SC\). (This list may be obtained by the function \texttt{multi_orbit}.) The \(d.i\) are integers s.t. \(\leq d.1 \leq c.2 \leq d.2 \leq \ldots \leq d.n\). Let \(SD\) be the product of the symmetric groups \(S_{d.1} \times S_{d.2} \times \cdots \times S_{d.n}\). The function \texttt{pui_direct} returns the first \(n\) power functions of \(SD(f)\) deduced from the power functions of \(SC(f)\), where \(n\) is the size of \(SD(f)\).
The result is in multi-contracted form w.r.t. $SD$, i.e. only one element is kept per orbit, under the action of $SD$.

(%i1) l: [[x, y], [a, b]]
(%o1) [[x, y], [a, b]]
(%i2) pui_direct (multi_orbit (a*x + b*y, l), l, [2, 2])
(%o2) [a x, 4 a b x y + a x]
(%i3) pui_direct (multi_orbit (a*x + b*y, l), l, [3, 2])
(%o3) [2 a x, 4 a b x y + 2 a x, 3 a b x y + 2 a x, 12 a b x y + 4 a b x y + 2 a x, 10 a b x y + 5 a b x y + 2 a x, 3 2 3 2 4 4 5 5, 4 2 4 2 5 5 6 6, 40 a b x y + 15 a b x y + 6 a b x y + 2 a x]
(%i4) pui_direct ([y + x + 2*c, y + x + 2*b, y + x + 2*a], [[x, y], [a, b, c]], [2, 3])
(%o4) [3 x + 2 a, 6 x y + 3 x + 4 a x + 4 a, 9 x y + 12 a x y + 3 x + 6 a x + 12 a x + 8 a]

30.2.4 Partitions

kostka (part_1, part_2) [Function]
written by P. Esperet, calculates the Kostka number of the partition part_1 and part_2.

(%i1) kostka ([3, 3, 3], [2, 2, 1, 1, 1])
(%o1) 6

lgtreillis (n, m) [Function]
returns the list of partitions of weight $n$ and length $m$.

(%i1) lgtreillis (4, 2)
(%o1) [[3, 1], [2, 2]]

Also see: ltreillis, treillis and treinat.

ltreillis (n, m) [Function]
returns the list of partitions of weight $n$ and length less than or equal to $m$.

(%i1) ltreillis (4, 2)
(%o1) [[4, 0], [3, 1], [2, 2]]

Also see: lgtreillis, treillis and treinat.
treillis \((n)\)  
returns all partitions of weight \(n\).

\[
\text{(%i1) treillis (4);}
\]
\[
\text{(%o1) \[\{4\}, \{3, 1\}, \{2, 2\}, \{2, 1, 1\}, \{1, 1, 1, 1\}\]}
\]
See also: \(\text{lgtreillis}, \text{ltreillis}\) and \(\text{treinat}\).

\(\text{treinat} (\text{part})\)
returns the list of partitions inferior to the partition \(\text{part}\) w.r.t. the natural order.

\[
\text{(%i1) treinat ([5]);}
\]
\[
\text{(%o1) \[\{5\}\]}
\]
\[
\text{(%i2) treinat ([1, 1, 1, 1, 1]);}
\]
\[
\text{(%o2) \[\{5\}, \{4, 1\}, \{3, 2\}, \{3, 1, 1\}, \{2, 2, 1\}, \{2, 1, 1, 1\}, \{1, 1, 1, 1, 1\}\]}
\]
\[
\text{(%i3) treinat ([3, 2]);}
\]
\[
\text{(%o3) \[\{5\}, \{4, 1\}, \{3, 2\}\]}
\]
See also: \(\text{lgtreillis}, \text{ltreillis}\) and \(\text{treillis}\).

30.2.5 Polynomials and their roots

\(\text{ele2polynome} (L, z)\)
returns the polynomial in \(z\) s.t. the elementary symmetric functions of its roots are in the list \(L = [n, e_1, \ldots, e_n]\), where \(n\) is the degree of the polynomial and \(e_i\) the \(i\)-th elementary symmetric function.

\[
\text{(%i1) ele2polynome ([2, e1, e2], z);}
\]
\[
\text{2}
\]
\[
\text{(%o1) \(z - e1 \, z + e2\)}
\]
\[
\text{(%i2) polynome2ele (x^7 - 14*x^5 + 56*x^3 - 56*x + 22, x);}
\]
\[
\text{([7, 0, -14, 0, 56, 0, -56, -22])}
\]
\[
\text{(%i3) ele2polynome ([7, 0, -14, 0, 56, 0, -56, -22], x);}
\]
\[
\text{[7 5 3}}
\]
\[
\text{(%o3) \(x - 14 \, x + 56 \, x - 56 \, x + 22\)}
\]
The inverse: \(\text{polynome2ele} (P, z)\).
Also see: \(\text{polynome2ele}, \text{pui2polynome}\).

\(\text{polynome2ele} (P, x)\)
gives the list \(l = [n, e_1, \ldots, e_n]\) where \(n\) is the degree of the polynomial \(P\) in the variable \(x\) and \(e_i\) is the \(i\)-th elementary symmetric function of the roots of \(P\).

\[
\text{(%i1) polynome2ele (x^7 - 14*x^5 + 56*x^3 - 56*x + 22, x);}
\]
\[
\text{([7, 0, -14, 0, 56, 0, -56, -22])}
\]
\[
\text{(%i2) ele2polynome ([7, 0, -14, 0, 56, 0, -56, -22], x);}
\]
\[
\text{[7 5 3}}
\]
\[
\text{(%o2) \(x - 14 \, x + 56 \, x - 56 \, x + 22\)}
\]
The inverse: \(\text{ele2polynome} (1, x)\)
**prodrac** \((L, k)\)  
*Function*  
\(L\) is a list containing the elementary symmetric functions on a set \(A\). **prodrac** returns the polynomial whose roots are the \(k\) by \(k\) products of the elements of \(A\).  
Also see **somrac**.

**pui2polynome** \((x, lpui)\)  
*Function*  
calculates the polynomial in \(x\) whose power functions of the roots are given in the list \(lpui\).

```plaintext
(%i1) pui;
(%o1) 1
(%i2) kill(labels);
(%o0) done
(%i1) polynome2ele (x^3 - 4*x^2 + 5*x - 1, x);
(%o1) [3, 4, 5, 1]
(%i2) ele2pui (3, %);
(%o2) [3, 4, 6, 7]
(%i3) pui2polynome (x, %);
3 2
(%o3) x - 4 x + 5 x - 1
```

See also: **polynome2ele**, **ele2polynome**.

**somrac** \((L, k)\)  
*Function*  
The list \(L\) contains elementary symmetric functions of a polynomial \(P\). The function computes the polynomial whose roots are the \(k\) by \(k\) distinct sums of the roots of \(P\).  
Also see **prodrac**.

### 30.2.6 Resolvents

**resolvante** \((P, x, f, [x_\_1,..., \_d])\)  
*Function*  
calculates the resolvent of the polynomial \(P\) in \(x\) of degree \(n \geq d\) by the function \(f\) expressed in the variables \(x_\_1, ..., x_\_d\). For efficiency of computation it is important to not include in the list \([x_\_1, ..., x_\_d]\) variables which do not appear in the transformation function \(f\).

To increase the efficiency of the computation one may set flags in **resolvante** so as to use appropriate algorithms:

If the function \(f\) is unitary:

- A polynomial in a single variable,
- linear,
- alternating,
- a sum,
- symmetric,
- a product,
- the function of the Cayley resolvent (usable up to degree 5)

\[
(x_1*x_2 + x_2*x_3 + x_3*x_4 + x_4*x_5 + x_5*x_1 -
(x_1*x_3 + x_3*x_5 + x_5*x_2 + x_2*x_4 + x_4*x_1))^2
\]

general,
the flag of resolvante may be, respectively:

- unitaire,
- lineaire,
- alternee,
- somme,
- produit,
- cayley,
- generale.

(%i1) resolvante: unitaire$
(%i2) resolvante (x^7 - 14*x^5 + 56*x^3 - 56*x + 22, x, x^3 - 1, [x]);
" resolvante unitaire " [7, 0, 28, 0, 168, 0, 1120, - 154, 7840, - 2772, 56448, - 33880, 413952, - 352352, 3076668, - 3363360, 23114112, - 30494464, 175230832, - 267412992, 1338886528, - 2292126760] [x - 1, x - 2 x + 1, x - 3 x + 3 x - 1, 12 9 6 3 15 12 9 6 3 x - 4 x + 6 x - 4 x + 1, x - 5 x + 10 x - 10 x + 5 x 18 15 12 9 6 3 - 1, x - 6 x + 15 x - 20 x + 15 x - 6 x + 1, 21 18 15 12 9 6 3 x - 7 x + 21 x - 35 x + 35 x - 21 x + 7 x - 1] [- 7, 1127, - 6139, 431767, - 5472047, 201692519, - 3603982011] 7 6 5 4 3 2 7 (o2) y + 7 y - 539 y - 1841 y + 51443 y + 315133 y + 376999 y + 125253
(%i3) resolvante: lineaire$
(%i4) resolvante (x^4 - 1, x, x1 + 2*x2 + 3*x3, [x1, x2, x3]);
" resolvante lineaire " 24 20 16 12 8 (%o4) y + 80 y + 7520 y + 1107200 y + 49475840 y 4 + 344489984 y + 655360000
(%i5) resolvante: general$
(%i6) resolvante (x^4 - 1, x, x1 + 2*x2 + 3*x3, [x1, x2, x3]);
" resolvante generale "
\[ 24 \quad 20 \quad 16 \quad 12 \quad 8 \]
(%o6) y + 80y + 7520y + 1107200y + 49475840y
\[ + 344489984y + 655360000 \]
(%i7) resolvante (x^4 - 1, x, x1 + 2*x2 + 3*x3, [x1, x2, x3, x4]);
" resolvante generale "
\[ 24 \quad 20 \quad 16 \quad 12 \quad 8 \]
(%o7) y + 80y + 7520y + 1107200y + 49475840y
\[ + 344489984y + 655360000 \]
(%i8) direct ([x^4 - 1], x, x1 + 2*x2 + 3*x3, [[x1, x2, x3]]);
" resolvante lineaire "
\[ 4 \]
(%o8) y - 1
(%i9) resolvante: lineaire$
(%i10) resolvante (x^4 - 1, x, x1 + x2 + x3, [x1, x2, x3]);
" resolvante symetrique "
\[ 4 \]
(%o10) y - 1
(%i11) resolvante: symetrique$
(%i12) resolvante (x^4 - 1, x, x1 + x2 + x3, [x1, x2, x3]);
" resolvante symetrique "
\[ 4 \]
(%o12) y - 1
(%i13) resolvante (x^4 + x + 1, x, x1 - x2, [x1, x2]);
" resolvante alternee "
\[ 6 \quad 2 \]
(%o13) y - 4y - 1
(%i14) resolvante: alternee$
(%i15) resolvante (x^4 + x + 1, x, x1 - x2, [x1, x2]);
" resolvante alternee "
\[ 12 \quad 8 \quad 6 \quad 4 \quad 2 \]
(%o15) y + 8y + 26y - 112y + 216y + 229
(%i16) resolvante: produit$
(\%i17) resolvante (x^7 - 7*x + 3, x, x1*x2*x3, [x1, x2, x3]);

" resolvante produit "

\[
\begin{array}{cccccccc}
35 & 33 & 29 & 28 & 27 & 26 \\
35 & 33 & 29 & 28 & 27 & 26 \\
\end{array}
\]

(\%o17) \begin{align*}
& y - 7 y - 1029 y + 135 y + 7203 y - 756 y \\
& + 1323 y + 352947 y - 46305 y - 2463339 y + 324135 y \\
& - 30618 y - 453789 y - 40246444 y + 282225202 y \\
& + 155098503 y + 12252303 y + 2893401 y \\
& + 6751269 y + 2657205 y - 94517766 y \\
& - 3720087 y + 26040609 y + 14348907
\end{align*}

(\%i18) resolvante: symetrique$

(\%i19) resolvante (x^7 - 7*x + 3, x, x1*x2*x3, [x1, x2, x3]);

" resolvante symetrique "

\[
\begin{array}{cccccccc}
35 & 33 & 29 & 28 & 27 & 26 \\
35 & 33 & 29 & 28 & 27 & 26 \\
\end{array}
\]

(\%o19) \begin{align*}
& y - 7 y - 1029 y + 135 y + 7203 y - 756 y \\
& + 1323 y + 352947 y - 46305 y - 2463339 y + 324135 y \\
& - 30618 y - 453789 y - 40246444 y + 282225202 y \\
& + 155098503 y + 12252303 y + 2893401 y \\
& + 6751269 y + 2657205 y - 94517766 y \\
& - 3720087 y + 26040609 y + 14348907
\end{align*}

(\%i20) resolvante: cayley$
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(%i21) resolvante (x^5 - 4*x^2 + x + 1, x, a, []);

" resolvante de Cayley "

6 5 4 3 2
(%o21) x - 40 x + 4080 x - 92928 x + 3772160 x + 37880832 x
+ 93392896

For the Cayley resolvent, the 2 last arguments are neutral and the input polynomial
must necessarily be of degree 5.

See also:
⟨undefined⟩ resolvante_bipartite, ⟨undefined⟩ resolvante_produit_sym,
⟨undefined⟩ resolvante_unitaire, ⟨undefined⟩ resolvante_alternee1, ⟨undefined⟩
resolvante_klein,
⟨undefined⟩ resolvante_klein3, ⟨undefined⟩ resolvante_vierer, ⟨undefined⟩ resolvante_diedrale.

resolvante_alternee1 (P, x) [Function]
calculates the transformation P(x) of degree n by the function \( \prod_{1 \leq i < j \leq n-1} (x_i - x_j). \)

See also:
⟨undefined⟩ resolvante_produit_sym, ⟨undefined⟩ resolvante_unitaire,
⟨undefined⟩ resolvante, ⟨undefined⟩ resolvante_klein, ⟨undefined⟩ resolvante_klein3,
⟨undefined⟩ resolvante_vierer, ⟨undefined⟩ resolvante_diedrale, ⟨undefined⟩
resolvante_bipartite.

resolvante_bipartite (P, x) [Function]
calculates the transformation of \( P(x) \) of even degree n by the function \( x_1 x_2 \cdots x_{n/2} + x_{n/2+1} \cdots x_n. \)

(%i11) resolvante_bipartite (x^6 + 108, x);

10 8 6 4
(%o11) y - 972 y + 314928 y - 34012224 y

See also:
⟨undefined⟩ resolvante_produit_sym, ⟨undefined⟩ resolvante_unitaire,
⟨undefined⟩ resolvante, ⟨undefined⟩ resolvante_klein, ⟨undefined⟩ resolvante_klein3,
⟨undefined⟩ resolvante_vierer, ⟨undefined⟩ resolvante_diedrale, ⟨undefined⟩
resolvante_alternee1.

resolvante_diedrale (P, x) [Function]
calculates the transformation of \( P(x) \) by the function \( x_1 x_2 + x_4 x_3. \)
(%i1) resolvante_diedrale (x^5 - 3*x^4 + 1, x);
   15 12 11 10 9 8 7
(%o1) x - 21 x - 81 x - 21 x + 207 x + 1134 x + 2331 x
   6 5 4 3 2
- 945 x - 4970 x - 18333 x - 29079 x - 20745 x - 25326 x
- 697
See also:
(undef) resolvanteProduitSym, (undef) resolvanteUnitaire,
(undef) resolvanteAlternee1, (undef) resolvanteKlein,
(undef) resolvanteKlein3,
(undef) resolvanteVierer, (undef) resolvante.

resolvanteKlein (P, x)  [Function]
calculates the transformation of P(x) by the function x_1 x_2 x_4 + x_4.
See also:
(undef) resolvanteProduitSym, (undef) resolvanteUnitaire,
(undef) resolvanteAlternee1, (undef) resolvante, (undef) resolvanteKlein3,
(undef) resolvanteVierer, (undef) resolvanteDiedrale.

resolvanteKlein3 (P, x)  [Function]
calculates the transformation of P(x) by the function x_1 x_2 x_4 + x_4.
See also:
(undef) resolvanteProduitSym, (undef) resolvanteUnitaire,
(undef) resolvanteAlternee1, (undef) resolvante, (undef) resolvanteKlein,
(undef) resolvanteVierer, (undef) resolvanteDiedrale.

resolvanteProduitSym (P, x)  [Function]
calculates the list of all product resolvents of the polynomial P(x).

(%i1) resolvanteProduitSym (x^5 + 3*x^4 + 2*x - 1, x);
   5 4 10 8 7 6 5
(%o1) [y + 3 y + 2 y - 1, y - 2 y - 21 y - 31 y - 14 y
   4 3 2 10 8 7 6 5
- y + 14 y + 3 y + 1, y + 3 y + 14 y - y - 14 y - 31 y
   3 2 5 4
- 21 y - 2 y + 1, y - 2 y - 3 y - 1, y - 1]
(%i2) resolvante: produit$
(%i3) resolvante (x^5 + 3*x^4 + 2*x - 1, x, a*b*c, [a, b, c]);
   10 8 7 6 5 4 3 2
" resolvante produit "
(%o3) y + 3 y + 14 y - y - 14 y - 31 y - 21 y - 2 y + 1
See also:
(resolvante, resolvante_unitaire, resolvante_alternee1, resolvante_klein, resolvante_klein3, resolvante_vierer, resolvante_diedrale).

(resolvante_unitaire(P, Q, x)) [Function]
computes the resolvent of the polynomial \( P(x) \) by the polynomial \( Q(x) \).

(resolvante_vierer(P, x)) [Function]
computes the transformation of \( P(x) \) by the function \( x_1 x_2 - x_3 x_4 \).

30.2.7 Miscellaneous

(multinomial(r, part)) [Function]
where \( r \) is the weight of the partition \( part \). This function returns the associate multinomial coefficient: if the parts of \( part \) are \( i_1, i_2, \ldots, i_k \), the result is \( r!/(i_1! \cdot i_2! \cdot \ldots \cdot i_k!) \).

(permut(L)) [Function]
returns the list of permutations of the list \( L \).
31 Groups

31.1 Functions and Variables for Groups

todd_coxeter

\texttt{todd\_coxeter (relations, subgroup)}

\texttt{todd\_coxeter (relations)}

Find the order of $G/H$ where $G$ is the Free Group modulo \textit{relations}, and $H$ is the subgroup of $G$ generated by \textit{subgroup}. \textit{subgroup} is an optional argument, defaulting to \texttt{[]}.. In doing this it produces a multiplication table for the right action of $G$ on $G/H$, where the cosets are enumerated $[H,Hg_2,Hg_3,...]$. This can be seen internally in the variable \texttt{todd\_coxeter\_state}.

Example:

\begin{verbatim}(\%)1 \texttt{symet(n):=create\_list(}
  \begin{verbatim}
  if (j - i) = 1 then (p(i,j))^3 else
  if (not i = j) then (p(i,j))^2 else
  p(i,i), j, 1, n-1, i, 1, j);\end{verbatim}\end{verbatim}

\begin{verbatim}(\%)2 \texttt{symet(n):=create\_list(if j - i = 1 then p(i, j)}
  \begin{verbatim}
else (if not i = j then p(i, j) else p(i, i)), j, 1, n - 1, i, 1, j)\end{verbatim}\end{verbatim}

\begin{verbatim}(\%)3 \texttt{p(i,j) := concat(x,i).concat(x,j);}
(\%)4 \texttt{p(i,j) := concat(x,i).concat(x,j)}
(\%)5 \texttt{symet(5);}
(\%)6 \texttt{symet(5);}
(\%)7 \texttt{[x1 , (x1 . x2) , x2 , (x1 . x3) , (x2 . x3) ,}
  \begin{verbatim}
  x3 , (x1 . x4) , (x2 . x4) , (x3 . x4) , x4 ]\end{verbatim}\end{verbatim}

\begin{verbatim}(\%)8 \texttt{todd\_coxeter(%o3);}
\end{verbatim}

\begin{verbatim}
Rows tried 426
(\%)9 \texttt{120}
(\%)10 \texttt{todd\_coxeter(%o3,[x1]);}
\end{verbatim}

\begin{verbatim}
Rows tried 213
(\%)11 \texttt{60}
(\%)12 \texttt{todd\_coxeter(%o3,[x1,x2]);}
\end{verbatim}

\begin{verbatim}
Rows tried 71
(\%)13 \texttt{20}
\end{verbatim}
32 Runtime Environment

32.1 Introduction for Runtime Environment

`maxima-init.mac` is a file which is loaded automatically when Maxima starts. You can use `maxima-init.mac` to customize your Maxima environment. `maxima-init.mac`, if it exists, is typically placed in the directory named by `maxima_userdir`, although it can be in any directory searched by the function `file_search`.

Here is an example `maxima-init.mac` file:

```maxima
setup_autoload ("specfun.mac", ultraspherical, assoc_legendre_p);
showtime:all;
```

In this example, `setup_autoload` tells Maxima to load the specified file (`specfun.mac`) if any of the functions (`ultraspherical`, `assoc_legendre_p`) are called but not yet defined. Thus you needn’t remember to load the file before calling the functions.

The statement `showtime: all` tells Maxima to set the `showtime` variable. The `maxima-init.mac` file can contain any other assignments or other Maxima statements.

32.2 Interrupts

The user can stop a time-consuming computation with the "C (control-C) character. The default action is to stop the computation and print another user prompt. In this case, it is not possible to restart a stopped computation.

If the Lisp variable `*debugger-hook*` is set to `nil`, by executing

```lisp
:lisp (setq *debugger-hook* nil)
```

then upon receiving "C, Maxima will enter the Lisp debugger, and the user may use the debugger to inspect the Lisp environment. The stopped computation can be restarted by entering `continue` in the Lisp debugger. The means of returning to Maxima from the Lisp debugger (other than running the computation to completion) is different for each version of Lisp.

On Unix systems, the character "Z (control-Z) causes Maxima to stop altogether, and control is returned to the shell prompt. The `fg` command causes Maxima to resume from the point at which it was stopped.

32.3 Functions and Variables for Runtime Environment

`maxima_tempdir`  
[System variable]  
`maxima_tempdir` names the directory in which Maxima creates some temporary files. In particular, temporary files for plotting are created in `maxima_tempdir`.

The initial value of `maxima_tempdir` is the user’s home directory, if Maxima can locate it; otherwise Maxima makes a guess about a suitable directory.

`maxima_tempdir` may be assigned a string which names a directory.

`maxima_userdir`  
[System variable]  
`maxima_userdir` names a directory which Maxima searches to find Maxima and Lisp files. (Maxima searches some other directories as well; `file_search_maxima` and `file_search_lisp` are the complete lists.)
The initial value of `maxima_userdir` is a subdirectory of the user’s home directory, if Maxima can locate it; otherwise Maxima makes a guess about a suitable directory. `maxima_userdir` may be assigned a string which names a directory. However, assigning to `maxima_userdir` does not automatically change `file_search_maxima` and `file_search_lisp`; those variables must be changed separately.

### Function: `room`

- `room ()`
- `room (true)`
- `room (false)`

Prints out a description of the state of storage and stack management in Maxima. `room` calls the Lisp function of the same name.

- `room ()` prints out a moderate description.
- `room (true)` prints out a verbose description.
- `room (false)` prints out a terse description.

### Function: `sstatus (keyword, item)`

When `keyword` is the symbol `feature`, `item` is put on the list of system features. After `sstatus (keyword, item)` is executed, `status (feature, item)` returns `true`. If `keyword` is the symbol `nofeature`, `item` is deleted from the list of system features. This can be useful for package writers, to keep track of what features they have loaded in.

See also `status`.

### Function: `status`

- `status (feature)`
- `status (feature, item)`

Returns information about the presence or absence of certain system-dependent features.

- `status (feature)` returns a list of system features. These include Lisp version, operating system type, etc. The list may vary from one Lisp type to another.
- `status (feature, item)` returns `true` if `item` is on the list of items returned by `status (feature)` and `false` otherwise. `status` quotes the argument `item`. The quote-quote operator `'` defeats quotation. A feature whose name contains a special character, such as a hyphen, must be given as a string argument. For example, `status (feature, "ansi-cl")`.

See also `sstatus`.

The variable `features` contains a list of features which apply to mathematical expressions. See `features` and `featurep` for more information.

### Function: `system (command)`

Executes `command` as a separate process. The command is passed to the default shell for execution. `system` is not supported by all operating systems, but generally exists in Unix and Unix-like environments.

Supposing `_hist.out` is a list of frequencies which you wish to plot as a bar graph using `xgraph`.

```
(%i1) with_stdout("_hist.out",
```
for i:1 thru length(hist) do (  
    print(i, hist[i]));
system("xgraph -bar -brw .7 -nl < _hist.out"));

In order to make the plot be done in the background (returning control to Maxima)  
and remove the temporary file after it is done do:

system("(xgraph -bar -brw .7 -nl < _hist.out; rm -f _hist.out)&")

time (%o1, %o2, %o3, ...)
[Function]

Returns a list of the times, in seconds, taken to compute the output lines %o1, %o2,  
%o3, ... The time returned is Maxima’s estimate of the internal computation time,  
not the elapsed time. time can only be applied to output line variables; for any other  
variables, time returns unknown.

Set showtime: true to make Maxima print out the computation time and elapsed  
time with each output line.

timedate
[Function]

timedate (T, tz_offset)
timedate (T)
timedate ()

timedate(T, tz_offset) returns a string representing the time T in the time zone  
tz_offset. The string format is YYYY-MM-DD HH:MM:SS.NNN[+|-]ZZ:ZZ (using as many  
digits as necessary to represent the fractional part) if T has a nonzero fractional part,  
or YYYY-MM-DD HH:MM:SS[+|-]ZZ:ZZ if its fractional part is zero.

T measures time, in seconds, since midnight, January 1, 1900, in the GMT time zone.  
tz_offset measures the offset of the time zone, in hours, east (positive) or west (neg-  
avtive) of GMT. tz_offset must be an integer, rational, or float between -24 and 24,  
inclusive. If tz_offset is not a multiple of 1/60, it is rounded to the nearest multiple  
of 1/60.

timedate(T) is equivalent to timedate(T, tz_offset) with tz_offset equal to the  
offset of the local time zone.

timedate() is equivalent to timedate(absolute_real_time()). That is, it returns  
the current time in the local time zone.

Example:
timedate with no argument returns a string representing the current time and date.

(Ok1) d : timedate ();
(Ok1) 2010-06-08 04:08:09+01:00
(Ok2) print ("timedate reports current time", d) $  
timedate reports current time 2010-06-08 04:08:09+01:00

timedate with an argument returns a string representing the argument.

(Ok1) timedate (0);
(Ok1) 1900-01-01 01:00:00+01:00
(Ok2) timedate (absolute_real_time () - 7*24*3600);
(Ok2) 2010-06-01 04:19:51+01:00

timedate with optional timezone offset.

(Ok1) timedate (1000000000, -9.5);
parse_timedate

Function

parse_timedate (S)

Parses a string S representing a date or date and time of day and returns the number of seconds since midnight, January 1, 1900 GMT. If there is a nonzero fractional part, the value returned is a rational number, otherwise, it is an integer. parse_timedate returns false if it cannot parse S according to any of the allowed formats.

The string S must have one of the following formats, optionally followed by a timezone designation:

- YYYY-MM-DD[ T]hh:mm:ss[,.]nnn
- YYYY-MM-DD[ T]hh:mm:ss
- YYYY-MM-DD

where the fields are year, month, day, hours, minutes, seconds, and fraction of a second, and square brackets indicate acceptable alternatives. The fraction may contain one or more digits.

Except for the fraction of a second, each field must have exactly the number of digits indicated: four digits for the year, and two for the month, day of the month, hours, minutes, and seconds.

A timezone designation must have one of the following forms:

- [+-]hh:mm
- [+-]hhmm
- [+-]hh
- Z

where hh and mm indicate hours and minutes east (+) or west (-) of GMT. The timezone may be from +24 hours (inclusive) to -24 hours (inclusive).

A literal character Z is equivalent to +00:00 and its variants, indicating GMT.

If no timezone is indicated, the time is assumed to be in the local time zone.

Any leading or trailing whitespace (space, tab, newline, and carriage return) is ignored, but any other leading or trailing characters cause parse_timedate to fail and return false.

See also timedate and absolute_real_time.

Examples:

Midnight, January 1, 1900, in the local time zone, in each acceptable format. The result is the number of seconds the local time zone is ahead (negative result) or behind (positive result) GMT. In this example, the local time zone is 8 hours behind GMT.

(\%i1) parse_timedate ("1900-01-01 00:00:00,000");
(\%o1) 28800
(\%i2) parse_timedate ("1900-01-01 00:00:00.000");
(\%o2) 28800
(\%i3) parse_timedate ("1900-01-01T00:00:00,000");
(\%o3) 28800
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(%i4) parse_timedate ("1900-01-01T00:00:00.000");
(%o4) 28800
(%i5) parse_timedate ("1900-01-01 00:00:00");
(%o5) 28800
(%i6) parse_timedate ("1900-01-01T00:00:00");
(%o6) 28800
(%i7) parse_timedate ("1900-01-01");
(%o7) 28800

Midnight, January 1, 1900, GMT, in different indicated time zones.

(%i8) parse_timedate ("1900-01-01 19:00:00+19:00");
(%o8) 0
(%i9) parse_timedate ("1900-01-01 07:00:00+07:00");
(%o9) 0
(%i10) parse_timedate ("1900-01-01 01:00:00+01:00");
(%o10) 0
(%i11) parse_timedate ("1900-01-01Z");
(%o11) 0
(%i12) parse_timedate ("1899-12-31 21:00:00-03:00");
(%o12) 0
(%i13) parse_timedate ("1899-12-31 13:00:00-11:00");
(%o13) 0
(%i14) parse_timedate ("1899-12-31 08:00:00-16:00");
(%o14) 0

encode_time

[Function]
encode_time (year, month, day, hours, minutes, seconds, tz_offset)

Given a time and date specified by year, month, day, hours, minutes, and seconds, encode_time returns the number of seconds (possibly including a fractional part) since midnight, January 1, 1900 GMT.

year must be an integer greater than or equal to 1899. However, 1899 is allowed only if the resulting encoded time is greater than or equal to 0.

month must be an integer from 1 to 12, inclusive.

day must be an integer from 1 to n, inclusive, where n is the number of days in the month specified by month.

hours must be an integer from 0 to 23, inclusive.

minutes must be an integer from 0 to 59, inclusive.

seconds must be an integer, rational, or float greater than or equal to 0 and less than 60. When seconds is not an integer, encode_time returns a rational, such that the fractional part of the return value is equal to the fractional part of seconds. Otherwise, seconds is an integer, and the return value is likewise an integer.

tz_offset measures the offset of the time zone, in hours, east (positive) or west (negative) of GMT. tz_offset must be an integer, rational, or float between -24 and 24, inclusive. If tz_offset is not a multiple of 1/3600, it is rounded to the nearest multiple of 1/3600.
If \( tz\_offset \) is not present, the offset of the local time zone is assumed. See also \( \text{decode\_time} \).

Examples:

\begin{verbatim}
(%i1) encode_time (1900, 1, 1, 0, 0, 0, 0);
(%o1) 0
(%i2) encode_time (1970, 1, 1, 0, 0, 0, 0);
(%o2) 2208988800
(%i3) encode_time (1970, 1, 1, 8, 30, 0, 8.5);
(%o3) 2208988800
(%i4) encode_time (1969, 12, 31, 16, 0, 0, -8);
(%o4) 2208988800
(%i5) encode_time (1969, 12, 31, 16, 0, 1/1000, -8);
(%o5) 2208988800001    1
-----------------
1000
(%i6) % - 2208988800;
(%o6) 1000
\end{verbatim}

\( \text{decode\_time} \) [Function]

\begin{verbatim}
\text{decode\_time \( (T, tz\_offset) \)} \text{decode\_time \( (T) \)}
\end{verbatim}

\( \text{decode\_time} \) Given the number of seconds (possibly including a fractional part) since midnight, January 1, 1900 GMT, returns the date and time as represented by a list comprising the year, month, day of the month, hours, minutes, seconds, and time zone offset. \( tz\_offset \) measures the offset of the time zone, in hours, east (positive) or west (negative) of GMT. \( tz\_offset \) must be an integer, rational, or float between -24 and 24, inclusive. If \( tz\_offset \) is not a multiple of \( 1/3600 \), it is rounded to the nearest multiple of \( 1/3600 \).

If \( tz\_offset \) is not present, the offset of the local time zone is assumed. See also \( \text{encode\_time} \).

Examples:

\begin{verbatim}
(%i1) decode_time (0, 0);
(%o1) \[1900, 1, 1, 0, 0, 0, 0]\n(%i2) decode_time (0);
(%o2) \[1899, 12, 31, 16, 0, 0, -8]\n(%i3) decode_time (2208988800, 9.25);
(%o3) \[1969, 12, 31, 16, 0, 1/1000, -8]\n\end{verbatim}

\begin{verbatim}
2208988800001    1
-----------------
1000
(%i6) % - 2208988800;
(%o6) 1000
\end{verbatim}
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(%o5) [1969, 12, 31, 18, 0, ----, - 6]
    1000
(%i6) decode_time (2208988800 + 1729/1000);
     1729
(%o6) [1969, 12, 31, 16, 0, ----, - 8]
    1000

**absolute_real_time ()**

[Function]

Returns the number of seconds since midnight, January 1, 1900 GMT. The return value is an integer.

See also `elapsed_real_time` and `elapsed_run_time`.

Example:

(%i1) absolute_real_time ();
(%o1) 3385045277
(%i2) 1900 + absolute_real_time () / (365.25 * 24 * 3600);  
(%o2) 2007.265612087104

**elapsed_real_time ()**

[Function]

Returns the number of seconds (including fractions of a second) since Maxima was most recently started or restarted. The return value is a floating-point number.

See also `absolute_real_time` and `elapsed_run_time`.

Example:

(%i1) elapsed_real_time ();
(%o1) 2.559324
(%i2) expand ((a + b)^500)$
(%i3) elapsed_real_time ();
(%o3) 7.552087

**elapsed_run_time ()**

[Function]

Returns an estimate of the number of seconds (including fractions of a second) which Maxima has spent in computations since Maxima was most recently started or restarted. The return value is a floating-point number.

See also `absolute_real_time` and `elapsed_real_time`.

Example:

(%i1) elapsed_run_time ();
(%o1) 0.04
(%i2) expand ((a + b)^500)$
(%i3) elapsed_run_time ();
(%o3) 1.26
33 Miscellaneous Options

33.1 Introduction to Miscellaneous Options
In this section various options are discussed which have a global effect on the operation of Maxima. Also various lists such as the list of all user defined functions, are discussed.

33.2 Share
The Maxima "share" directory contains programs and other files of interest to Maxima users, but not part of the core implementation of Maxima. These programs are typically loaded via `load` or `setup_autoload`.

`:lisp *maxima-sharedir*` displays the location of the share directory within the user’s file system.

`printfile ("share.usg")` prints an out-of-date list of share packages. Users may find it more informative to browse the share directory using a file system browser.

33.3 Functions and Variables for Miscellaneous Options

askexp

When `asksign` is called, `askexp` is the expression `asksign` is testing.

At one time, it was possible for a user to inspect `askexp` by entering a Maxima break with control-A.

`gensumnum`

Default value: 0

`gensumnum` is the numeric suffix used to generate the next variable of summation. If it is set to `false` then the index will consist only of `genindex` with no numeric suffix.

gensym

`gensym ()` creates and returns a fresh symbol.

The name of the new symbol is the concatenation of a prefix, which defaults to "g", and a suffix, which is an integer that defaults to the value of an internal counter.

If `x` is supplied, and is a string, then that string is used as a prefix instead of "g" for this call to `gensym` only.

If `x` is supplied, and is a nonnegative integer, then that integer, instead of the value of the internal counter, is used as the suffix for this call to `gensym` only.

If and only if no explicit suffix is supplied, the internal counter is incremented after it is used.
Examples:

(\%i1) gensym();
\%o1) g887
(\%i2) gensym("new");
\%o2) new888
(\%i3) gensym(123);
\%o3) g123

packagefile

[Option variable]

Default value: \texttt{false}

Package designers who use \texttt{save} or \texttt{translate} to create packages (files) for others to use may want to set \texttt{packagefile: true} to prevent information from being added to Maxima’s information-lists (e.g. \texttt{values, functions}) except where necessary when the file is loaded in. In this way, the contents of the package will not get in the user’s way when he adds his own data. Note that this will not solve the problem of possible name conflicts. Also note that the flag simply affects what is output to the package file. Setting the flag to \texttt{true} is also useful for creating Maxima init files.

remvalue

[Function]

\texttt{remvalue (name\_1, \ldots, name\_n)}
\texttt{remvalue remvalue (all)}

Removes the values of user variables \texttt{name\_1, \ldots, name\_n} (which can be subscripted) from the system.

\texttt{remvalue (all)} removes the values of all variables in \texttt{values}, the list of all variables given names by the user (as opposed to those which are automatically assigned by Maxima).

See also \texttt{values}.

rncombine (expr)

[Function]

Transforms \texttt{expr} by combining all terms of \texttt{expr} that have identical denominators or denominators that differ from each other by numerical factors only. This is slightly different from the behavior of \texttt{combine}, which collects terms that have identical denominators.

Setting \texttt{pfeformat: true} and using \texttt{combine} yields results similar to those that can be obtained with \texttt{rncombine}, but \texttt{rncombine} takes the additional step of cross-multiplying numerical denominator factors. This results in neater forms, and the possibility of recognizing some cancellations.

\texttt{load("rncomb")} loads this function.

setup_autoload (filename, function\_1, \ldots, function\_n)

[Function]

Specifies that if any of \texttt{function\_1, \ldots, function\_n} are referenced and not yet defined, \texttt{filename} is loaded via \texttt{load}. \texttt{filename} usually contains definitions for the functions specified, although that is not enforced.

\texttt{setup_autoload} does not work for \texttt{memoizing functions}.

\texttt{setup_autoload} quotes its arguments.

Example:

(\%i1) legendre\_p (1, \%pi);
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(%o1) legendre_p(1, %pi)
(%i2) setup_autoload ("specfun.mac", legendre_p, ultraspherical);
(%o2) done
(%i3) ultraspherical (2, 1/2, %pi);
Warning - you are redefining the Macsyma function ultraspherical
Warning - you are redefining the Macsyma function legendre_p

2
3 (%pi - 1) + 3 (%pi - 1) + 1
2

(%o3) \frac{3 \left(\pi - 1\right)}{2} + 3 \left(\pi - 1\right) + 1

(%i4) legendre_p (1, %pi);
(%o4) %pi
(%i5) legendre_q (1, %pi);

%pi + 1
%pi log(------)
1 - %pi

(%o5) \frac{\pi + 1}{\pi \log\left(\frac{\pi}{\pi - 1}\right)} - 1

2

tcl_output

 [Function]
tcl_output (list, i0, skip)
tcl_output (list, i0)
tcl_output ([list_1, ..., list_n], i)

Prints elements of a list enclosed by curly braces \{\}, suitable as part of a program in the Tcl/Tk language.

tcl_output (list, i0, skip) prints list, beginning with element i0 and printing elements i0 + skip, i0 + 2 skip, etc.

tcl_output (list, i0) is equivalent to tcl_output (list, i0, 2).

tcl_output ([list_1, ..., list_n], i) prints the i’th elements of list_1, ..., list_n.

Examples:

(%i1) tcl_output ([1, 2, 3, 4, 5, 6], 1, 3)$

\{1.000000000
4.000000000
\}
(%i2) tcl_output ([1, 2, 3, 4, 5, 6], 2, 3)$

\{2.000000000
5.000000000
\}
(%i3) tcl_output ([3/7, 5/9, 11/13, 13/17], 1)$

\{(\text{RAT SIMP} 3 7) (\text{RAT SIMP} 11 13)\}
(%i4) tcl_output ([x1, y1, x2, y2, x3, y3], 2)$

\{$x1 \ x2 \ x3 \ y1 \ y2 \ y3\}
} (\%i5) tcl_output ([[1, 2, 3], [11, 22, 33]], 1)$

{SIMP 1.000000000 11.00000000}

}
34 Rules and Patterns

34.1 Introduction to Rules and Patterns

This section describes user-defined pattern matching and simplification rules. There are two groups of functions which implement somewhat different pattern matching schemes. In one group are `tellsimp`, `tellsimpafter`, `defmatch`, `defrule`, `apply1`, `applyb1`, and `apply2`. In the other group are `let` and `letsimp`. Both schemes define patterns in terms of pattern variables declared by `matchdeclare`.

Pattern-matching rules defined by `tellsimp` and `tellsimpafter` are applied automatically by the Maxima simplifier. Rules defined by `defmatch`, `defrule`, and `let` are applied by an explicit function call.

There are additional mechanisms for rules applied to polynomials by `tellrat`, and for commutative and noncommutative algebra in `affine` package.

34.2 Functions and Variables for Rules and Patterns

`apply1(expr, rule_1, ..., rule_n)`  
[Function]  
Repeatedly applies `rule_1` to `expr` until it fails, then repeatedly applies the same rule to all subexpressions of `expr`, left to right, until `rule_1` has failed on all subexpressions. Call the result of transforming `expr` in this manner `expr_2`. Then `rule_2` is applied in the same fashion starting at the top of `expr_2`. When `rule_n` fails on the final subexpression, the result is returned.

`maxapplydepth` is the depth of the deepest subexpressions processed by `apply1` and `apply2`.

See also `applyb1`, `apply2` and `let`.

`apply2(expr, rule_1, ..., rule_n)`  
[Function]  
If `rule_1` fails on a given subexpression, then `rule_2` is repeatedly applied, etc. Only if all rules fail on a given subexpression is the whole set of rules repeatedly applied to the next subexpression. If one of the rules succeeds, then the same subexpression is reprocessed, starting with the first rule.

`maxapplydepth` is the depth of the deepest subexpressions processed by `apply1` and `apply2`.

See also `apply1` and `let`.

`applyb1(expr, rule_1, ..., rule_n)`  
[Function]  
Repeatedly applies `rule_1` to the deepest subexpression of `expr` until it fails, then repeatedly applies the same rule one level higher (i.e., larger subexpressions), until `rule_1` has failed on the top-level expression. Then `rule_2` is applied in the same fashion to the result of `rule_1`. After `rule_n` has been applied to the top-level expression, the result is returned.

`applyb1` is similar to `apply1` but works from the bottom up instead of from the top down.

`maxapplyheight` is the maximum height which `applyb1` reaches before giving up.

See also `apply1`, `apply2` and `let`. 
current_let_rule_package

[Option variable]

Default value: default_let_rule_package

current_let_rule_package is the name of the rule package that is used by functions in the let package (letsimp, etc.) if no other rule package is specified. This variable may be assigned the name of any rule package defined via the let command.

If a call such as letsimp (expr, rule_pkg_name) is made, the rule package rule_pkg_name is used for that function call only, and the value of current_let_rule_package is not changed.

default_let_rule_package

[Option variable]

Default value: default_let_rule_package

default_let_rule_package is the name of the rule package used when one is not explicitly set by the user with let or by changing the value of current_let_rule_package.

defmatch

[Function]

defmatch (progname, pattern, x_1, . . ., x_n)
defmatch (progname, pattern)

Defines a function progname(expr, x_1, . . ., x_n) which tests expr to see if it matches pattern.

pattern is an expression containing the pattern arguments x_1, . . ., x_n (if any) and some pattern variables (if any). The pattern arguments are given explicitly as arguments to defmatch while the pattern variables are declared by the matchdeclare function. Any variable not declared as a pattern variable in matchdeclare or as a pattern argument in defmatch matches only itself.

The first argument to the created function progname is an expression to be matched against the pattern and the other arguments are the actual arguments which correspond to the dummy variables x_1, . . ., x_n in the pattern.

If the match is successful, progname returns a list of equations whose left sides are the pattern arguments and pattern variables, and whose right sides are the subexpressions which the pattern arguments and variables matched. The pattern variables, but not the pattern arguments, are assigned the subexpressions they match. If the match fails, progname returns false.

A literal pattern (that is, a pattern which contains neither pattern arguments nor pattern variables) returns true if the match succeeds.

See also matchdeclare, defrule, tellsimp and tellsimpafter.

Examples:

Define a function linearp(expr, x) which tests expr to see if it is of the form a*x + b such that a and b do not contain x and a is nonzero. This match function matches expressions which are linear in any variable, because the pattern argument x is given to defmatch.

(%i1) matchdeclare (a, lambda ([e], e#0 and freeof(x, e)), b, freeof(x));
(%o1) done
(%i2) defmatch (linearp, a*x + b, x);
(%o2) linearp
Define a function `linearp(expr)` which tests `expr` to see if it is of the form `a*x + b` such that `a` and `b` do not contain `x` and `a` is nonzero. This match function only matches expressions linear in `x`, not any other variable, because no pattern argument is given to `defmatch`.

Define a function `checklimits(expr)` which tests `expr` to see if it is a definite integral.
(%i8) 'integrate (sin(t), t, %pi + x, 2*%pi + x);
          x + 2 %pi
       / 
      [ 
        ]
    (%o8) I sin(t) dt 
       / 
    x + %pi

(%i9) checklimits (%);
( %o9) [ b = x + 2 %pi, a = x + %pi, x = t, f = sin(t)]

defrule (rule_name, pattern, replacement) [Function]
Defines and names a replacement rule for the given pattern. If the rule named rule
name is applied to an expression (by apply1, applyb1, or apply2), every subex-
pression matching the pattern will be replaced by the replacement. All variables in
the replacement which have been assigned values by the pattern match are assigned
those values in the replacement which is then simplified.

The rules themselves can be treated as functions which transform an expression by
one operation of the pattern match and replacement. If the match fails, the rule
function returns false.

disprule (rule_name_1, ..., rule_name_2) [Function]
disprule (all)
Display rules with the names rule_name_1, ..., rule_name_n, as returned by defrule,
tellsimp, or tellsimpafter, or a pattern defined by defmatch. Each rule is dis-
played with an intermediate expression label (%t).

disprule (all) displays all rules.

disprule quotes its arguments. disprule returns the list of intermediate expression
labels corresponding to the displayed rules.

See also letrules, which displays rules defined by let.

Examples:
(%i1) tellsimpafter (foo (x, y), bar (x) + baz (y));
( %o1) [foorule1, false]
(%i2) tellsimpafter (x + y, special_add (x, y));
( %o2) [+rule1, simplus]
(%i3) defmatch (quux, mumble (x));
( %o3)
quux
(%i4) disprule (foorule1, +%rule1, quux);
( %t4) foorule1 : foo(x, y) -> baz(y) + bar(x)
( %t5) +rule1 : y + x -> special_add(x, y)
( %t6)
quux : mumble(x) -> []
( %o6) [%t4, %t5, %t6]
Chapter 34: Rules and Patterns

(\%i7) ev(%);
(\%o7) [foorule1 : foo(x, y) -> baz(y) + bar(x),
 +rule1 : y + x -> special_add(x, y), quux : mumble(x) -> []]

let

let (prod, repl, predname, arg_1, ..., arg_n)
let ([prod, repl, predname, arg_1, ..., arg_n], package_name)
 Defines a substitution rule for \texttt{letsimp} such that \texttt{prod} is replaced by \texttt{repl}. \texttt{prod} is a product of positive or negative powers of the following terms:

- Atoms which \texttt{letsimp} will search for literally unless previous to calling \texttt{letsimp} the \texttt{matchdeclare} function is used to associate a predicate with the atom. In this case \texttt{letsimp} will match the atom to any term of a product satisfying the predicate.
- Kernels such as \texttt{sin(x)}, n!, f(x,y), etc. As with atoms above \texttt{letsimp} will look for a literal match unless \texttt{matchdeclare} is used to associate a predicate with the argument of the kernel.

A term to a positive power will only match a term having at least that power. A term to a negative power on the other hand will only match a term with a power at least as negative. In the case of negative powers in \texttt{prod} the switch \texttt{letrat} must be set to \texttt{true}. See also \texttt{letrat}.

If a predicate is included in the \texttt{let} function followed by a list of arguments, a tentative match (i.e. one that would be accepted if the predicate were omitted) is accepted only if \texttt{predname (arg_1', ..., arg_n')} evaluates to \texttt{true} where \texttt{arg_i'} is the value matched to \texttt{arg_i}. The \texttt{arg_i} may be the name of any atom or the argument of any kernel appearing in \texttt{prod}. \texttt{repl} may be any rational expression. If any of the atoms or arguments from \texttt{prod} appear in \texttt{repl} the appropriate substitutions are made.

The global flag \texttt{letrat} controls the simplification of quotients by \texttt{letsimp}. When \texttt{letrat} is \texttt{false}, \texttt{letsimp} simplifies the numerator and denominator of \texttt{expr} separately, and does not simplify the quotient. Substitutions such as \texttt{n!/n} goes to \texttt{(n-1)!} then fail. When \texttt{letrat} is \texttt{true}, then the numerator, denominator, and the quotient are simplified in that order.

These substitution functions allow you to work with several rule packages at once. Each rule package can contain any number of \texttt{let} rules and is referenced by a user-defined name. The command \texttt{let ([prod, repl, predname, arg_1, ..., arg_n], package_name)} adds the rule \texttt{predname} to the rule package \texttt{package_name}.

The command \texttt{letsimp (expr, package_name)} applies the rules in \texttt{package_name}. \texttt{letsimp (expr, package_name1, package_name2, ...)} is equivalent to \texttt{letsimp (expr, package_name1)} followed by \texttt{letsimp (%1, package_name2), ...}

\texttt{current_let_rule_package} is the name of the rule package that is presently being used. This variable may be assigned the name of any rule package defined via the \texttt{let} command. Whenever any of the functions comprising the \texttt{let} package are called with no package name, the package named by \texttt{current_let_rule_package} is used. If a call such as \texttt{letsimp (expr, rule_pkg_name)} is made, the rule package \texttt{rule_pkg_name} is used for that \texttt{letsimp} command only, and \texttt{current_let_rule_package} is not changed. If not otherwise specified, \texttt{current_let_rule_package} defaults to \texttt{default_let_rule_package}.
(%i1) matchdeclare ([a, a1, a2], true)$
(%i2) oneless (x, y) := is (x = y-1)$
(%i3) let (a1*a2!, a1!, oneless, a2, a1);
    a1 a2! --> a1! where oneless(a2, a1)
(%i4) letrat: true$
(%i5) let (a1!/a1, (a1-1)!);  
   a1!
    --- --> (a1 - 1)!
   a1
(%i6) letsimp (n*m!*(n-1)!/m);
(%i7) let (sin(a)^2, 1 - cos(a)^2);
       2    2
sin (a) --> 1 - cos (a)
(%i8) letsimp (sin(x)^4);
       4    2
cos (x) - 2 cos (x) + 1

letrat              [Option variable]

Default value: false

When letrat is false, letsimp simplifies the numerator and denominator of a ratio separately, and does not simplify the quotient.

When letrat is true, the numerator, denominator, and their quotient are simplified in that order.

(%i11) matchdeclare (n, true)$
(%i12) let (n!/n, (n-1)!);
       n!
       -- --> (n - 1)!
       n
(%i13) letrat: false$
(%i14) letsimp (a!/a);
       a!
       --
       a
(%i15) letrat: true$
(%i16) letsimp (a!/a);
(%o6) (a - 1)!

letrules          [Function]

letrules ()
letrules (package_name)

Displays the rules in a rule package. letrules () displays the rules in the current rule package. letrules (package_name) displays the rules in package_name.

The current rule package is named by current_let_rule_package. If not otherwise specified, current_let_rule_package defaults to default_let_rule_package.

See also disprule, which displays rules defined by tellsimp and tellsimpafter.
**letsimp**

```lisp
letsimp (expr)
letsimp (expr, package_name)
letsimp (expr, package_name_1, ..., package_name_n)
```

Repeatedly applies the substitution rules defined by `let` until no further change is made to `expr`.

`letsimp (expr)` uses the rules from `current_let_rule_package`.

`letsimp (expr, package_name)` uses the rules from `package_name` without changing `current_let_rule_package`.

`letsimp (expr, package_name_1, ..., package_name_n)` is equivalent to `letsimp (expr, package_name_1)`, followed by `letsimp (%, package_name_2)`, and so on.

See also `let`. For other ways to do substitutions see also `subst`, `psubst`, `at` and `ratsubst`.

```lisp
(%i1) e0: e(k) = -(9*y(k))/(5*z)-u(k-1)/(5*z)+(4*y(k))/(5*z^2) + (3*u(k-1))/(5*z^2)+y(k)-(2*u(k-1))/5;
2 u(k - 1) 3 u(k - 3) 4 y(k - 2)
--- + y(k) + ---------- + ---------- + y(k)
5 5

(%o1) e(k) = (- -------) - ------- + ------- + --------- + y(k)
5 z 5 z 2 2

5 z 5 z
2 u(k - 1)

(%i2) matchdeclare(h,any)$
(%i3) let(u(h)/z,u(h-1));
(%o3) u(h) ---- --> u(h - 1)
 z
(%i4) let(y(h)/z, y(h-1));
(%o4) y(h) ---- --> y(h - 1)
 z
(%i5) e1:letsimp(e0);
(%o5) e(k) = (- ----------) + y(k) + --------- + ---------
5 5
u(k - 2) 9 y(k - 1)

+ (- ----------) + (- ----------)
5 5
```

**let_rule_packages**

Default value: `[default_let_rule_package]`

`let_rule_packages` is a list of all user-defined let rule packages plus the default package `default_let_rule_package`.

**matchdeclare (a_1, pred_1, ..., a_n, pred_n)**

Associates a predicate `pred_k` with a variable or list of variables `a_k` so that `a_k` matches expressions for which the predicate returns anything other than `false`. 
A predicate is the name of a function, or a lambda expression, or a function call or lambda call missing the last argument, or \texttt{true} or \texttt{all}. Any expression matches \texttt{true} or \texttt{all}. If the predicate is specified as a function call or lambda call, the expression to be tested is appended to the list of arguments; the arguments are evaluated at the time the match is evaluated. Otherwise, the predicate is specified as a function name or lambda expression, and the expression to be tested is the sole argument. A predicate function need not be defined when \texttt{matchdeclare} is called; the predicate is not evaluated until a match is attempted.

A predicate may return a Boolean expression as well as \texttt{true} or \texttt{false}. Boolean expressions are evaluated by \texttt{is} within the constructed rule function, so it is not necessary to call \texttt{is} within the predicate.

If an expression satisfies a match predicate, the match variable is assigned the expression, except for match variables which are operands of addition \texttt{+} or multiplication \texttt{*}. Only addition and multiplication are handled specially; other n-ary operators (both built-in and user-defined) are treated like ordinary functions.

In the case of addition and multiplication, the match variable may be assigned a single expression which satisfies the match predicate, or a sum or product (respectively) of such expressions. Such multiple-term matching is greedy: predicates are evaluated in the order in which their associated variables appear in the match pattern, and a term which satisfies more than one predicate is taken by the first predicate which it satisfies. Each predicate is tested against all operands of the sum or product before the next predicate is evaluated. In addition, if 0 or 1 (respectively) satisfies a match predicate, and there are no other terms which satisfy the predicate, 0 or 1 is assigned to the match variable associated with the predicate.

The algorithm for processing addition and multiplication patterns makes some match results (for example, a pattern in which a "match anything" variable appears) dependent on the ordering of terms in the match pattern and in the expression to be matched. However, if all match predicates are mutually exclusive, the match result is insensitive to ordering, as one match predicate cannot accept terms matched by another.

Calling \texttt{matchdeclare} with a variable \texttt{a} as an argument changes the \texttt{matchdeclare} property for \texttt{a}, if one was already declared; only the most recent \texttt{matchdeclare} is in effect when a rule is defined. Later changes to the \texttt{matchdeclare} property (via \texttt{matchdeclare} or \texttt{remove}) do not affect existing rules.

\texttt{propvars (matchdeclare)} returns the list of all variables for which there is a \texttt{matchdeclare} property. \texttt{printprops (a, matchdeclare)} returns the predicate for variable \texttt{a}. \texttt{printprops (all, matchdeclare)} returns the list of predicates for all \texttt{matchdeclare} variables. \texttt{remove (a, matchdeclare)} removes the \texttt{matchdeclare} property from \texttt{a}.

The functions \texttt{defmatch}, \texttt{defrule}, \texttt{tellsimp}, \texttt{tellsimpafter}, and \texttt{let} construct rules which test expressions against patterns.

\texttt{matchdeclare} quotes its arguments. \texttt{matchdeclare} always returns \texttt{done}.

Examples:

A predicate is the name of a function, or a lambda expression, or a function call or lambda call missing the last argument, or \texttt{true} or \texttt{all}.
(%i1) matchdeclare (aa, integerp);
(%o1) done
(%i2) matchdeclare (bb, lambda ([x], x > 0));
(%o2) done
(%i3) matchdeclare (cc, freeof (%e, %pi, %i));
(%o3) done
(%i4) matchdeclare (dd, lambda ([x, y], gcd (x, y) = 1) (1728));
(%o4) done
(%i5) matchdeclare (ee, true);
(%o5) done
(%i6) matchdeclare (ff, all);
(%o6) done

If an expression satisfies a match predicate, the match variable is assigned the expression.

(%i1) matchdeclare (aa, integerp, bb, atom);
(%o1) done
(%i2) defrule (r1, bb^aa, ["integer" = aa, "atom" = bb]);
    aa
r1 : bb -> [integer = aa, atom = bb]
(%i3) r1 (%pi^8);
[integer = 8, atom = %pi]

In the case of addition and multiplication, the match variable may be assigned a single expression which satisfies the match predicate, or a sum or product (respectively) of such expressions.

(%i1) matchdeclare (aa, atom, bb, lambda ([x], not atom(x)));   
(%o1) done
(%i2) defrule (r1, aa + bb, ["all atoms" = aa, "all nonatoms" =  
    bb]);
    aa
r1 : bb + aa -> [all atoms = aa, all nonatoms = bb]
(%i3) r1 (%pi^8);
    %pi
[integer = 8, atom = %pi]

When matching arguments of + and *, if all match predicates are mutually exclusive, the match result is insensitive to ordering, as one match predicate cannot accept terms matched by another.

(%i1) matchdeclare (aa, atom, bb, lambda ([x], not atom(x)));   
(%o1) done
(%i2) defrule (r1, aa + bb, ["all atoms" = aa, "all nonatoms" =  
    bb]);
    aa
r1 : bb + aa -> [all atoms = aa, all nonatoms = bb]
(%i3) r1 (8 + a*b + sin(x));
    a
[all atoms = 8, all nonatoms = (b + a) sin(x)]
(%i4) defrule (r2, aa * bb, ["all atoms" = aa, "all nonatoms" =  
    bb]);
    aa
r2 : aa bb -> [all atoms = aa, all nonatoms = bb]
(%i5) r2 (8 * (a + b) * sin(x));
    a
[all atoms = 8, all nonatoms = (b + a) sin(x)]
(%i3) r1 (8 + a*b + %pi + sin(x) - c + 2^n);

(%o3) [all atoms = %pi + 8, all nonatoms = sin(x) + 2 - c + a b]

(%i4) defrule (r2, aa * bb, ["all atoms" = aa, "all nonatoms" = bb]);

(%o4) r2 : aa bb -> [all atoms = aa, all nonatoms = bb]

(%i5) r2 (8 * (a + b) * %pi * sin(x) / c * 2^n);

n + 3
(b + a) 2 sin(x)

(%o5) [all atoms = %pi, all nonatoms = ---------------------]
c

The functions propvars and printprops return information about match variables.

(%i1) matchdeclare ([aa, bb, cc], atom, [dd, ee], integerp);

(%o1) done

(%i2) matchdeclare (ff, floatnump, gg, lambda ([x], x > 100));

(%o2) done

(%i3) propvars (matchdeclare);

(%o3) [aa, bb, cc, dd, ee, ff, gg]

(%i4) printprops (ee, matchdeclare);

(%o4) [integerp(ee)]

(%i5) printprops (gg, matchdeclare);

(%o5) [lambda([x], x > 100, gg)]

(%i6) printprops (all, matchdeclare);

(%o6) [lambda([x], x > 100, gg), floatnump(ff), integerp(ee),
        integerp(dd), atom(cc), atom(bb), atom(aa)]

maxapplydepth

Default value: 10000

maxapplydepth is the maximum depth to which apply1 and apply2 will delve.

maxapplyheight

Default value: 10000

maxapplyheight is the maximum height to which applyb1 will reach before giving up.

remlet

remlet (prod, name)
remlet ()
remlet (all)
remlet (all, name)

Deletes the substitution rule, prod --> repl, most recently defined by the let function. If name is supplied the rule is deleted from the rule package name.

remlet() and remlet(all) delete all substitution rules from the current rule package. If the name of a rule package is supplied, e.g. remlet (all, name), the rule package name is also deleted.

If a substitution is to be changed using the same product, remlet need not be called, just redefine the substitution using the same product (literally) with the let function.
and the new replacement and/or predicate name. Should `remlet (prod)` now be called the original substitution rule is revived.

See also `remrule`, which removes a rule defined by `tellsimp` or `tellsimpafter`.

**remrule**

```
remrule (op, rulename)
remrule (op, all)
```

Removes rules defined by `tellsimp` or `tellsimpafter`.

- `remrule (op, rulename)` removes the rule with the name `rulename` from the operator `op`. When `op` is a built-in or user-defined operator (as defined by `infix`, `prefix`, etc.), `op` and `rulename` must be enclosed in double quote marks.

- `remrule (op, all)` removes all rules for the operator `op`.

See also `remlet`, which removes a rule defined by `let`.

**Examples:**

```
(%i1) tellsimp (foo (aa, bb), bb - aa);
(%o1) [foorule1, false]
(%i2) tellsimpafter (aa + bb, special_add (aa, bb));
(%o2) [+rule1, plus]
(%i3) infix ("@@");
(%o3) @@
(%i4) tellsimp (aa @@ bb, bb/aa);
(%o4) [@@rule1, false]
(%i5) tellsimpafter (quux (%pi, %e), %pi - %e);
(%o5) [quuxrule1, false]
(%i6) tellsimpafter (quux (%e, %pi), %pi + %e);
(%o6) [quuxrule2, quuxrule1, false]
(%i7) [foo (aa, bb), aa + bb, aa @@ bb, quux (%pi, %e), quux (%e, %pi)];
        bb
(%o7) [bb - aa, special_add(aa, bb), --, %pi - %e, %pi + %e]
        aa
(%i8) remrule (foo, foorule1);
(%o8) foo
(%i9) remrule ("+", +rule1);
(%o9) +
(%i10) remrule ("@@", @@rule1);
(%o10) @@
(%i11) remrule (quux, all);
(%o11) quux
(%i12) [foo (aa, bb), aa + bb, aa @@ bb, quux (%pi, %e), quux (%e, %pi)];
(%o12) [foo(aa, bb), bb + aa, aa @@ bb, quux(%pi, %e), quux(%e, %pi)]
```

**tellsimp (pattern, replacement)**

```
```

is similar to `tellsimpafter` but places new information before old so that it is applied before the built-in simplification rules.
tellsimp is used when it is important to modify the expression before the simplifier works on it, for instance if the simplifier "knows" something about the expression, but what it returns is not to your liking. If the simplifier "knows" something about the main operator of the expression, but is simply not doing enough for you, you probably want to use tellsimpafter.

The pattern may not be a sum, product, single variable, or number.

The system variable rules is the list of rules defined by defrule, defmatch, tellsimp, and tellsimpafter.

Examples:

```maxima
(%i1) matchdeclare (x, freeof (%i));
(%o1) done

(%i2) %iargs: false$
(%i3) tellsimp (sin(%i*x), %i*sinh(x));
(%o3) [sinrule1, simp-%sin]

(%i4) trigexpand (sin (%i*y + x));
(%o4) sin(x) cos(%i y) + %i cos(x) sinh(y)

(%i5) %iargs: true$
(%i6) errcatch(0^0);

0 has been generated
(%o6) []

(%i7) ev (tellsimp (0^0, 1), simp: false);
(%o7) [^rule1, simpexpt]

(%i8) 0^0;
(%o8) 1

(%i9) remrule (^, %th(2)[1]);
(%o9) ^

(%i10) tellsimp (sin(x)^2, 1 - cos(x)^2);
(%o10) [=rule2, simpexpt]

(%i11) (1 + sin(x))^2;
(%o11) 2

(%i12) expand (%); 2
(%o12) 2 sin(x) - cos (x) + 2

(%i13) sin(x)^2;
(%o13) 2

(%i14) kill (rules);
(%o14) done

(%i15) matchdeclare (a, true);
(%o15) done

(%i16) tellsimp (sin(a)^2, 1 - cos(a)^2);
(%o16) [=rule3, simpexpt]

(%i17) sin(y)^2;
(%o17) 2
```
\[(\%o17) \quad 1 - \cos (y)\]

**tellsimpafter (pattern, replacement)**

[Function]

Defines a simplification rule which the Maxima simplifier applies after built-in simplification rules. `pattern` is an expression, comprising pattern variables (declared by `matchdeclare`) and other atoms and operators, considered literals for the purpose of pattern matching. `replacement` is substituted for an actual expression which matches `pattern`; pattern variables in `replacement` are assigned the values matched in the actual expression.

`pattern` may be any nonatomic expression in which the main operator is not a pattern variable; the simplification rule is associated with the main operator. The names of functions (with one exception, described below), lists, and arrays may appear in `pattern` as the main operator only as literals (not pattern variables); this rules out expressions such as `aa(x)` and `bb[y]` as patterns, if `aa` and `bb` are pattern variables. Names of functions, lists, and arrays which are pattern variables may appear as operators other than the main operator in `pattern`.

There is one exception to the above rule concerning names of functions. The name of a subscripted function in an expression such as `aa[x](y)` may be a pattern variable, because the main operator is not `aa` but rather the Lisp atom `mqapply`. This is a consequence of the representation of expressions involving subscripted functions.

Simplification rules are applied after evaluation (if not suppressed through quotation or the flag `noeval`). Rules established by `tellsimpafter` are applied in the order they were defined, and after any built-in rules. Rules are applied bottom-up, that is, applied first to subexpressions before application to the whole expression. It may be necessary to repeatedly simplify a result (for example, via the quote-quote operator `'` or the flag `infeval`) to ensure that all rules are applied.

Pattern variables are treated as local variables in simplification rules. Once a rule is defined, the value of a pattern variable does not affect the rule, and is not affected by the rule. An assignment to a pattern variable which results from a successful rule match does not affect the current assignment (or lack of it) of the pattern variable. However, as with all atoms in Maxima, the properties of pattern variables (as declared by `put` and related functions) are global.

The rule constructed by `tellsimpafter` is named after the main operator of `pattern`. Rules for built-in operators, and user-defined operators defined by `infix`, `prefix`, `postfix`, `matchfix`, and `nofix`, have names which are Lisp identifiers. Rules for other functions have names which are Maxima identifiers.

The treatment of noun and verb forms is slightly confused. If a rule is defined for a noun (or verb) form and a rule for the corresponding verb (or noun) form already exists, the newly-defined rule applies to both forms (noun and verb). If a rule for the corresponding verb (or noun) form does not exist, the newly-defined rule applies only to the noun (or verb) form.

The rule constructed by `tellsimpafter` is an ordinary Lisp function. If the name of the rule is `$foorule1`, the construct `:lisp (trace $foorule1)` traces the function, and `:lisp (symbol-function 'f$oorule1)` displays its definition.

`tellsimpafter` quotes its arguments. `tellsimpafter` returns the list of rules for the main operator of `pattern`, including the newly established rule.
See also `matchdeclare`, `defmatch`, `defrule`, `tellsimp`, `let`, `kill`, `remrule` and `clear_rules`.

Examples:

`pattern` may be any nonatomic expression in which the main operator is not a pattern variable.

(%i1) matchdeclare (aa, atom, [ll, mm], listp, xx, true)$
(%i2) tellsimpafter (sin (ll), map (sin, ll));
   [sinrule1, simp-%sin]
(%i3) sin ([1/6, 1/4, 1/3, 1/2, 1]*%pi);
   1 1 sqrt(3)
   [1, -------, -------, 1, 0]
   2 sqrt(2) 2
(%i4) tellsimpafter (ll^mm, map ("^", ll, mm));
   [^rule1, simpexpt]
(%i5) [a, b, c]^ [1, 2, 3];
   2 3
   [a, b, c]
(%i6) tellsimpafter (foo (aa (xx)), aa (foo (xx)));
   [foorule1, false]
(%i7) foo (bar (u - v));
   bar(foo(u - v))

Rules are applied in the order they were defined. If two rules can match an expression, the rule which was defined first is applied.

(%i1) matchdeclare (aa, integerp);
   done
(%i2) tellsimpafter (foo (aa), bar_1 (aa));
   [foorule1, false]
(%i3) tellsimpafter (foo (aa), bar_2 (aa));
   [foorule2, foorule1, false]
(%i4) foo (42);
   bar_1(42)
(%i5) bb: 12345;
   12345
(%i6) foo (42, %e);
   bar(aa = 42, bb = %e)
(%i7) bb;
   12345

Pattern variables are treated as local variables in simplification rules. (Compare to `defmatch`, which treats pattern variables as global variables.)

(%i1) matchdeclare (aa, integerp, bb, atom);
   done
(%i2) tellsimpafter (foo(aa, bb), bar('aa=aa, 'bb=bb));
   [foorule1, false]
(%i3) bb: 12345;
   12345
(%i4) foo (42, %e);
   bar(aa = 42, bb = %e)
(%i5) bb;
   12345
As with all atoms, properties of pattern variables are global even though values are local. In this example, an assignment property is declared via `define_variable`. This is a property of the atom `bb` throughout Maxima.

```
(%i1) matchdeclare (aa, integerp, bb, atom);
(%o1) done
(%i2) tellsimpafter (foo(aa, bb), bar('aa=aa, 'bb=bb));
(%o2) [foorule1, false]
(%i3) foo (42, %e);
(%o3) bar(aa = 42, bb = %e)
(%i4) define_variable (bb, true, boolean);
(%o4) true
(%i5) foo (42, %e);
translator: bb was declared with mode boolean, but it has value:
%e
-- an error. To debug this try: debugmode(true);
```

Rules are named after main operators. Names of rules for built-in and user-defined operators are Lisp identifiers, while names for other functions are Maxima identifiers.

```
(%i1) tellsimpafter (foo (%pi + %e), 3*%pi);
(%o1) [foorule1, false]
(%i2) tellsimpafter (foo (%pi * %e), 17*%e);
(%o2) [foorule2, foorule1, false]
(%i3) tellsimpafter (foo (%i - %e), -42*%i);
(%o3) [foorule3, foorule2, foorule1, false]
(%i4) tellsimpafter (foo (9) + foo (13), quux (22));
(%o4) [+rule1, simplus]
(%i5) tellsimpafter (foo (9) * foo (13), blurf (22));
(%o5) [*rule1, simptimes]
(%i6) tellsimpafter (foo (9) ^ foo (13), mumble (22));
(%o6) [^rule1, simpexpt]
(%i7) rules;
(%o7) [foorule1, foorule2, foorule3, +rule1, *rule1, ^rule1]
(%i8) foorule_name: first (%o1);
(%o8) foorule1
(%i9) plusrule_name: first (%o4);
(%o9) +rule1
(%i10) remrule (foo, foorule1);
(%o10) foo
(%i11) remrule ("~", ?\~rule1);
(%o11) ~
(%i12) rules;
(%o12) [foorule2, foorule3, +rule1, *rule1]
```

A worked example: anticommutative multiplication.

```
(%i1) gt (i, j) := integerp(j) and i < j;
(%o1) gt(i, j) := integerp(j) and (i < j)
(%i2) matchdeclare (i, integerp, j, gt(i));
(%o2) done
```
(%i3) tellsimpafter (s[i]**2, 1);
(%o3) ["^rule1, simpncexpt]
(%i4) tellsimpafter (s[i] . s[j], -s[j] . s[i]);
(%o4) [".rule1, simpncnct]
(%i5) s[i] . (s[1] + s[2]);
(%o5) s . (s + s )

(%i6) expand (%);
(%o6) 1 - s . s

(%i7) factor (expand (sum (s[i], i, 0, 9)**5));
(%o7) 100 \( s + s + s + s + s + s + s + s + s + s \)

clear_rules ()
[Function]
Executes kill (rules) and then resets the next rule number to 1 for addition +,
multiplication *, and exponentiation ^.
35 Sets

35.1 Introduction to Sets

Maxima provides set functions, such as intersection and union, for finite sets that are defined by explicit enumeration. Maxima treats lists and sets as distinct objects. This feature makes it possible to work with sets that have members that are either lists or sets.

In addition to functions for finite sets, Maxima provides some functions related to combinatorics; these include the Stirling numbers of the first and second kind, the Bell numbers, multinomial coefficients, partitions of nonnegative integers, and a few others. Maxima also defines a Kronecker delta function.

35.1.1 Usage

To construct a set with members \(a_1, \ldots, a_n\), write \(\text{set}(a_1, \ldots, a_n)\) or \\{a_1, \ldots, a_n\}; to construct the empty set, write \(\text{set}()\) or \\{}. In input, \(\text{set}(\ldots)\) and \\{ \ldots \} are equivalent. Sets are always displayed with curly braces.

If a member is listed more than once, simplification eliminates the redundant member.

\[
\begin{align*}
(\%i1) & \quad \text{set}(); \\
(\%o1) & \quad \{} \\
(\%i2) & \quad \text{set}(a, b, a); \\
(\%o2) & \quad \{a, b\} \\
(\%i3) & \quad \text{set}(a, \text{set}(b)); \\
(\%o3) & \quad \{a, \{b\}\} \\
(\%i4) & \quad \text{set}(a, [b]); \\
(\%o4) & \quad \{a, [b]\} \\
(\%i5) & \quad \{}; \\
(\%o5) & \quad \{} \\
(\%i6) & \quad \{a, b, a\}; \\
(\%o6) & \quad \{a, b\} \\
(\%i7) & \quad \{a, \{b\}\}; \\
(\%o7) & \quad \{a, \{b\}\} \\
(\%i8) & \quad \{a, [b]\}; \\
(\%o8) & \quad \{a, [b]\}
\end{align*}
\]

Two would-be elements \(x\) and \(y\) are redundant (i.e., considered the same for the purpose of set construction) if and only if \(\text{is}(x = y)\) yields \text{true}. Note that \(\text{is(}\text{equal}(x, y)\)) can yield \text{true} while \(\text{is}(x = y)\) yields \text{false}; in that case the elements \(x\) and \(y\) are considered distinct.

\[
\begin{align*}
(\%i1) & \quad x: a/c + b/c; \\
(\%o1) & \quad \begin{array}{ccc}
& b & a \\
- & + & - \\
c & c & c
end{array} \\
(\%i2) & \quad y: a/c + b/c; \\
(\%o2) & \quad \begin{array}{ccc}
& b & a \\
- & + & - \\
c & c & c
end{array}
\end{align*}
\]
(\%i3) z: (a + b)/c;

(\%o3) \frac{b + a}{c}

(\%i4) is (x = y);

(\%o4) true

(\%i5) is (y = z);

(\%o5) false

(\%i6) is (equal (y, z));

(\%o6) true

(\%i7) y - z;

(\%o7) \frac{b + a}{c} - \frac{b}{c} + \frac{a}{c}

(\%i8) ratsimp (%);

(\%o8) 0

(\%i9) \{x, y, z\};

(\%o9) \{\frac{b + a}{c}, \frac{-b}{c}, \frac{a}{c}\}

To construct a set from the elements of a list, use setify.

(\%i1) setify ([b, a]);

(\%o1) \{a, b\}

Set members x and y are equal provided is(x = y) evaluates to true. Thus rat(x) and x are equal as set members; consequently,

(\%i1) \{x, rat(x)\};

(\%o1) \{x\}

Further, since is((x - 1)*(x + 1) = x^2 - 1) evaluates to false, (x - 1)*(x + 1) and x^2 - 1 are distinct set members; thus

(\%i1) \{(x - 1)*(x + 1), x^2 - 1\};

(\%o1) \{(x - 1) (x + 1), x - 1\}

To reduce this set to a singleton set, apply rat to each set member:

(\%i1) \{(x - 1)*(x + 1), x^2 - 1\};

(\%o1) \{(x - 1) (x + 1), x - 1\}

(\%i2) map (rat, %);

(\%o2) \{x - 1\}

To remove redundancies from other sets, you may need to use other simplification functions. Here is an example that uses trigsimp:

(\%i1) \{1, \cos(x)^2 + \sin(x)^2\};

(\%o1) \{1, \sin (x) + \cos (x)\}

(\%i2) map (trigsimp, \%);
A set is simplified when its members are non-redundant and sorted. The current version of the set functions uses the Maxima function `orderlessp` to order sets; however, future versions of the set functions might use a different ordering function.

Some operations on sets, such as substitution, automatically force a re-simplification; for example,

```maxima
(%i1) s: \{a, b, c\}$
(%i2) subst (c=a, s);
(%o2) \{a, b\}
(%i3) subst ([a=x, b=x, c=x], s);
(%o3) \{x\}
(%i4) map (lambda ([x], x^2), set (-1, 0, 1));
(%o4) \{0, 1\}
```

Maxima treats lists and sets as distinct objects; functions such as `union` and `intersection` complain if any argument is not a set. If you need to apply a set function to a list, use the `setify` function to convert it to a set. Thus

```maxima
(%i1) union ([1, 2], \{a, b\});
Function union expects a set, instead found [1,2]
-- an error. Quitting. To debug this try debugmode(true);
(%i2) union (setify ([1, 2]), \{a, b\});
(%o2) \{1, 2, a, b\}
```

To extract all set elements of a set `s` that satisfy a predicate `f`, use `subset(s, f)`. (A predicate is a boolean-valued function.) For example, to find the equations in a given set that do not depend on a variable `z`, use

```maxima
(%i1) subset ({x + y + z, x - y + 4, x + y - 5}, lambda ([e], freeof (z, e)));
(%o1) \{- y + x + 4, y + x - 5\}
```

The section Section 35.2 [Functions and Variables for Sets], page 608, has a complete list of the set functions in Maxima.

### 35.1.2 Set Member Iteration

There two ways to to iterate over set members. One way is the use `map`; for example:

```maxima
(%i1) map (f, \{a, b, c\});
(%o1) \{f(a), f(b), f(c)\}
```

The other way is to use `for` `x` `in` `s` do

```maxima
(%i1) s: \{a, b, c\};
(%o1) \{a, b, c\}
(%i2) for si in s do print (concat (si, 1));
a1
b1
c1
done
```

The Maxima functions `first` and `rest` work correctly on sets. Applied to a set, `first` returns the first displayed element of a set; which element that is may be implementation-dependent. If `s` is a set, then `rest(s)` is equivalent to `disjoin(first(s), s)`. Currently,
there are other Maxima functions that work correctly on sets. In future versions of the set functions, \texttt{first} and \texttt{rest} may function differently or not at all.

Maxima’s \texttt{orderless} and \texttt{ordergreat} mechanisms are incompatible with the set functions. If you need to use either \texttt{orderless} or \texttt{ordergreat}, call those functions before constructing any sets, and do not call \texttt{unorder}.

35.1.3 Authors
Stavros Macrakis of Cambridge, Massachusetts and Barton Willis of the University of Nebraska at Kearney (UNK) wrote the Maxima set functions and their documentation.

35.2 Functions and Variables for Sets

\texttt{adjoin (x, a)}

Returns the union of the set \(a\) with \(\{x\}\).

\texttt{adjoin} complains if \(a\) is not a literal set.

\texttt{adjoin(x, a)} and \texttt{union(set(x), a)} are equivalent; however, \texttt{adjoin} may be somewhat faster than \texttt{union}.

See also \texttt{disjoin}.

Examples:

\begin{verbatim}
(%i1) adjoin (c, {a, b});
(%o1) {a, b, c}
(%i2) adjoin (a, {a, b});
(%o2) {a, b}
\end{verbatim}

\texttt{belln (n)}

Represents the \(n\)-th Bell number. \texttt{belln(n)} is the number of partitions of a set with \(n\) members.

For nonnegative integers \(n\), \texttt{belln(n)} simplifies to the \(n\)-th Bell number. \texttt{belln} does not simplify for any other arguments.

\texttt{belln} distributes over equations, lists, matrices, and sets.

Examples:

\texttt{belln} applied to nonnegative integers.

\begin{verbatim}
(%i1) makelist (belln (i), i, 0, 6);
(%o1) [1, 1, 2, 5, 15, 52, 203]
(%i2) is (cardinality (set_partitions ({})) = belln (0));
(%o2) true
(%i3) is (cardinality (set_partitions ({1, 2, 3, 4, 5, 6})) = belln (6));
(%o3) true
\end{verbatim}

\texttt{belln} applied to arguments which are not nonnegative integers.

\begin{verbatim}
(%i1) [belln (x), belln (sqrt(3)), belln (-9)];
(%o1) [belln(x), belln(sqrt(3)), belln(- 9)]
\end{verbatim}
cardinality (a)

Returns the number of distinct elements of the set a. cardinality ignores redundant elements even when simplification is disabled.

Examples:

(\%i1) cardinality ({});
(\%o1) 0
(\%i2) cardinality ({a, a, b, c});
(\%o2) 3
(\%i3) simp : false;
(\%o3) false
(\%i4) cardinality ({a, a, b, c});
(\%o4) 3

cartesian_product (b_1, ... , b_n)

Returns a set of lists of the form [x_1, ..., x_n], where x_1, ..., x_n are elements of the sets b_1, ... , b_n, respectively.

See also cartesian_product_list.

Examples:

(\%i1) cartesian_product ({0, 1});
(\%o1) {[0], [1]}
(\%i2) cartesian_product ({0, 1}, {0, 1});
(\%o2) {[0, 0], [0, 1], [1, 0], [1, 1]}
(\%i3) cartesian_product ({x}, {y}, {z});
(\%o3) {[x, y, z]}
(\%i4) cartesian_product ({x}, {-1, 0, 1});
(\%o4) {[x, -1], [x, 0], [x, 1]}

cartesian_product_list (b_1, ... , b_n)

Returns a list of lists of the form [x_1, ..., x_n], where x_1, ..., x_n are elements of the lists b_1, ... , b_n, respectively, comprising all possible combinations of the elements of b_1, ... , b_n.

The list returned by cartesian_product_list is equivalent to the following recursive definition. Let L be the list returned by cartesian_product_list(b_2, ... , b_n). Then cartesian_product_list(b_1, b_2, ..., b_n) (i.e., b_1 in addition to b_2, ..., b_n) returns a list comprising each element of L appended to the first element of b_1, each element of L appended to the second element of b_1, each element of L appended to the third element of b_1, etc. The order of the list returned by cartesian_product_list(b_1, b_2, ..., b_n) may therefore be summarized by saying the lesser indices (1, 2, 3, ...) vary more slowly than the greater indices.

The list returned by cartesian_product_list contains duplicate elements if any argument b_1, ..., b_n contains duplicates. In this respect, cartesian_product_list differs from cartesian_product, which returns no duplicates. Also, the ordering of the list returned cartesian_product_list is determined by the order of the elements of b_1, ..., b_n. Again, this differs from cartesian_product, which returns a set (with order determined by orderlessp).
The length of the list returned by \maxa{cartesian_product_list} is equal to the product of the lengths of the arguments \( b_1, \ldots, b_n \).

See also \maxa{cartesian_product}.

\maxa{cartesian_product_list} complains if any argument is not a list.

Examples:

\maxa{cartesian_product_list} returns a list of lists comprising all possible combinations.

\begin{verbatim}
(%i1) cartesian_product_list ([0, 1]);
(%o1) [[0], [1]]
(%i2) cartesian_product_list ([0, 1], [0, 1]);
(%o2) [[0, 0], [0, 1], [1, 0], [1, 1]]
(%i3) cartesian_product_list ([x], [y], [z]);
(%o3) [[x, y, z]]
(%i4) cartesian_product_list ([x], [-1, 0, 1]);
(%o4) [[x, -1], [x, 0], [x, 1]]
(%i5) cartesian_product_list ([a, h, e], [c, b, 4]);
(%o5) [[a, c], [a, b], [a, 4], [h, c], [h, b], [h, 4], [e, c],
    [e, b], [e, 4]]
\end{verbatim}

The order of the list returned by \maxa{cartesian_product_list} may be summarized by saying the lesser indices vary more slowly than the greater indices.

\begin{verbatim}
(%i1) cartesian_product_list ([1, 2, 3], [a, b], [i, ii]);
(%o1) [[1, a, i], [1, a, ii], [1, b, i], [1, b, ii], [2, a, i],
    [2, a, ii], [2, b, i], [2, b, ii], [3, a, i], [3, a, ii],
    [3, b, i], [3, b, ii]]
\end{verbatim}

The list returned by \maxa{cartesian_product_list} contains duplicate elements if any argument contains duplicates.

\begin{verbatim}
(%i1) cartesian_product_list ([e, h], [3, 7, 3]);
(%o1) [[e, 3], [e, 7], [e, 3], [h, 3], [h, 7], [h, 3]]
\end{verbatim}

The length of the list returned by \maxa{cartesian_product_list} is equal to the product of the lengths of the arguments.

\begin{verbatim}
(%i1) foo: cartesian_product_list ([1, 1, 2, 2, 3], [h, z, h]);
(%o1) [[1, h], [1, z], [1, h], [1, z], [1, h], [2, h],
    [2, z], [2, h], [2, z], [2, h], [3, h], [3, z], [3, h]]
(%i2) is (length (foo) = 5*3);
(%o2) true
\end{verbatim}

\maxa{disjoin} \hspace{1em} \textbf{[Function]}

Returns the set \( a \) without the member \( x \). If \( x \) is not a member of \( a \), return \( a \) unchanged.

\maxa{disjoin} complains if \( a \) is not a literal set.

\maxa{disjoin(x, a)}, \maxa{delete(x, a)}, and \maxa{setdifference(a, set(x))} are all equivalent. Of these, \maxa{disjoin} is generally faster than the others.

Examples:

\begin{verbatim}
(%i1) disjoin (a, {a, b, c, d});
(%o1) {b, c, d}
\end{verbatim}
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(%i2) disjoin (a + b, {5, z, a + b, %pi});
(%o2) {5, %pi, z}
(%i3) disjoin (a - b, {5, z, a + b, %pi});
(%o3) {5, %pi, b + a, z}

disjointp (a, b)
Returns true if and only if the sets a and b are disjoint.
disjointp complains if either a or b is not a literal set.
Examples:
(%i1) disjointp ({a, b, c}, {1, 2, 3});
(%o1) true
(%i2) disjointp ({a, b, 3}, {1, 2, 3});
(%o2) false

divisors (n)
Represents the set of divisors of n.
divisors(n) simplifies to a set of integers when n is a nonzero integer. The set of
divisors includes the members 1 and n. The divisors of a negative integer are the
divisors of its absolute value.
divisors distributes over equations, lists, matrices, and sets.
Examples:
We can verify that 28 is a perfect number: the sum of its divisors (except for itself)
is 28.

(%i1) s: divisors(28);
(%o1) {1, 2, 4, 7, 14, 28}
(%i2) lreduce (+, args(s)) - 28;
(%o2) 28
divisors is a simplifying function. Substituting 8 for a in divisors(a) yields the
divisors without reevaluating divisors(8).

(%i1) divisors (a);
(%o1) divisors(a)
(%i2) subst (8, a, %);
(%o2) {1, 2, 4, 8}
divisors distributes over equations, lists, matrices, and sets.

(%i1) divisors (a = b);
(%i1) divisors(a) = divisors(b)
(%i2) divisors ([a, b, c]);
(%i2) [divisors(a), divisors(b), divisors(c)]
(%i3) divisors (matrix ([a, b], [c, d]));
(%o3) [ [ divisors(a) divisors(b) ]
     [       ]
     [ divisors(c) divisors(d) ]
(%i4) divisors ({a, b, c});
(%i4) {divisors(a), divisors(b), divisors(c)}
elementp (x, a)
Returns true if and only if x is a member of the set a.

elementp complains if a is not a literal set.

Examples:
(\%i1) elementp (sin(1), {sin(1), sin(2), sin(3)});
(\%o1) true
(\%i2) elementp (sin(1), {cos(1), cos(2), cos(3)});
(\%o2) false

emptyp (a)
Return true if and only if a is the empty set or the empty list.

Examples:
(\%i1) map (emptyp, [{}, []]);
(\%o1) [true, true]
(\%i2) map (emptyp, [a + b, {{}}, %pi]);
(\%o2) [false, false, false]

equiv_classes (s, F)
Returns a set of the equivalence classes of the set s with respect to the equivalence relation F.

F is a function of two variables defined on the Cartesian product of s with s. The return value of F is either true or false, or an expression expr such that is(expr) is either true or false.

When F is not an equivalence relation, equiv_classes accepts it without complaint, but the result is generally incorrect in that case.

Examples:
The equivalence relation is a lambda expression which returns true or false.
(\%i1) equiv_classes ({1, 1.0, 2, 2.0, 3, 3.0},
lambda ([x, y], is (equal (x, y))));
(\%o1) {{1, 1.0}, {2, 2.0}, {3, 3.0}}
The equivalence relation is the name of a relational function which is evaluates to true or false.
(\%i1) equiv_classes ({1, 1.0, 2, 2.0, 3, 3.0}, equal);
(\%o1) {{1, 1.0}, {2, 2.0}, {3, 3.0}}
The equivalence classes are numbers which differ by a multiple of 3.
(\%i1) equiv_classes ({1, 2, 3, 4, 5, 6, 7},
lambda ([x, y], remainder (x - y, 3) = 0));
(\%o1) {{1, 4, 7}, {2, 5}, {3, 6}}

every (f, s)
every (f, L_1, ..., L_n)
Returns true if the predicate f is true for all given arguments.
Given one set as the second argument, \( \text{every}(f, s) \) returns true if \( \text{is}(f(a_i)) \) returns true for all \( a_i \) in \( s \). every may or may not evaluate \( f \) for all \( a_i \) in \( s \). Since sets are unordered, every may evaluate \( f(a_i) \) in any order.

Given one or more lists as arguments, \( \text{every}(f, L_1, \ldots, L_n) \) returns true if \( \text{is}(f(x_1, \ldots, x_n)) \) returns true for all \( x_1, \ldots, x_n \) in \( L_1, \ldots, L_n \), respectively. every may or may not evaluate \( f \) for every combination \( x_1, \ldots, x_n \). every evaluates lists in the order of increasing index.

Given an empty set {} or empty lists [] as arguments, every returns true.

When the global flag maperror is true, all lists \( L_1, \ldots, L_n \) must have equal lengths. When maperror is false, list arguments are effectively truncated to the length of the shortest list.

Return values of the predicate \( f \) which evaluate (via is) to something other than true or false are governed by the global flag prederror. When prederror is true, such values are treated as false, and the return value from every is false. When prederror is false, such values are treated as unknown, and the return value from every is unknown.

Examples:

every applied to a single set. The predicate is a function of one argument.

\[
\begin{align*}
(\%i1) & \quad \text{every (integerp, \{1, 2, 3, 4, 5, 6\});} \\
(\%o1) & \quad \text{true} \\
(\%i2) & \quad \text{every (atom, \{1, 2, sin(3), 4, 5 + y, 6\});} \\
(\%o2) & \quad \text{false}
\end{align*}
\]

every applied to two lists. The predicate is a function of two arguments.

\[
\begin{align*}
(\%i1) & \quad \text{every ("=", \[a, b, c\], \[a, b, c\]);} \\
(\%o1) & \quad \text{true} \\
(\%i2) & \quad \text{every ("#", \[a, b, c\], \[a, b, c\]);} \\
(\%o2) & \quad \text{false}
\end{align*}
\]

Return values of the predicate \( f \) which evaluate to something other than true or false are governed by the global flag prederror.

\[
\begin{align*}
(\%i1) & \quad \text{prederror : false;} \\
(\%o1) & \quad \text{false} \\
(\%i2) & \quad \text{map (lambda ([a, b], is (a < b)), \[x, y, z\],} \\
& \quad \quad \quad \quad \quad \quad \quad [x^2, y^2, z^2]); \\
(\%o2) & \quad \text{[unknown, unknown, unknown]} \\
(\%i3) & \quad \text{every ("<", \[x, y, z\], \[x^2, y^2, z^2\]);} \\
(\%o3) & \quad \text{unknown} \\
(\%i4) & \quad \text{prederror : true;} \\
(\%o4) & \quad \text{true} \\
(\%i5) & \quad \text{every ("<", \[x, y, z\], \[x^2, y^2, z^2\]);} \\
(\%o5) & \quad \text{false}
\end{align*}
\]
extremal_subset

extremal_subset (s, f, max)

Returns the subset of s for which the function f takes on maximum or minimum values.

extremal_subset (s, f, max) returns the subset of the set or list s for which the
real-valued function f takes on its maximum value.

extremal_subset (s, f, min) returns the subset of the set or list s for which the
real-valued function f takes on its minimum value.

Examples:

(\(\texttt{(%i1)}\)) extremal_subset ({-2, -1, 0, 1, 2}, abs, max);
(\(\texttt{(%o1)}\)) \{-2, 2\}

(\(\texttt{(%i2)}\)) extremal_subset ({sqrt(2), 1.57, \%pi/2}, sin, min);
(\(\texttt{(%o2)}\)) \{sqrt(2)\}

flatten (expr)

Collects arguments of subexpressions which have the same operator as expr and
constructs an expression from these collected arguments.

Subexpressions in which the operator is different from the main operator of expr are
copied without modification, even if they, in turn, contain some subexpressions in
which the operator is the same as for expr.

It may be possible for flatten to construct expressions in which the number of
arguments differs from the declared arguments for an operator; this may provoke an
error message from the simplifier or evaluator. flatten does not try to detect such
situations.

Expressions with special representations, for example, canonical rational expressions
(CRE), cannot be flattened; in such cases, flatten returns its argument unchanged.

Examples:

Applied to a list, flatten gathers all list elements that are lists.

(\(\texttt{(%i1)}\)) flatten ([a, b, [c, [d, e], f], [[g, h]], i, j));
(\(\texttt{(%o1)}\)) [a, b, c, d, e, f, g, h, i, j]

Applied to a set, flatten gathers all members of set elements that are sets.

(\(\texttt{(%i1)}\)) flatten ({a, {b}, {{c}}});
(\(\texttt{(%o1)}\)) \{a, b, c\}

(\(\texttt{(%i2)}\)) flatten ({a, {[a], {a}}});
(\(\texttt{(%o2)}\)) \{a, [a]\}

flatten is similar to the effect of declaring the main operator n-ary. However,flatten has no effect on subexpressions which have an operator different from the
main operator, while an n-ary declaration affects those.

(\(\texttt{(%i1)}\)) expr: flatten (f (g (f (f (x)))));
(\(\texttt{(%o1)}\)) f(g(f(f(x))))

(\(\texttt{(%i2)}\)) declare (f, nary);
(\(\texttt{(%o2)}\)) done

(\(\texttt{(%i3)}\)) ev (expr);
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(%o3) \quad f(g(f(x)))

`flatten` treats subscripted functions the same as any other operator.

(%i1) flatten (f[5] (f[5] (x, y), z));
(%o1) \quad f (x, y, z)

It may be possible for `flatten` to construct expressions in which the number of arguments differs from the declared arguments for an operator;

(%i1) 'mod (5, 'mod (7, 4));
(%o1) \quad \text{mod}(5, \text{mod}(7, 4))
(%i2) flatten (%);
(%o2) \quad \text{mod}(5, 7, 4)
(%i3) ''%, nouns;
Wrong number of arguments to mod
-- an error. Quitting. To debug this try debugmode(true);

`full_listify (a)`
Replaces every set operator in `a` by a list operator, and returns the result. `full_listify` replaces set operators in nested subexpressions, even if the main operator is not `set`.

`listify` replaces only the main operator.

Examples:

(%i1) full_listify ({a, b, {c, {d, e, f}, g}});
(%o1) \quad [a, b, [c, [d, e, f], g]]
(%i2) full_listify (F (G ({a, b, H({c, d, e})})));
(%o2) \quad F(G([a, b, H([c, d, e])]))

`fullsetify (a)`
When `a` is a list, replaces the list operator with a set operator, and applies `fullsetify` to each member which is a set. When `a` is not a list, it is returned unchanged.

`setify` replaces only the main operator.

Examples:

In line (%o2), the argument of `f` isn’t converted to a set because the main operator of `f([b])` isn’t a list.

(%i1) fullsetify ([a, [a]]);
(%o1) \quad \{a, \{a\}\}
(%i2) fullsetify ([a, f([b])]);
(%o2) \quad \{a, f([b])\}

`identity (x)`
Returns `x` for any argument `x`.

Examples:

`identity` may be used as a predicate when the arguments are already Boolean values.

(%i1) every (identity, [true, true]);
(%o1) \quad \text{true}
integer_partitions

integer_partitions (n)
integer_partitions (n, len)
Returns integer partitions of \( n \), that is, lists of integers which sum to \( n \).

integer_partitions (n) returns the set of all partitions of the integer \( n \). Each partition is a list sorted from greatest to least.

integer_partitions (n, len) returns all partitions that have length \( len \) or less; in this case, zeros are appended to each partition with fewer than \( len \) terms to make each partition have exactly \( len \) terms. Each partition is a list sorted from greatest to least.

A list \([a_1, ..., a_m]\) is a partition of a nonnegative integer \( n \) when (1) each \( a_i \) is a nonzero integer, and (2) \( a_1 + ... + a_m = n \). Thus 0 has no partitions.

Examples:

\((%i1)\) integer_partitions (3);
\(%o1\) \{[1, 1, 1], [2, 1], [3]\}
\((%i2)\) s: integer_partitions (25)$
\(%i3\) cardinality (s);
\(%o3\) 1958
\((%i4)\) map (lambda ([x], apply (“+”, x)), s);
\(%o4\) \{25\}
\((%i5)\) integer_partitions (5, 3);
\(%o5\) \{[2, 2, 1], [3, 1, 1], [3, 2, 0], [4, 1, 0], [5, 0, 0]\}
\((%i6)\) integer_partitions (5, 2);
\(%o6\) \{[3, 2], [4, 1], [5, 0]\}

To find all partitions that satisfy a condition, use the function subset; here is an example that finds all partitions of 10 that consist of prime numbers.

\((%i1)\) s: integer_partitions (10)$
\(%o1\) cardinality (s);
\(%o2\) 42
\((%i3)\) xprimep(x) := integerp(x) and (x > 1) and primep(x)$
\((%i4)\) subset (s, lambda ([x], every (xprimep, x)));
\(%o4\) \{[2, 2, 2, 2, 2], [3, 3, 2, 2], [5, 3, 2], [5, 5], [7, 3]\}

intersect (a_1, ..., a_n)

intersect is the same as intersection, which see.

intersection (a_1, ..., a_n)

Returns a set containing the elements that are common to the sets \( a_1 \) through \( a_n \).

intersection complains if any argument is not a literal set.

Examples:

\((%i1)\) S_1 : {a, b, c, d};
\(%o1\) \{a, b, c, d\}
\((%i2)\) S_2 : {d, e, f, g};
\(%o2\) \{d, e, f, g\}
\((%i3)\) S_3 : {c, d, e, f};
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(%o3) \{c, d, e, f\}
(%i4) S_4 : \{u, v, w\};
(%o4) \{u, v, w\}
(%i5) intersection (S_1, S_2);
(%o5) \{d\}
(%i6) intersection (S_2, S_3);
(%o6) \{d, e, f\}
(%i7) intersection (S_1, S_2, S_3);
(%o7) \{d\}
(%i8) intersection (S_1, S_2, S_3, S_4);
(%o8) \{

**kron_delta** \((x_1, x_2, \ldots, x_p)\)

represents the Kronecker delta function.

**kron_delta** simplifies to 1 when \(x_i\) and \(y_j\) are equal for all pairs of arguments, and it simplifies to 0 when \(x_i\) and \(y_j\) are not equal for some pair of arguments. Equality is determined using \text{is}\(\text{(equal}(x_i, x_j))\) and inequality by \text{is}\(\text{(notequal}(x_i, x_j))\). For exactly one argument, **kron_delta** signals an error.

Examples:

(%i11) kron_delta(a,a);
(%o11) 1
(%i12) kron_delta(a,b,a,b);
(%o12) kron_delta(a, b)
(%i13) kron_delta(a,a,b,a+1);
(%o13) 0
(%i14) assume(equal(x,y));
(%o14) [equal(x, y)]
(%i15) kron_delta(x,y);
(%o15) 1

**listify** \((a)\)

returns a list containing the members of \(a\) when \(a\) is a set. Otherwise, **listify** returns \(a\).

**full_listify** replaces all set operators in \(a\) by list operators.

Examples:

(%i11) listify({a, b, c, d});
(%o11) [a, b, c, d]
(%i12) listify(F({a, b, c, d}));
(%o12) F({a, b, c, d})

**makeset** \((expr, x, s)\)

returns a set with members generated from the expression \(expr\), where \(x\) is a list of variables in \(expr\), and \(s\) is a set or list of lists. To generate each set member, \(expr\) is evaluated with the variables \(x\) bound in parallel to a member of \(s\).

Each member of \(s\) must have the same length as \(x\). The list of variables \(x\) must be a list of symbols, without subscripts. Even if there is only one symbol, \(x\) must be a list of one element, and each member of \(s\) must be a list of one element.
See also `makelist`.

Examples:

```lisp
(%i1) makeset (i/j, [i, j], [[1, a], [2, b], [3, c], [4, d]]);
     1  2  3  4
(%o1)   {-, -, -, -}
        a b c d

(%i2) S : {x, y, z}$
(%i3) S3 : cartesian_product (S, S, S);
{x, x, x}, {x, x, y}, {x, x, z}, {x, y, x}, {x, y, y},
{x, y, z}, {y, x, x}, {y, x, y}, {y, x, z}, {y, y, x},
{y, y, y}, {y, y, z}, {y, z, x}, {y, z, y}, {z, x, x},
{z, x, y}, {z, y, x}, {z, y, y}, {z, z, x}, {z, z, y},
{z, z, z}

(%i4) makeset (i + j + k, [i, j, k], S3);
{3 x, 3 y, y + 2 x, 2 y + x, 3 z, z + 2 x, z + y + x,
     2 z + y, 2 z + x, 2 z + y}

(%i5) makeset (sin(x), [x], {[1], [2], [3]});
{sin(1), sin(2), sin(3)}
```

`moebius(n)`

[Function]

Represents the Moebius function.

When $n$ is product of $k$ distinct primes, `moebius(n)` simplifies to $(-1)^k$; when $n = 1$, it simplifies to 1; and it simplifies to 0 for all other positive integers.

`moebius` distributes over equations, lists, matrices, and sets.

Examples:

```lisp
(%i1) moebius (1);
(%o1) 1
(%i2) moebius (2 * 3 * 5);
(%o2) - 1
(%i3) moebius (11 * 17 * 29 * 31);
(%o3) 1
(%i4) moebius (2^32);
(%o4) 0
(%i5) moebius (n);
(%o5) moebius(n)
(%i6) moebius (n = 12);
(%o6) moebius(n) = 0
(%i7) moebius ([11, 11 * 13, 11 * 13 * 15]);
(%o7) [- 1, 1, 1]
(%i8) moebius (matrix ([[11, 12], [13, 14]]));
    [ - 1  0 ]
    [       ]
    [ - 1  1 ]
(%i9) moebius ({21, 22, 23, 24});
(%o9) {- 1, 0, 1}
```
multinomial_coeff

multinomial_coeff (a_1, ..., a_n)

Returns the multinomial coefficient.

When each a_k is a nonnegative integer, the multinomial coefficient gives the number of ways of placing a_1 + ... + a_n distinct objects into n boxes with a_k elements in the k'th box. In general, multinomial_coeff (a_1, ..., a_n) evaluates to (a_1 + ... + a_n)!/(a_1! ... a_n!).

multinomial_coeff() (with no arguments) evaluates to 1.

minfactorial may be able to simplify the value returned by multinomial_coeff.

Examples:

(%i1) multinomial_coeff (1, 2, x);
(x + 3)!
(%o1) -------
   2 x!

(%i2) minfactorial (%);
(x + 1) (x + 2) (x + 3)
(%o2) -----------------------
      2

(%i3) multinomial_coeff (-6, 2);
(- 4)!
(%o3) -------
    2 (- 6)!

(%i4) minfactorial (%);
(%o4) 10

num_distinct_partitions

num_distinct_partitions (n)

num_distinct_partitions (n, list)

Returns the number of distinct integer partitions of n when n is a nonnegative integer. Otherwise, num_distinct_partitions returns a noun expression.

num_distinct_partitions(n, list) returns a list of the number of distinct partitions of 1, 2, 3, ..., n.

A distinct partition of n is a list of distinct positive integers k_1, ..., k_m such that n = k_1 + ... + k_m.

Examples:

(%i1) num_distinct_partitions (12);
(%o1) 15

(%i2) num_distinct_partitions (12, list);
(%o2) [1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10, 12, 15]

(%i3) num_distinct_partitions (n);
(%o3) num_distinct_partitions(n)
num_partitions

num_partitions (n)
num_partitions (n, list)

Returns the number of integer partitions of \( n \) when \( n \) is a nonnegative integer. Otherwise, num_partitions returns a noun expression.

num_partitions(n, list) returns a list of the number of integer partitions of 1, 2, 3, ..., \( n \).

For a nonnegative integer \( n \), num_partitions(n) is equal to cardinality(integer_partitions(n)); however, num_partitions does not actually construct the set of partitions, so it is much faster.

Examples:

(%i1) num_partitions (5) = cardinality (integer_partitions (5));
(%o1) 7 = 7
(%i2) num_partitions (8, list);
(%o2) [1, 1, 2, 3, 5, 7, 11, 15, 22]
(%i3) num_partitions (n);
(%o3) num_partitions(n)

partition_set (a, f)

Partitions the set \( a \) according to the predicate \( f \).

partition_set returns a list of two sets. The first set comprises the elements of \( a \) for which \( f \) evaluates to false, and the second comprises any other elements of \( a \).

partition_set does not apply is to the return value of \( f \).

partition_set complains if \( a \) is not a literal set.

See also subset.

Examples:

(%i1) partition_set ({2, 7, 1, 8, 2, 8}, evenp);
(%o1) {{1, 7}, {2, 8}}
(%i2) partition_set ({x, rat(y), rat(y) + z, 1},
   lambda ([x], ratp(x)));
(%o2)/R/ [{1, x}, {y, y + z}]

permutations (a)

Returns a set of all distinct permutations of the members of the list or set \( a \). Each permutation is a list, not a set.

When \( a \) is a list, duplicate members of \( a \) are included in the permutations.

permutations complains if \( a \) is not a literal list or set.

See also random_permutation.

Examples:

(%i1) permutations ([a, a]);
(%o1) {{[a, a]}
(%i2) permutations ([a, a, b]);
(%o2) {[a, a, b], [a, b, a], [b, a, a]}
**powerset**

- **powerset (a)**
- **powerset (a, n)**

  Returns the set of all subsets of a, or a subset of that set.

  **powerset(a)** returns the set of all subsets of the set a. **powerset(a)** has \(2^{\text{cardinality}(a)}\) members.

  **powerset(a, n)** returns the set of all subsets of a that have cardinality n.

  **powerset** complains if a is not a literal set, or if n is not a nonnegative integer.

  Examples:
  
  ```
  (%i1) powerset ({a, b, c});
  (%o1) {{}, {a}, {a, b}, {a, b, c}, {a, c}, {b}, {b, c}, {c}}
  (%i2) powerset ({w, x, y, z}, 4);
  (%o2) {{w, x, y, z}}
  (%i3) powerset ({w, x, y, z}, 3);
  (%o3) {{w, x, y}, {w, x, z}, {w, y, z}, {x, y, z}}
  (%i4) powerset ({w, x, y, z}, 2);
  (%o4) {{w, x}, {w, y}, {w, z}, {x, y}, {x, z}, {y, z}}
  (%i5) powerset ({w, x, y, z}, 1);
  (%o5) {{w}, {x}, {y}, {z}}
  (%i6) powerset ({w, x, y, z}, 0);
  (%o6) {{}}
  ```

**random_permutation (a)**

  Returns a random permutation of the set or list a, as constructed by the Knuth shuffle algorithm.

  The return value is a new list, which is distinct from the argument even if all elements happen to be the same. However, the elements of the argument are not copied.

  Examples:
  
  ```
  (%i1) random_permutation ([a, b, c, 1, 2, 3]);
  (%o1) [c, 1, 2, 3, a, b]
  (%i2) random_permutation ([a, b, c, 1, 2, 3]);
  (%o2) [b, 3, 1, c, a, 2]
  (%i3) random_permutation ([x + 1, y + 2, z + 3]);
  (%o3) [y + 2, z + 3, x + 1]
  (%i4) random_permutation ([x + 1, y + 2, z + 3]);
  (%o4) [x + 1, y + 2, z + 3]
  ```

**setdifference (a, b)**

  Returns a set containing the elements in the set a that are not in the set b.

  **setdifference** complains if either a or b is not a literal set.

  Examples:
  
  ```
  (%i1) S_1 : {a, b, c, x, y, z};
  (%o1) {a, b, c, x, y, z}
  (%i2) S_2 : {aa, bb, c, x, y, zz};
  (%o2) {aa, bb, c, x, y, zz}
  ```
(%i3) setdifference (S_1, S_2);  
(%o3) {a, b, z}  
(%i4) setdifference (S_2, S_1);  
(%o4) {aa, bb, zz}  
(%i5) setdifference (S_1, S_1);  
(%o5) {}  
(%i6) setdifference (S_1, {});  
(%o6) {a, b, c, x, y, z}  
(%i7) setdifference ({}, S_1);  
(%o7) {}  

setequalp (a, b)  
[Function]  
Returns \texttt{true} if sets \(a\) and \(b\) have the same number of elements and \(\text{is}(x = y)\) is \texttt{true} for \(x\) in the elements of \(a\) and \(y\) in the elements of \(b\), considered in the order determined by \texttt{listify}. Otherwise, \texttt{setequalp} returns \texttt{false}.

Examples:

(%i1) setequalp ({1, 2, 3}, {1, 2, 3});  
(%o1) true  
(%i2) setequalp ({a, b, c}, {1, 2, 3});  
(%o2) false  
(%i3) setequalp ({x^2 - y^2}, {(x + y) * (x - y)});  
(%o3) false

setify (a)  
[Function]  
Constructs a set from the elements of the list \(a\). Duplicate elements of the list \(a\) are deleted and the elements are sorted according to the predicate \texttt{orderlessp}.

\texttt{setify} complains if \(a\) is not a literal list.

Examples:

(%i1) setify ([1, 2, 3, a, b, c]);  
(%o1) {1, 2, 3, a, b, c}  
(%i2) setify ([a, b, c, a, b, c]);  
(%o2) {a, b, c}  
(%i3) setify ([7, 13, 11, 1, 3, 9, 5]);  
(%o3) {1, 3, 5, 7, 9, 11, 13}

setp (a)  
[Function]  
Returns \texttt{true} if and only if \(a\) is a Maxima set.

\texttt{setp} returns \texttt{true} for unsimplified sets (that is, sets with redundant members) as well as simplified sets.

\texttt{setp} is equivalent to the Maxima function \texttt{setp(a) := not atom(a) and op(a) = 'set}.

Examples:

(%i1) simp : false;  
(%o1) false  
(%i2) {a, a, a};  
(%o2) {a, a, a}
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(%i3) setp (%);
(%o3) true

set_partitions
  set_partitions (a)
  set_partitions (a, n)
Returns the set of all partitions of a, or a subset of that set.

set_partitions(a, n) returns a set of all decompositions of a into n nonempty disjoint subsets.

set_partitions(a) returns the set of all partitions.

stirling2 returns the cardinality of the set of partitions of a set.

A set of sets \( P \) is a partition of a set \( S \) when
1. each member of \( P \) is a nonempty set,
2. distinct members of \( P \) are disjoint,
3. the union of the members of \( P \) equals \( S \).

Examples:
The empty set is a partition of itself, the conditions 1 and 2 being vacuously true.

(%i1) set_partitions ({});
(%o1) {{}}

The cardinality of the set of partitions of a set can be found using \texttt{stirling2}.

(%i1) s: \{0, 1, 2, 3, 4, 5\}$
(%i2) p: set_partitions (s, 3)$
(%i3) cardinality(p) = stirling2 (6, 3);
(%o3) 90 = 90

Each member of \( p \) should have \( n = 3 \) members; let’s check.

(%i1) s: \{0, 1, 2, 3, 4, 5\}$
(%i2) p: set_partitions (s, 3)$
(%i3) map (cardinality, p);
(%o3) \{3\}

Finally, for each member of \( p \), the union of its members should equal \( s \); again let’s check.

(%i1) s: \{0, 1, 2, 3, 4, 5\}$
(%i2) p: set_partitions (s, 3)$
(%i3) map (lambda ([x], apply (union, listify (x))), p);
(%o3) \{{0, 1, 2, 3, 4, 5}\}

some
  some (f, a)
  some (f, L_1, ..., L_n)
Returns \texttt{true} if the predicate \( f \) is \texttt{true} for one or more given arguments.

Given one set as the second argument, \texttt{some}(f, s) returns \texttt{true} if \texttt{is}(f(a_i)) returns \texttt{true} for one or more \( a_i \) in \( s \). some may or may not evaluate \( f \) for all \( a_i \) in \( s \). Since sets are unordered, some may evaluate \( f(a_i) \) in any order.
Given one or more lists as arguments, \( \text{some}(f, L_1, \ldots, L_n) \) returns \text{true} if \( \text{is}(f(x_1, \ldots, x_n)) \) returns \text{true} for one or more \( x_1, \ldots, x_n \) in \( L_1, \ldots, L_n \), respectively. \text{some} may or may not evaluate \( f \) for some combinations \( x_1, \ldots, x_n \). \text{some} evaluates lists in the order of increasing index.

Given an empty set \{\} or empty lists [] as arguments, \text{some} returns \text{false}.

When the global flag \text{maperror} is \text{true}, all lists \( L_1, \ldots, L_n \) must have equal lengths. When \text{maperror} is \text{false}, list arguments are effectively truncated to the length of the shortest list.

Return values of the predicate \( f \) which evaluate (via \text{is}) to something other than \text{true} or \text{false} are governed by the global flag \text{prederror}. When \text{prederror} is \text{true}, such values are treated as \text{false}. When \text{prederror} is \text{false}, such values are treated as \text{unknown}.

Examples:

\text{some} applied to a single set. The predicate is a function of one argument.

\begin{verbatim}
(%i1) some (integerp, {1, 2, 3, 4, 5, 6});  
(%o1) true  
(%i2) some (atom, {1, 2, sin(3), 4, 5 + y, 6});  
(%o2) true
\end{verbatim}

\text{some} applied to two lists. The predicate is a function of two arguments.

\begin{verbatim}
(%i1) some ("=", [a, b, c], [a, b, c]);  
(%o1) true  
(%i2) some ("#", [a, b, c], [a, b, c]);  
(%o2) false
\end{verbatim}

Return values of the predicate \( f \) which evaluate to something other than \text{true} or \text{false} are governed by the global flag \text{prederror}.

\begin{verbatim}
(%i1) prederror : false;  
(%o1) false  
(%i2) map (lambda ([a, b], is (a < b)), [x, y, z],  
    [x^2, y^2, z^2]);  
(%o2) [unknown, unknown, unknown]  
(%i3) some ("<", [x, y, z], [x^2, y^2, z^2]);  
(%o3) unknown  
(%i4) some ("<", [x, y, z], [x^2, y^2, z + 1]);  
(%o4) true  
(%i5) prederror : true;  
(%o5) true  
(%i6) some ("<", [x, y, z], [x^2, y^2, z^2]);  
(%o6) false  
(%i7) some ("<", [x, y, z], [x^2, y^2, z + 1]);  
(%o7) true
\end{verbatim}

\text{stirling1} \( (n, m) \)

Represents the Stirling number of the first kind.

When \( n \) and \( m \) are nonnegative integers, the magnitude of \text{stirling1} \( (n, m) \) is the number of permutations of a set with \( n \) members that have \( m \) cycles.
stirling1 is a simplifying function. Maxima knows the following identities:

1. \( \text{stirling1}(1, k) = \text{kron}\_\text{delta}(1, k), k \geq 0 \) (see https://dlmf.nist.gov/26.8.E2)

2. \( \text{stirling1}(n, n) = 1, n \geq 0 \) (see https://dlmf.nist.gov/26.8.E1)

3. \( \text{stirling1}(n, n - 1) = -\text{binomial}(n, 2), n \geq 1 \) (see https://dlmf.nist.gov/26.8.E16)

4. \( \text{stirling1}(n, 0) = \text{kron}\_\text{delta}(n, 0), n \geq 0 \) (see https://dlmf.nist.gov/26.8.E14 and https://dlmf.nist.gov/26.8.E1)

5. \( \text{stirling1}(n, 1) = (-1)^{n-1}(n-1)!, n \geq 1 \) (see https://dlmf.nist.gov/26.8.E14)

6. \( \text{stirling1}(n, k) = 0, n \geq 0 \) and \( k > n \).

These identities are applied when the arguments are literal integers or symbols declared as integers, and the first argument is nonnegative. \text{stirling1} does not simplify for non-integer arguments.

Examples:

\begin{verbatim}
(%i1) declare (n, integer)$
(%i2) assume (n >= 0)$
(%i3) stirling1 (n, n);
(%o3) 1
\end{verbatim}

\text{stirling2 (n, m)}

Represents the Stirling number of the second kind.

When \( n \) and \( m \) are nonnegative integers, \text{stirling2 (n, m)} is the number of ways a set with cardinality \( n \) can be partitioned into \( m \) disjoint subsets.

\text{stirling2} is a simplifying function. Maxima knows the following identities.

1. \( \text{stirling2}(n, 0) = 1, n \geq 1 \) (see https://dlmf.nist.gov/26.8.E17 and \text{stirling2}(0, 0) = 1)

2. \( \text{stirling2}(n, n) = 1, n \geq 0 \), (see https://dlmf.nist.gov/26.8.E4)

3. \( \text{stirling2}(n, 1) = 1, n \geq 1 \), (see https://dlmf.nist.gov/26.8.E17 and \text{stirling2}(0, 1) = 0)

4. \( \text{stirling2}(n, 2) = 2^{n-1} - 1, n \geq 1 \), (see https://dlmf.nist.gov/26.8.E17)

5. \( \text{stirling2}(n, n - 1) = \text{binomial}(n, 2), n \geq 1 \) (see https://dlmf.nist.gov/26.8.E16)

6. \( \text{stirling2}(n, k) = 0, n \geq 0 \) and \( k > n \).

These identities are applied when the arguments are literal integers or symbols declared as integers, and the first argument is nonnegative. \text{stirling2} does not simplify for non-integer arguments.

Examples:

\begin{verbatim}
(%i1) declare (n, integer)$
(%i2) assume (n >= 0)$
(%i3) stirling2 (n, n);
(%o3) 1
\end{verbatim}
stirling2 does not simplify for non-integer arguments.

(%i1) stirling2 (%pi, %pi);
(%o1) stirling2(%pi, %pi)

subset (a, f)  
Returns the subset of the set a that satisfies the predicate f.
subset returns a set which comprises the elements of a for which f returns anything other than false. subset does not apply is to the return value of f.
subset complains if a is not a literal set.
See also partition_set.
Examples:

(%i1) subset ({1, 2, x, x + y, z, x + y + z}, atom);
(%o1) {1, 2, x, z}
(%i2) subset ({1, 2, 7, 8, 9, 14}, evenp);
(%o2) {2, 8, 14}

subsetp (a, b)  
Returns true if and only if the set a is a subset of b.
subsetp complains if either a or b is not a literal set.
Examples:

(%i1) subsetp ({1, 2, 3}, {a, 1, b, 2, c, 3});
(%o1) true
(%i2) subsetp ({a, 1, b, 2, c, 3}, {1, 2, 3});
(%o2) false

symmdifference (a_1, . . ., a_n)  
Returns the symmetric difference of sets a_1, . . ., a_n.
Given two arguments, symmdifference (a, b) is the same as union (setdifference (a, b), setdifference (b, a)).
symmdifference complains if any argument is not a literal set.
Examples:

(%i1) S_1 : {a, b, c};
(%o1) {a, b, c}
(%i2) S_2 : {1, b, c};
(%o2) {1, b, c}
(%i3) S_3 : {a, b, z};
(%o3) {a, b, z}
(%i4) symmdifference ();
(%o4) {}  
(%i5) symmdifference (S_1);
(%o5) {a, b, c}
(%i6) symmdifference (S_1, S_2);
(%o6) {1, a}
(%i7) symmdifference (S_1, S_2, S_3);
(%o7) {1, b, z}
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union (a_1, \ldots, a_n)

Returns the union of the sets a_1 through a_n.

union() (with no arguments) returns the empty set.

union complains if any argument is not a literal set.

Examples:

(%i1) S_1 : {a, b, c + d, %e};
(%o1) {a, b, c + d, %e}
(%i2) S_2 : {%pi, %i, %e, c + d};
(%o2) {a, b, c + d}
(%i3) S_3 : {17, 29, 1729, %pi, %i};
(%o3) {17, 29, 1729, %i, %pi}
(%i4) union ();
(%o4) {}
(%i5) union (S_1);
(%o5) {a, b, c + d}
(%i6) union (S_1, S_2);
(%o6) {a, b, c + d}
(%i7) union (S_1, S_2, S_3);
(%o7) {a, b, c + d, 17, 29, 1729, %i, %pi, %e}
(%i8) symmdifference ({}, S_1, S_2, S_3);
(%o8) {1, b, z}
36 Function Definition

36.1 Introduction to Function Definition

36.2 Function

36.2.1 Ordinary functions

To define a function in Maxima you use the := operator. E.g.

```
f(x) := sin(x)
```

defines a function f. Anonymous functions may also be created using lambda. For example

```
lambda ([i, j], ...)
```
can be used instead of f where

```
f(i, j) := block ([], ...);
map (lambda ([i], i+1), l)
```

would return a list with 1 added to each term.

You may also define a function with a variable number of arguments, by having a final argument which is assigned to a list of the extra arguments:

```
(a1) f ([u]) := u;
(a2) f ([u]) := u
(a3) f (1, 2, 3, 4);
(a4) [1, 2, 3, 4]
(a5) f (a, b, [u]) := [a, b, u];
(a6) f(a, b, [u]) := [a, b, u]
(a7) f (1, 2, 3, 4, 5, 6);
(a8) [1, 2, [3, 4, 5, 6]]
```

The right hand side of a function is an expression. Thus if you want a sequence of expressions, you do

```
f(x) := (expr1, expr2, ..., exprn);
```

and the value of exprn is what is returned by the function.

If you wish to make a return from some expression inside the function then you must use block and return.

```
block ([], expr1, ..., if (a > 10) then return(a), ..., exprn)
```

is itself an expression, and so could take the place of the right hand side of a function definition. Here it may happen that the return happens earlier than the last expression.

The first [] in the block, may contain a list of variables and variable assignments, such as [a: 3, b, c: []], which would cause the three variables a, b, and c to not refer to their global values, but rather have these special values for as long as the code executes inside the block, or inside functions called from inside the block. This is called dynamic binding, since the variables last from the start of the block to the time it exits. Once you return from the block, or throw out of it, the old values (if any) of the variables will be restored.
It is certainly a good idea to protect your variables in this way. Note that the assignments in the block variables, are done in parallel. This means, that if you had used c: a in the
above, the value of c would have been the value of a at the time you just entered the block, but before a was bound. Thus doing something like

\[
\text{block ([a: a], expr1, ... a: a+3, ...}, \text{ exprn)}
\]

will protect the external value of a from being altered, but would let you access what that value was. Thus the right hand side of the assignments, is evaluated in the entering context, before any binding occurs. Using just \text{block ([x], ...)} would cause the x to have itself as value, just as if it would have if you entered a fresh Maxima session.

The actual arguments to a function are treated in exactly same way as the variables in a block. Thus in

\[
f(x) := (\text{expr1, ...}, \text{ exprn});
\]

and

\[
f(1);
\]

we would have a similar context for evaluation of the expressions as if we had done

\[
\text{block ([x: 1], expr1, ...}, \text{ exprn)}
\]

Inside functions, when the right hand side of a definition, may be computed at runtime, it is useful to use \text{define} and possibly \text{buildq}.

### 36.2.2 Memoizing Functions

A \textit{memoizing function} caches the result the first time it is called with a given argument, and returns the stored value, without recomputing it, when that same argument is given. Memoizing functions are often called \textit{array function} and are in fact handled like arrays in many ways:

The names of memoizing functions are appended to the global list \texttt{arrays} (not the global list \texttt{functions}). \texttt{arrayinfo} returns the list of arguments for which there are stored values, and \texttt{listarray} returns the stored values. \texttt{dispfun} and \texttt{fundef} return the array function definition.

\texttt{arraymake} constructs an array function call, analogous to \texttt{funmake} for ordinary functions. \texttt{arrayapply} applies an array function to its arguments, analogous to \texttt{apply} for ordinary functions. There is nothing exactly analogous to \texttt{map} for array functions, although \texttt{map(lambda([x], a[x]), L)} or \texttt{makelist(a[x], x, L)}, where L is a list, are not too far off the mark.

\texttt{remarray} removes an array function definition (including any stored function values), analogous to \texttt{remfunction} for ordinary functions.

\texttt{kill(a[x])} removes the value of the array function a stored for the argument x; the next time a is called with argument x, the function value is recomputed. However, there is no way to remove all of the stored values at once, except for \texttt{kill(a)} or \texttt{remarray(a)}, which also remove the function definition.

**Examples**

If evaluating the function needs much time and only a limited number of points is ever evaluated (which means not much time is spent looking up results in a long list of cached results) Memoizing functions can speed up calculations considerably.

\[
(\%i1) \text{showtime: true}$
\]

Evaluation took 0.0000 seconds (0.0000 elapsed) using 0 bytes.
(\%i2) a[x] := float(sum(sin(x*t), t, 1, 10000));
Evaluation took 0.0000 seconds (0.0000 elapsed) using 0 bytes.
(\%o2) a := float(sum(sin(x t), t, 1, 10000))
\phantom{x}
(\%i3) a[1];
Evaluation took 5.1250 seconds (5.1260 elapsed) using 775.250 MB.
(\%o3) \phantom{x} 1.633891021792447
(\%i4) a[1];
Evaluation took 0.0000 seconds (0.0000 elapsed) using 0 bytes.
(\%o4) \phantom{x} 1.633891021792447

As the memoizing function is only evaluated once for each input value changes in variables
the memoizing function uses are not considered for values that are already cached:

(\%i1) a[x] := b*x;
(\%o1) a := b x
\phantom{x}
(\%i2) b:1;
(\%o2) 1
(\%i3) a[2];
(\%o3) 2
(\%i4) b:2;
(\%o4) 2
(\%i5) a[1];
(\%o5) 2
(\%i6) a[2];
(\%o6) 2

36.3 Macros

\texttt{buildq} \ (L, expr) \quad [\text{Function}]

Substitutes variables named by the list \( L \) into the expression \( \text{expr} \), in parallel, without evaluating \( \text{expr} \). The resulting expression is simplified, but not evaluated, after \texttt{buildq} carries out the substitution.

The elements of \( L \) are symbols or assignment expressions \texttt{symbol: value}, evaluated in parallel. That is, the binding of a variable on the right-hand side of an assignment is the binding of that variable in the context from which \texttt{buildq} was called, not the binding of that variable in the variable list \( L \). If some variable in \( L \) is not given an explicit assignment, its binding in \texttt{buildq} is the same as in the context from which \texttt{buildq} was called.

Then the variables named by \( L \) are substituted into \( \text{expr} \) in parallel. That is, the substitution for every variable is determined before any substitution is made, so the substitution for one variable has no effect on any other.

If any variable \( x \) appears as \texttt{splice} \ (x) in \( \text{expr} \), then \( x \) must be bound to a list, and the list is spliced (interpolated) into \( \text{expr} \) instead of substituted.

Any variables in \( \text{expr} \) not appearing in \( L \) are carried into the result verbatim, even if they have bindings in the context from which \texttt{buildq} was called.
Examples

a is explicitly bound to x, while b has the same binding (namely 29) as in the calling context, and c is carried through verbatim. The resulting expression is not evaluated until the explicit evaluation '%'.

(%i1) (a: 17, b: 29, c: 1729)$
(%i2) buildq ([a: x, b], a + b + c);
(x + c + 29)
(%i3) '%;
(%o3) x + 1758

e is bound to a list, which appears as such in the arguments of foo, and interpolated into the arguments of bar.

(%i1) buildq ([e: [a, b, c]], foo (x, e, y));
foo(x, [a, b, c], y)
(%i2) buildq ([e: [a, b, c]], bar (x, splice (e), y));
bar(x, a, b, c, y)

The result is simplified after substitution. If simplification were applied before substitution, these two results would be the same.

(%i1) buildq ([e: [a, b, c]], splice (e) + splice (e));
2 c + 2 b + 2 a
(%i2) buildq ([e: [a, b, c]], 2 * splice (e));
2 a b c

The variables in L are bound in parallel; if bound sequentially, the first result would be foo (b, b). Substitutions are carried out in parallel; compare the second result with the result of subst, which carries out substitutions sequentially.

(%i1) buildq ([a: b, b: a], foo (a, b));
foo(b, a)
(%i2) buildq ([u: v, v: w, w: x, x: y, y: z, z: u],
bar (u, v, w, x, y, z));
bar(v, w, x, y, z, u)
(%i3) subst ([u=v, v=w, w=x, x=y, y=z, z=u],
bar (u, v, w, x, y, z));
bar(u, u, u, u, u, u)

Construct a list of equations with some variables or expressions on the left-hand side and their values on the right-hand side. macroexpand shows the expression returned by show_values.

(%i1) show_values ([L]) ::= buildq ([L], map ("=",'L, L));
(%o1) show_values([L]) ::= buildq([L], map("=",'L, L))
(%i2) (a: 17, b: 29, c: 1729)$
(%i3) show_values (a, b, c - a - b);
[a = 17, b = 29, c - b - a = 1683]
(%i4) macroexpand (show_values (a, b, c - a - b));
(map(=, '(a, b, c - b - a), [a, b, c - b - a]))

Given a function of several arguments, create another function for which some of the arguments are fixed.
(%i1) curry (f, [a]) :=
   buildq ([f, a], lambda ([[x]], apply (f, append (a, x))))$
(%i2) by3 : curry ("*", 3);
(%o2) lambda([\[x\]], apply(*, append([3], x)))
(%i3) by3 (a + b);
(%o3) 3 (b + a)

macroexpand (expr) [Function]
Returns the macro expansion of expr without evaluating it, when expr is a macro
function call. Otherwise, macroexpand returns expr.
If the expansion of expr yields another macro function call, that macro function call
is also expanded.
macroexpand quotes its argument. However, if the expansion of a macro function call
has side effects, those side effects are executed.
See also ::=, macros, and macroexpand1..
Examples
(%i1) g (x) ::= x / 99;
   x
(%o1) g(x) ::= --
       99
(%i2) h (x) ::= buildq ([x], g (x - a));
(%o2) h(x) ::= buildq([x], g(x - a))
(%i3) a: 1234;
(%o3) 1234
(%i4) macroexpand (h (y));
   y - a
(%o4) ------
       99
(%i5) h (y);
   y - 1234
(%o5) ------
       99

macroexpand1 (expr) [Function]
Returns the macro expansion of expr without evaluating it, when expr is a macro
function call. Otherwise, macroexpand1 returns expr.
macroexpand1 quotes its argument. However, if the expansion of a macro function
call has side effects, those side effects are executed.
If the expansion of expr yields another macro function call, that macro function call
is not expanded.
See also ::=, macros, and macroexpand.
Examples
(%i1) g (x) ::= x / 99;
   x
(%o1) g(x) ::= --
       99
\begin{verbatim}
(%i2) h (x) ::= buildq ([x], g (x - a));
(%o2) h(x) ::= buildq([x], g(x - a))
(%i3) a: 1234;  
(%o3) 1234
(%i4) macroexpand1 (h (y));
(%o4) g(y - a)
(%i5) h (y);
  y - 1234
(%o5) ---------
    99
\end{verbatim}

**macros**

[Global variable]

Default value: []

`macros` is the list of user-defined macro functions. The macro function definition operator `::=` puts a new macro function onto this list, and `kill`, `remove`, and `remfunction` remove macro functions from the list.

See also `infolists`.

**splice (a)**

[Function]

Splices (interpolates) the list named by the atom `a` into an expression, but only if `splice` appears within `buildq`; otherwise, `splice` is treated as an undefined function. If appearing within `buildq` as `a` alone (without `splice`), `a` is substituted (not interpolated) as a list into the result. The argument of `splice` can only be an atom; it cannot be a literal list or an expression which yields a list.

Typically `splice` supplies the arguments for a function or operator. For a function `f`, the expression `f (splice (a))` within `buildq` expands to `f (a[1], a[2], a[3], ...). For an operator `o`, the expression "o" (splice (a)) within `buildq` expands to "o" (a[1], a[2], a[3], ...), where `o` may be any type of operator (typically one which takes multiple arguments). Note that the operator must be enclosed in double quotes ".

Examples

\begin{verbatim}
(%i11) buildq ([x: [1, %pi, z - y]], foo (splice (x)) / length (x));
  foo(1, %pi, z - y)
(%o11) -----------------------
          length([1, %pi, z - y])
(%i12) buildq ([x: [1, %pi]], "/" (splice (x)));
  1
(%o12) ---
    %pi
(%i13) matchfix ("<>", "<>");
(%o13) <>
(%i14) buildq ([x: [1, %pi, z - y]], ">" (splice (x)));
(%o14) <>1, %pi, z - y>
\end{verbatim}
36.4 Functions and Variables for Function Definition

apply \((F, [x_1, \ldots, x_n])\)

Constructs and evaluates an expression \(F(arg_1, \ldots, arg_n)\).

apply does not attempt to distinguish a memoizing function from an ordinary function: when \(F\) is the name of a memoizing function, apply evaluates \(F(...)\) (that is, a function call with parentheses instead of square brackets). arrayapply evaluates a function call with square brackets in this case.

See also funmake and args.

Examples:

apply evaluates its arguments. In this example, \(\text{min}\) is applied to the value of \(L\).

\(\%i1\) \(L : [1, 5, -10.2, 4, 3];\)
\(\%o1\) \([1, 5, -10.2, 4, 3]\)
\(\%i2\) apply \((\text{min}, L)\);
\(\%o2\) \(-10.2\)

apply evaluates arguments, even if the function \(F\) quotes them.

\(\%i1\) \(F (x) := x / 1729;\)
\(\%o1\) \(F(x) := \frac{x}{1729}\)
\(\%i2\) fname : \(F;\)
\(\%o2\) \(F\)
\(\%i3\) dispfun \((F)\);
\(\%t3\) \(F(x) := \frac{x}{1729}\)
\(\%o3\) \([\%t3]\)
\(\%i4\) dispfun \((\text{name})\);
fundef: no such function: \(\text{name}\)
-- an error. To debug this try: debugmode(true);
\(\%i5\) apply \((\text{dispfun}, [\text{name}])\);
\(\%t5\) \(F(x) := \frac{x}{1729}\)
\(\%o5\) \([\%t5]\)

apply evaluates the function name \(F\). Single quote ' defeats evaluation. demoivre is the name of a global variable and also a function.

\(\%i1\) demoivre;
\(\%o1\) \(\text{false}\)
\(\%i2\) demoivre \((\text{exp} \((\%i * x)\))\);
\(\%o2\) \(\%i \sin(x) + \cos(x)\)
apply (demoivre, [exp (%i * x)]);
apply: found false where a function was expected.
-- an error. To debug this try: debugmode(true);
apply ('demoivre, [exp (%i * x)]);
%i sin(x) + cos(x)

How to convert a nested list into a matrix:

a: [[1,2], [3,4]]
apply(matrix, a);
[ 1 2 ]
[ 3 4 ]

block

block ([v_1, ..., v_m], expr_1, ..., expr_n)
block (expr_1, ..., expr_n)

The function block allows to make the variables v_1, ..., v_m to be local for a sequence of commands. If these variables are already bound block saves the current values of the variables v_1, ..., v_m (if any) upon entry to the block, then unbinds the variables so that they evaluate to themselves; The local variables may be bound to arbitrary values within the block but when the block is exited the saved values are restored, and the values assigned within the block are lost.

If there is no need to define local variables then the list at the beginning of the block command may be omitted. In this case if neither return nor go are used block behaves similar to the following construct:

(expr_1, expr_2, ..., expr_n);

expr_1, ..., expr_n will be evaluated in sequence and the value of the last expression will be returned. The sequence can be modified by the go, throw, and return functions. The last expression is expr_n unless return or an expression containing throw is evaluated.

The declaration local (v_1, ..., v_m) within block saves the properties associated with the symbols v_1, ..., v_m, removes any properties before evaluating other expressions, and restores any saved properties on exit from the block. Some declarations are implemented as properties of a symbol, including :=, array, dependencies, atvalue, matchdeclare, atomgrad, constant, nonscalar, assume, and some others. The effect of local is to make such declarations effective only within the block; otherwise declarations within a block are actually global declarations.

block may appear within another block. Local variables are established each time a new block is evaluated. Local variables appear to be global to any enclosed blocks. If a variable is non-local in a block, its value is the value most recently assigned by an enclosing block, if any; otherwise, it is the value of the variable in the global environment. This policy may coincide with the usual understanding of "dynamic scope".

The value of the block is the value of the last statement or the value of the argument to the function return which may be used to exit explicitly from the block. The
function \texttt{go} may be used to transfer control to the statement of the block that is tagged with the argument to \texttt{go}. To tag a statement, precede it by an atomic argument as another statement in the block. For example: \texttt{block ([x], x:1, loop, x: x+1, ..., \texttt{go(loop)}, ...).} The argument to \texttt{go} must be the name of a tag appearing within the block. One cannot use \texttt{go} to transfer to a tag in a block other than the one containing the \texttt{go}.

Blocks typically appear on the right side of a function definition but can be used in other places as well.

See also \texttt{return} and \texttt{go}.

\texttt{break (expr_1, ..., expr_n)}

Evaluates and prints \texttt{expr_1}, ..., \texttt{expr_n} and then causes a Maxima break at which point the user can examine and change his environment. Upon typing \texttt{exit}; the computation resumes.

\texttt{catch (expr_1, ..., expr_n)}

Evaluates \texttt{expr_1}, ..., \texttt{expr_n} one by one; if any leads to the evaluation of an expression of the form \texttt{throw (arg)}, then the value of the \texttt{catch} is the value of \texttt{throw (arg)}, and no further expressions are evaluated. This "non-local return" thus goes through any depth of nesting to the nearest enclosing \texttt{catch}. If there is no \texttt{catch} enclosing a \texttt{throw}, an error message is printed.

If the evaluation of the arguments does not lead to the evaluation of any \texttt{throw} then the value of \texttt{catch} is the value of \texttt{expr_n}.

\begin{verbatim}
(%i1) lambda ([x], if x < 0 then throw(x) else f(x))$
(%i2) g(1) := catch (map ('%, 1))$
(%i3) g ([1, 2, 3, 7])
(%i4) g ([1, 2, -3, 7])
\end{verbatim}

The function \texttt{g} returns a list of \texttt{f} of each element of \texttt{l} if \texttt{l} consists only of non-negative numbers; otherwise, \texttt{g} "catches" the first negative element of \texttt{l} and "throws" it up.

\texttt{compfile (filename, f_1, ..., f_n, functions)}

Translates Maxima functions into Lisp and writes the translated code into the file \texttt{filename}.

\texttt{compfile (filename, all)}

Translates all user-defined functions.

The Lisp translations are not evaluated, nor is the output file processed by the Lisp compiler. \texttt{translate} creates and evaluates Lisp translations. \texttt{compfile} translates Maxima into Lisp, and then executes the Lisp compiler.

See also \texttt{translate}, \texttt{translate_file}, and \texttt{compile_file}. 

compile

compile (f_1, ..., f_n)
compile (functions)
compile (all)

Translates Maxima functions \( f_1, ..., f_n \) into Lisp, evaluates the Lisp translations, and calls the Lisp function COMPILE on each translated function. compile returns a list of the names of the compiled functions.

compile (all) or compile (functions) compiles all user-defined functions.

compile quotes its arguments; the quote-quote operator " evaluates to defeat quotation.

 Compiling a function to native code can mean a big increase in speed and might cause the memory footprint to reduce drastically. Code tends to be especially effective when the flexibility it needs to provide is limited. If compilation doesn’t provide the speed that is needed a few ways to limit the code’s functionality are the following:

- If the function accesses global variables the complexity of the function can be drastically be reduced by limiting these variables to one data type, for example using mode_declare or a statement like the following one: put(x_1, bigfloat, numerical_type)

- The compiler might warn about undeclared variables if text could either be a named option to a command or (if they are assigned a value to) the name of a variable. Prepending the option with a single quote ' tells the compiler that the text is meant as an option.

define

define (f(x_1, ..., x_n), expr)
define (f[x_1, ..., x_n], expr)
define (f[x_1, ..., x_n](y_1, ..., y_m), expr)
define (funmake (f, [x_1, ..., x_n]), expr)
define (arraymake (f, [x_1, ..., x_n]), expr)
define (ev (expr_1), expr_2)

Defines a function named \( f \) with arguments \( x_1, ..., x_n \) and function body \( expr \).

define always evaluates its second argument (unless explicitly quoted). The function so defined may be an ordinary Maxima function (with arguments enclosed in parentheses) or a memoizing function (with arguments enclosed in square brackets).

When the last or only function argument \( x_n \) is a list of one element, the function defined by define accepts a variable number of arguments. Actual arguments are assigned one-to-one to formal arguments \( x_1, ..., x_{(n - 1)} \), and any further actual arguments, if present, are assigned to \( x_n \) as a list.

When the first argument of define is an expression of the form \( f(x_1, ..., x_n) \) or \( f[x_1, ..., x_n] \), the function arguments are evaluated but \( f \) is not evaluated, even if there is already a function or variable by that name.

When the first argument is an expression with operator funmake, arraymake, or ev, the first argument is evaluated; this allows for the function name to be computed, as well as the body.

All function definitions appear in the same namespace; defining a function \( f \) within another function \( g \) does not automatically limit the scope of \( f \) to \( g \). However, local(f)
makes the definition of function \( f \) effective only within the block or other compound expression in which \texttt{local} appears.

If some formal argument \( x_k \) is a quoted symbol (after evaluation), the function defined by \texttt{define} does not evaluate the corresponding actual argument. Otherwise all actual arguments are evaluated.

See also \texttt{:=} and \texttt{::=}.

Examples:

\texttt{define} always evaluates its second argument (unless explicitly quoted).

\begin{verbatim}
(%i1) expr : cos(y) - sin(x);
    cos(y) - sin(x)
(%o1)                          
(%i2) define (F1 (x, y), expr);
    F1(x, y) := cos(y) - sin(x)
(%o2)                          
(%i3) F1 (a, b);
    cos(b) - sin(a)
(%o3)                          
(%i4) F2 (x, y) := expr;
    F2(x, y) := expr
(%o4)                          
(%i5) F2 (a, b);
    cos(y) - sin(x)
(%o5)                          
\end{verbatim}

The function defined by \texttt{define} may be an ordinary Maxima function or a \texttt{memoizing function}.

\begin{verbatim}
(%i1) define (G1 (x, y), x*y - y*x);
    G1(x, y) := x . y - y . x
(%o1)                          
(%i2) define (G2 \[x, y\], x*y - y*x);
    G2 := x . y - y . x
(%i3) G1 (a, b);
    x, y
(%o3)                          
\end{verbatim}

When the last or only function argument \( x_n \) is a list of one element, the function defined by \texttt{define} accepts a variable number of arguments.

\begin{verbatim}
(%i1) define (H ([L]), '(apply ("+", L)));
    H([L]) := apply("+", L)
(%o1)                          
(%i2) H (a, b, c);
    c + b + a
(%o2)                          
\end{verbatim}

When the first argument is an expression with operator \texttt{funmake}, \texttt{arraymake}, or \texttt{ev}, the first argument is evaluated.

\begin{verbatim}
(%i1) [F : I, u : x];
    [I, x]
(%o1)                       
(%i2) funmake (F, [u]);
    I(x)
(%o2)                       
(%i3) define (funmake (F, [u]), cos(u) + 1);
    I(x) := cos(x) + 1
(%o3)                       
(%i4) define (arraymake (F, [u]), cos(u) + 1);
    I := cos(x) + 1
(%o4)                       
(%i5) define (foo (x, y), bar (y, x));
    foo(x, y) := bar(y, x)
(%o5)                       
\end{verbatim}
(\%i6) define (ev (foo (x, y)), sin(x) - cos(y));
(\%o6) bar(y, x) := sin(x) - cos(y)

define_variable (name, default_value, mode) [Function]
Introduces a global variable into the Maxima environment. define_variable is useful in user-written packages, which are often translated or compiled as it gives the compiler hints of the type ("mode") of a variable and therefore avoids requiring it to generate generic code that can deal with every variable being an integer, float, maxima object, array etc.

define_variable carries out the following steps:

1. mode_declare (name, mode) declares the mode ("type") of name to the translator which can considerably speed up compiled code as it allows having to create generic code. See mode_declare for a list of the possible modes.
2. If the variable is unbound, default_value is assigned to name.
3. Associates name with a test function to ensure that name is only assigned values of the declared mode.

The value_check property can be assigned to any variable which has been defined via define_variable with a mode other than any. The value_check property is a lambda expression or the name of a function of one variable, which is called when an attempt is made to assign a value to the variable. The argument of the value_check function is the would-be assigned value.

define_variable evaluates default_value, and quotes name and mode. define_variable returns the current value of name, which is default_value if name was unbound before, and otherwise it is the previous value of name.

Examples:

foo is a Boolean variable, with the initial value true.

(\%i11) define_variable (foo, true, boolean);
(\%o11) true
(\%i12) foo;
(\%o12) true
(\%i13) foo: false;
(\%o13) false
(\%i14) foo: %pi;

translator: foo was declared with mode boolean, but it has value: %pi

-- an error. To debug this try: debugmode(true);
(\%i15) foo;
(\%o15) false

bar is an integer variable, which must be prime.

(\%i1) define_variable (bar, 2, integer);
(\%o1) 2
(\%i2) qput (bar, prime_test, value_check);
(\%o2) prime_test
dispfun (f_1, ..., f_n)
dispfun (all)

Displays the definition of the user-defined functions f_1, ..., f_n. Each argument may be the name of a macro (defined with ::=), an ordinary function (defined with := or define), an array function (defined with := or define, but enclosing arguments in square brackets [ ]), a subscripted function (defined with := or define, but enclosing some arguments in square brackets and others in parentheses ( )), one of a family of subscripted functions selected by a particular subscript value, or a subscripted function defined with a constant subscript.

dispfun (all) displays all user-defined functions as given by the functions, arrays, and macros lists, omitting subscripted functions defined with constant subscripts.
dispfun creates an intermediate expression label (\%t1, \%t2, etc.) for each displayed function, and assigns the function definition to the label. In contrast, fundef returns the function definition.

dispfun quotes its arguments; the quote-quote operator `'` defeats quotation. dispfun returns the list of intermediate expression labels corresponding to the displayed functions.

Examples:

(%i1) m(x, y) ::= x^(-y);
   \(- y\)
(%o1) m(x, y) ::= x

(%i2) f(x, y) ::= x^(-y);
   \(- y\)
(%o2) f(x, y) ::= x

(%i3) g[x, y] ::= x^(-y);
   \(- y\)
(%o3) g := x

(%i4) h[x](y) ::= x^(-y);
   \(- y\)
(%o4) h (y) ::= x

(%i5) i[8](y) ::= 8^(-y);
   \(- y\)
(%o5) i (y) ::= 8
\begin{verbatim}
(%i6) dispfun (m, f, g, h, h[5], h[10], i[8]);
   - y
m(x, y) ::= x

(%t6)  
   - y
f(x, y) ::= x

(%t7)  
   - y
g := x
   x, y

(%t8)  
   - y
h (y) ::= x
   x

(%t9)  
   - y
h (y) ::= --
   5 y
   5

(%t10)  
   - y
h (y) ::= ---
   10 y
   10

(%t11)  
   - y
i (y) ::= 8
   8

(%o12) [%t6, %t7, %t8, %t9, %t10, %t11, %t12]

(%i13) ''%;
   - y - y - y

(%o13) [m(x, y) ::= x , f(x, y) ::= x , g ::= x ,
   x, y
   - y 1 1 - y
h (y) ::= x , h (y) ::= --, h (y) ::= ---, i (y) ::= 8 ]
   x 5 y 10 y 8
   5 10
\end{verbatim}

**fullmap** \((f, \text{expr}_1, \ldots)\)  
[Function]

Similar to **map**, but **fullmap** keeps mapping down all subexpressions until the main operators are no longer the same.

**fullmap** is used by the Maxima simplifier for certain matrix manipulations; thus, Maxima sometimes generates an error message concerning **fullmap** even though **fullmap** was not explicitly called by the user.

Examples:
(%i1) a + b * c;
(%o1) b c + a
(%i2) fullmap (g, %);
(%o2) g(b) g(c) + g(a)
(%i3) map (g, %th(2));
(%o3) g(b c) + g(a)

fullmap (f, list_1, . . .)    [Function]

Similar to fullmap, but fullmapl only maps onto lists and matrices.

Example:

(%i1) fullmapl ("+", [3, [4, 5]], [[a, 1], [0, -1.5]]);
(%o1) [[a + 3, 4], [4, 3.5]]

functions    [System variable]

Default value: []

functions is the list of ordinary Maxima functions in the current session. An ordinary
function is a function constructed by define or := and called with parentheses (). A
function may be defined at the Maxima prompt or in a Maxima file loaded by load
or batch.

Memoizing functions (called with square brackets, e.g., F[x]) and subscripted func-
tions (called with square brackets and parentheses, e.g., F[x](y)) are listed by the
global variable arrays, and not by functions.

Lisp functions are not kept on any list.

Examples:

(%i1) F_1 (x) := x - 100;
(%o1) F_1(x) := x - 100
(%i2) F_2 (x, y) := x / y;
(%o2) F_2(x, y) := -
    x
    y
(%i3) define (F_3 (x), sqrt (x));
(%o3) F_3(x) := sqrt(x)
(%i4) G_1 [x] := x - 100;
(%o4) G_1 := x - 100
(%i5) G_2 [x, y] := x / y;
(%o5) G_2 := -
    x, y
    y
(%i6) define (G_3 [x], sqrt (x));
(%o6) G_3 := sqrt(x)
(%i7) H_1 [x] (y) := x^y;
(%o7) H_1(y) := x
    x
Chapter 36: Function Definition

(%i8) functions;
(%o8) [F_1(x), F_2(x, y), F_3(x)]

(%i9) arrays;
(%o9) [G_1, G_2, G_3, H_1]

fundef (f)

Returns the definition of the function f.

The argument may be

• the name of a macro (defined with ::=),
• an ordinary function (defined with := or define),
• a memoizing function (defined with := or define, but enclosing arguments in square brackets []),
• a subscripted function (defined with := or define, but enclosing some arguments in square brackets and others in parentheses ()),
• one of a family of subscripted functions selected by a particular subscript value,
• or a subscripted function defined with a constant subscript.

fundef quotes its argument; the quote-quote operator ''' defeats quotation.

fundef (f) returns the definition of f. In contrast, dispfun (f) creates an intermediate expression label and assigns the definition to the label.

funmake (F, [arg_1, ..., arg_n])

Returns an expression F(arg_1, ..., arg_n). The return value is simplified, but not evaluated, so the function F is not called, even if it exists.

funmake does not attempt to distinguish memoizing functions from ordinary functions; when F is the name of a memoizing function, funmake returns F(...) (that is, a function call with parentheses instead of square brackets). arraymake returns a function call with square brackets in this case.

funmake evaluates its arguments.

See also apply and args.

Examples:

funmake applied to an ordinary Maxima function.

(%i1) F (x, y) := y^2 - x^2;
(%o1) F(x, y) := y^2 - x^2

(%i2) funmake (F, [a + 1, b + 1]);
(%o2) F(a + 1, b + 1)

(%o3) (b + 1)^2 - (a + 1)^2

funmake applied to a macro.

(%i1) G (x) ::= (x - 1)/2;
(%o1) G(x) ::= (x - 1)/2

(%o2) G(x) ::= (x - 1)/2
(\%i2) funmake (G, [u]);
(\%o2) G(u)
(\%i3) 'u - 1;
(\%o3) ------
      2

\textbf{funmake} applied to a subscripted function.

(\%i1) H [a] (x) := (x - 1)^a;
(\%o1) a
       H (x) := (x - 1)
(\%i2) funmake (H [n], [%e]);
(\%o2) n
      lambda([x], (x - 1)^n)
(\%i3) 'n;
(\%o3) (%e - 1)
(\%i4) funmake ('(H [n]), [%e]);
(\%o4) n
      H (%e)
(\%i5) 'n;
(\%o5) (%e - 1)

\textbf{funmake} applied to a symbol which is not a defined function of any kind.

(\%i1) funmake (A, [u]);
(\%o1) A(u)
(\%i2) 'A(u);
(\%o2) A(u)

\textbf{funmake} evaluates its arguments, but not the return value.

(\%i1) det(a,b,c) := b^2 -4*a*c;
(\%o1) 2
       det(a, b, c) := b - 4 a c
(\%i2) (x : 8, y : 10, z : 12);
(\%o2) 12
(\%i3) f : det;
(\%o3) det
(\%i4) funmake (f, [x, y, z]);
(\%o4) det(8, 10, 12)
(\%i5) 'f;
(\%o5) - 284

Maxima simplifies \textbf{funmake}'s return value.

(\%i1) funmake (sin, [%pi / 2]);
(\%o1) 1
lambda

lambda ([x_1, . . ., x_m], expr_1, . . ., expr_n)
lambda ([L], expr_1, . . ., expr_n)
lambda ([x_1, . . ., x_m, L], expr_1, . . ., expr_n)

Defines and returns a lambda expression (that is, an anonymous function). The
function may have required arguments x_1, . . ., x_m and/or optional arguments L,
which appear within the function body as a list. The return value of the function
is expr_n. A lambda expression can be assigned to a variable and evaluated like an
ordinary function. A lambda expression may appear in some contexts in which a
function name is expected.

When the function is evaluated, unbound local variables x_1, . . ., x_m are created.
lambda may appear within block or another lambda; local variables are established
each time another block or lambda is evaluated. Local variables appear to be global
to any enclosed block or lambda. If a variable is not local, its value is the value most
recently assigned in an enclosing block or lambda, if any, otherwise, it is the value
of the variable in the global environment. This policy may coincide with the usual
understanding of "dynamic scope".

After local variables are established, expr_1 through expr_n are evaluated in turn. The
special variable %%, representing the value of the preceding expression, is recognized.
throw and catch may also appear in the list of expressions.

return cannot appear in a lambda expression unless enclosed by block, in which case
return defines the return value of the block and not of the lambda expression, unless
the block happens to be expr_n. Likewise, go cannot appear in a lambda expression
unless enclosed by block.

lambda quotes its arguments; the quote-quote operator '' defeats quotation.

Examples:

- A lambda expression can be assigned to a variable and evaluated like an ordinary
  function.

  (%i1) f: lambda ([x], x^2);
      2
  lambda([x], x )
  (f)(a);
  2
  a

- A lambda expression may appear in contexts in which a function evaluation is
  expected.

  (%i1) lambda ([x], x^2) (a);
      2
  a
  (%i2) apply (lambda ([x], x^2), [a]);
      2
  a
  (%i3) map (lambda ([x], x^2), [a, b, c, d, e]);
      2 2 2 2 2
  [a, b, c, d, e]
• Argument variables are local variables. Other variables appear to be global variables. Global variables are evaluated at the time the lambda expression is evaluated, unless some special evaluation is forced by some means, such as "'".

\[
\begin{align*}
\text{(\%i1)} & \quad \text{a:}\ \%\pi$
\text{(\%i2)} & \quad \text{b:}\ \%e$
\text{(\%i3)} & \quad \text{g: lambda ([a], a*b)};
\text{(\%o3)} & \quad \text{lambda([a], a b)}
\text{(\%i4)} & \quad \text{b:}\ \%\gamma$
\text{(\%i5)} & \quad \text{g(i/2)};
\text{(\%o5)} & \quad \text{\%gamma}
\text{(\%i6)} & \quad \text{g2: lambda ([a], a*'b)};
\text{(\%o6)} & \quad \text{lambda([a], a \%gamma)}
\text{(\%i7)} & \quad \text{b:}\ \%e$
\text{(\%i8)} & \quad \text{g2(1/2)};
\text{(\%o8)} & \quad \text{\%gamma}
\end{align*}
\]

• Lambda expressions may be nested. Local variables within the outer lambda expression appear to be global to the inner expression unless masked by local variables of the same names.

\[
\begin{align*}
\text{(\%i1)} & \quad \text{h: lambda ([a, b], h2: lambda ([a], a*b), h2(1/2))};
\text{(\%o1)} & \quad \text{lambda([a, b], h2 : lambda([a], a b), h2(-))}
\text{(\%i2)} & \quad \text{h(%pi, \%gamma)};
\text{(\%o2)} & \quad \text{\%gamma}
\end{align*}
\]

• Since \texttt{lambda} quotes its arguments, lambda expression i below does not define a "multiply by a" function. Such a function can be defined via \texttt{buildq}, as in lambda expression i2 below.

\[
\begin{align*}
\text{(\%i1)} & \quad \text{i: lambda ([a], lambda ([x], a*x))};
\text{(\%o1)} & \quad \text{lambda([a], lambda([x], a x))}
\text{(\%i2)} & \quad \text{i(1/2)};
\text{(\%o2)} & \quad \text{lambda([x], a x)}
\text{(\%i3)} & \quad \text{i2: lambda([a], buildq([a: a], lambda([x], a*x)))};
\text{(\%o3)} & \quad \text{lambda([a], buildq([a : a], lambda([x], a x)))}
\text{(\%i4)} & \quad \text{i2(1/2)};
\text{(\%o4)} & \quad \text{lambda([x], (-) x)}
\end{align*}
\]
A lambda expression may take a variable number of arguments, which are indicated by \([L]\) as the sole or final argument. The arguments appear within the function body as a list.

\[
\begin{align*}
\text{(i1)} & \ f : \text{lambda} \ (\text{[aa, bb, [cc]], aa * cc + bb}); \\
\text{(o1)} & \ \text{lambda}([\text{aa, bb, [cc]}], \text{aa cc + bb}) \\
\text{(i2)} & \ f (\text{foo, %i, 17, 29, 256}); \\
\text{(o2)} & \ [17 \text{ foo + %i, 29 foo + %i, 256 foo + %i}] \\
\text{(i3)} & \ g : \text{lambda} \ (\text{[[aa]], apply (+, aa)}); \\
\text{(o3)} & \ \text{lambda}([[\text{aa}]], \text{apply} (+, \text{aa})) \\
\text{(i4)} & \ g (17, 29, x, y, z, %e); \\
\text{(o4)} & \ z + y + x + %e + 46
\end{align*}
\]

\(\text{local} \ (v_1, \ldots, v_n)\)  

Saves the properties associated with the symbols \(v_1, \ldots, v_n\), removes any properties before evaluating other expressions, and restores any saved properties on exit from the block or other compound expression in which local appears.

Some declarations are implemented as properties of a symbol, including \(:=\), array, dependencies, atvalue, matchdeclare, atomgrad, constant, nonscalar, assume, and some others. The effect of local is to make such declarations effective only within the block or other compound expression in which local appears; otherwise such declarations are global declarations.

local can only appear in block or in the body of a function definition or lambda expression, and only one occurrence is permitted in each.

local quotes its arguments. local returns done.

Example:

A local function definition.

\[
\begin{align*}
\text{(i1)} & \ \text{foo} (x) := 1 - x; \\
\text{(o1)} & \ \text{foo(x)} := 1 - x \\
\text{(i2)} & \ \text{foo} (100); \\
\text{(o2)} & \ - 99 \\
\text{(i3)} & \ \text{block} \ (\text{local} \ (\text{foo}), \text{foo} (x) := 2 * x, \text{foo} (100)); \\
\text{(o3)} & \ 200 \\
\text{(i4)} & \ \text{foo} (100); \\
\text{(o4)} & \ - 99
\end{align*}
\]

\(\text{macroexpansion}\)  

[Option variable]  

Default value: false

macroexpansion controls whether the expansion (that is, the return value) of a macro function is substituted for the macro function call. A substitution may speed up subsequent expression evaluations, at the cost of storing the expansion.

false The expansion of a macro function is not substituted for the macro function call.
expand  The first time a macro function call is evaluated, the expansion is stored. The expansion is not recomputed on subsequent calls; any side effects (such as print or assignment to global variables) happen only when the macro function call is first evaluated. Expansion in an expression does not affect other expressions which have the same macro function call.

displace  The first time a macro function call is evaluated, the expansion is substituted for the call, thus modifying the expression from which the macro function was called. The expansion is not recomputed on subsequent calls; any side effects happen only when the macro function call is first evaluated. Expansion in an expression does not affect other expressions which have the same macro function call.

Examples

When macroexpansion is false, a macro function is called every time the calling expression is evaluated, and the calling expression is not modified.

(%i1) f(x) := h(x) / g(x);
   h(x)
(%o1) f(x) := ----
       g(x)
(%i2) g(x) ::= block (print("x + 99 is equal to", x),
                   return (x + 99));
   x + 99
(%o2) g(x) ::= block(print("x + 99 is equal to", x),
                  return(x + 99))
(%i3) h(x) ::= block (print("x - 99 is equal to", x),
                   return (x - 99));
   x - 99
(%o3) h(x) ::= block(print("x - 99 is equal to", x),
                   return(x - 99))
(%i4) macroexpansion: false;
(%o4) false
(%i5) f (a * b);
   x - 99 is equal to x
   x + 99 is equal to x
a b - 99
(%o5) ----------
a b + 99
(%i6) dispfun (f);
   h(x)
(%t6) f(x) := ----
       g(x)
(%o6) [%t6]
(%i7) f (a * b);
   x - 99 is equal to x
   x + 99 is equal to x
a b - 99
(%o7) ----------
a b + 99
When macroexpansion is **expand**, a macro function is called once, and the calling expression is not modified.

```
(%i1) f(x) := h(x) / g(x);
   h(x)
(%o1) f(x) := ----
   g(x)
(%i2) g(x) ::= block (print ("x + 99 is equal to", x),
   return (x + 99));
   g(x) ::= block(print("x + 99 is equal to", x),
   return(x + 99))
(%i3) h(x) ::= block (print ("x - 99 is equal to", x),
   return (x - 99));
   h(x) ::= block(print("x - 99 is equal to", x),
   return(x - 99))
(%i4) macroexpansion: expand;
(%o4) expand
(%i5) f(a * b);
   x - 99 is equal to x
   x + 99 is equal to x
   a b - 99
   --------
   a b + 99
(%i6) dispfun (f);
   mmacroexpanded(x - 99, h(x))
   mmacroexpanded(x + 99, g(x))
(%o6) [%t6]
(%i7) f(a * b);
   a b - 99
   --------
   a b + 99
```

When macroexpansion is **displace**, a macro function is called once, and the calling expression is modified.

```
(%i1) f(x) := h(x) / g(x);
   h(x)
(%o1) f(x) := ----
   g(x)
(%i2) g(x) ::= block (print ("x + 99 is equal to", x),
   return (x + 99));
   g(x) ::= block(print("x + 99 is equal to", x),
   return(x + 99))
(%i3) h(x) ::= block (print ("x - 99 is equal to", x),
   return (x - 99));
   h(x) ::= block(print("x - 99 is equal to", x),
   return(x - 99))
```
(%i4) macroexpansion: displace;
(%o4) displace
(%i5) f (a * b);
x - 99 is equal to x
x + 99 is equal to x

a b - 99
--------
a b + 99

(%i6) dispfun (f);
x - 99

f(x) := 

x + 99

(%o6) [%t6]
(%i7) f (a * b);

a b - 99
--------
a b + 99

mode_declare (y_1, mode_1, ..., y_n, mode_n) [Function]
modeDeclare (y_1, mode_1, ..., y_n, mode_n) [Function]

A mode_declare informs the compiler which type (lisp programmers name the type: "mode") a function parameter or its return value will be of. This can greatly boost the efficiency of the code the compiler generates: Without knowing the type of all variables and knowing the return value of all functions a function uses in advance very generic (and thus potentially slow) code needs to be generated.

The arguments of mode_declare are pairs consisting of a variable (or a list of variables all having the same mode) and a mode. Available modes ("types") are:

- array: an declared array (see the detailed description below)
- boolean: true or false
- integer: integers (including arbitrary-size integers)
- fixnum: integers (excluding arbitrary-size integers)
- float: machine-size floating-point numbers
- real: machine-size floating-point or integer
- number: Numbers
- any: any kind of object (useful for arrays of any)

A function parameter named a can be declared as an array filled with elements of the type t the following way:

mode_declare (a, array(t, dim1, dim2, ...))

If none of the elements of the array a needs to be checked if it still doesn’t contain a value additional code can be omitted by declaring this fact, too:

mode_declare (a, array (t, complete, dim1, dim2, ...))

The complete has no effect if all array elements are of the type fixnum or float: Machine-sized numbers inevitably contain a value (and will automatically be initialized to 0 in most lisp implementations).
Another way to tell that all entries of the array \( a \) are of the type ("mode") \( m \) and have been assigned a value to would be:

\[
\text{mode\_declare} \ (\text{completearray} \ (a), \ m) \]

Numeric code using arrays might run faster still if the size of the array is known at compile time, as well, as in:

\[
\text{mode\_declare} \ (\text{completearray} \ (a \ [10, \ 10]), \ \text{float})
\]

for a floating point number array named \( a \) which is \( 10 \times 10 \).

\text{mode\_declare} also can be used in order to declare the type of the result of a function. In this case the function compilation needs to be preceded by another \text{mode\_declare} statement. For example the expression,

\[
\text{mode\_declare} \ [[\text{function} \ (f_1, \ f_2, \ ...)], \ \text{fixnum}]
\]

declares that the values returned by \( f_1, f_2, \ldots \) are single-word integers.

\text{mode\_declare} is a synonym for \text{mode\_declare}.

If the type of function parameters and results doesn’t match the declaration by \text{mode\_declare} the function may misbehave or a warning or an error might occur, see \text{mode\_checkp}, \text{mode\_check\_errorp} and \text{mode\_check\_warnp}.

See \text{mode\_identity} for declaring the type of lists and \text{define\_variable} for declaring the type of all global variables compiled code uses, as well.

Example:

\[
\begin{align*}
(\%i1) \ & \ \text{square\_float}(f):= ( \\
& \quad \text{mode\_declare}(f, \text{float}), \\
& \quad f^*f) \\
(\%o1) \ & \ \text{square\_float}(f) := (\text{mode\_declare}(f, \text{float}), f \ f) \\
(\%i2) \ & \ \text{mode\_declare}([[\text{function}(f)], \text{float}]); \\
(\%o2) \ & \ [[\text{function}(f)]] \\
(\%i3) \ & \ \text{compile} \ (\text{square\_float}); \\
(\%o3) \ & \ [\text{square\_float}] \\
(\%i4) \ & \ \text{square\_float}(100.0); \\
(\%o4) \ & \ 10000.0
\end{align*}
\]

\text{mode\_checkp}

Default value: \text{true}

When \text{mode\_checkp} is \text{true}, \text{mode\_declare} does not only define which type a variable will be of so the compiler can generate more efficient code, but will also create a runtime warning if the variable isn’t of the variable type the code was compiled to deal with.

\[
\begin{align*}
(\%i1) \ & \ \text{mode\_checkp}:\text{true}; \\
(\%o1) \ & \ \text{true} \\
(\%i2) \ & \ \text{square}(f):= ( \\
& \quad \text{mode\_declare}(f, \text{float}), \\
& \quad f^\text{\ caret}2); \\
(\%o2) \ & \ \text{square}(f) := (\text{mode\_declare}(f, \text{float}), f )
\end{align*}
\]
\[(\%i3)\] \(\text{compile(square);}\)

\[(\%o3)\] \([	ext{square}]\)

\[(\%i4)\] \(\text{square(2.3);}\)

\[(\%o4)\] \(5.289999999999999\)

\[(\%i5)\] \(\text{square(4);}\)

Maxima encountered a Lisp error:

The value

4

is not of type

\text{DOUBLE-FLOAT}

when binding $F$

Automatically continuing.
To enable the Lisp debugger set *debugger-hook* to nil.

\text{mode\textunderscore check\textunderscore errorp} \quad \text{[Option variable]}

Default value: false

When \text{mode\textunderscore check\textunderscore errorp} is true, \text{mode\textunderscore declare} calls error.

\text{mode\textunderscore check\textunderscore warnp} \quad \text{[Option variable]}

Default value: true

When \text{mode\textunderscore check\textunderscore warnp} is true, mode errors are described.

\text{mode\textunderscore identity} (arg\_1, arg\_2) \quad \text{[Function]}

\text{mode\textunderscore identity} works similar to \text{mode\textunderscore declare}, but is used for informing the compiler that a thing like a \text{macro} or a list operation will only return a specific type of object. The purpose of doing so is that maxima supports many objects: Machine integers, arbitrary length integers, equations, machine floats, big floats, which means that for everything that deals with return values of operations that can result in any object the compiler needs to output generic (and therefore potentially slow) code.

The first argument to \text{mode\textunderscore identity} is the type of return value something will return (for possible types see \text{mode\textunderscore declare}). (i.e., one of \text{float}, \text{fixnum}, \text{number}, \text{The second argument is the expression that will return an object of this type.}

If the the return value of this expression is of a type the code was not compiled for error or warning is signalled.

If you knew that \text{first} (1) returned a number then you could write

\text{mode\textunderscore identity} (number, first (1)).

However, if you need this construct more often it would be more efficient to define a function that returns a number fist:

\text{firstnumb} (x) ::= \text{buildq} ([x], \text{mode\textunderscore identity} (number, first(x)));

\text{compile(firstnumb)}

\text{firstnumb} now can be used every time you need the first element of a list that is guaranteed to be filled with numbers.
remfunction

remfunction(f_1, \ldots, f_n)
remfunction(all)

Unbinds the function definitions of the symbols \( f_1, \ldots, f_n \). The arguments may be the names of ordinary functions (created by \(::=\) or \texttt{define}) or macro functions (created by \( ::=\)).

\texttt{remfunction (all)} unbinds all function definitions.

\texttt{remfunction} quotes its arguments.

\texttt{remfunction} returns a list of the symbols for which the function definition was unbound. \texttt{false} is returned in place of any symbol for which there is no function definition.

\texttt{remfunction} does not apply to \texttt{memoizing functions} or subscripted functions. \texttt{remarray} applies to those types of functions.

\texttt{savedef}

\texttt{.savedef\]}

\texttt{Default value: true}

When \texttt{savedef} is \texttt{true}, the Maxima version of a user function is preserved when the function is translated. This permits the definition to be displayed by \texttt{dispfun} and allows the function to be edited.

When \texttt{savedef} is \texttt{false}, the names of translated functions are removed from the \texttt{functions} list.

\texttt{translate}

\texttt{translate(f_1, \ldots, f_n)}
\texttt{translate(functions)}
\texttt{translate(all)}

Translates the user-defined functions \( f_1, \ldots, f_n \) from the Maxima language into Lisp and evaluates the Lisp translations. Typically the translated functions run faster than the originals.

\texttt{translate (all)} or \texttt{translate (functions)} translates all user-defined functions.

Functions to be translated should include a call to \texttt{mode\_declare} at the beginning when possible in order to produce more efficient code. For example:

\[
f(x_1, x_2, \ldots) := \text{block}([v_1, v_2, \ldots],\)
\[
\quad \text{mode\_declare}(v_1, \text{mode\_1}, v_2, \text{mode\_2}, \ldots), \ldots)
\]

where the \( x_1, x_2, \ldots \) are the parameters to the function and the \( v_1, v_2, \ldots \) are the local variables.

The names of translated functions are removed from the \texttt{functions} list if \texttt{savedef} is \texttt{false} (see below) and are added to the \texttt{props} lists.

Functions should not be translated unless they are fully debugged.

Expressions are assumed simplified; if they are not, correct but non-optimal code gets generated. Thus, the user should not set the \texttt{simp} switch to \texttt{false} which inhibits simplification of the expressions to be translated.

The switch \texttt{translate}, if \texttt{true}, causes automatic translation of a user’s function to Lisp.
Note that translated functions may not run identically to the way they did before translation as certain incompatibilities may exist between the Lisp and Maxima versions. Principally, the \texttt{rat} function with more than one argument and the \texttt{ratvars} function should not be used if any variables are \texttt{mode\_declare}'d canonical rational expressions (CRE). Also the \texttt{prederror: false} setting will not translate.

\texttt{savedef} - if \texttt{true} will cause the Maxima version of a user function to remain when the function is \texttt{translate}'d. This permits the definition to be displayed by \texttt{dispfun} and allows the function to be edited.

\texttt{transrun} - if \textbf{false} will cause the interpreted version of all functions to be run (provided they are still around) rather than the translated version.

The result returned by \texttt{translate} is a list of the names of the functions translated.

\begin{verbatim}
translate_file
translate_file (maxima_filename)
translate_file (maxima_filename, lisp_filename)
\end{verbatim}

Translates a file of Maxima code into a file of Lisp code. \texttt{translate_file} returns a list of three filenames: the name of the Maxima file, the name of the Lisp file, and the name of file containing additional information about the translation. \texttt{translate_file} evaluates its arguments.

\texttt{translate_file ("foo.mac"); load("foo.LISP")} is the same as the command \texttt{batch ("foo.mac")}, except for certain restrictions, the use of ' ' and \%, for example. \texttt{translate_file (maxima_filename)} translates a Maxima file \texttt{maxima\_filename} into a similarly-named Lisp file. For example, \texttt{foo.mac} is translated into \texttt{foo.LISP}. The Maxima filename may include a directory name or names, in which case the Lisp output file is written to the same directory from which the Maxima input comes.

\texttt{translate_file (maxima_filename, lisp_filename)} translates a Maxima file \texttt{maxima\_filename} into a Lisp file \texttt{lisp\_filename}. \texttt{translate_file} ignores the filename extension, if any, of \texttt{lisp\_filename}; the filename extension of the Lisp output file is always LISP. The Lisp filename may include a directory name or names, in which case the Lisp output file is written to the specified directory.

\texttt{translate_file} also writes a file of translator warning messages of various degrees of severity. The filename extension of this file is UNLISP. This file may contain valuable information, though possibly obscure, for tracking down bugs in translated code. The UNLISP file is always written to the same directory from which the Maxima input comes.

\texttt{translate_file} emits Lisp code which causes some declarations and definitions to take effect as soon as the Lisp code is compiled. See \texttt{compile\_file} for more on this topic.

See also
\begin{verbatim}
tr_array_as_ref
(undefined) tr_bound_function_applyp,
tr_exponent
(undefined) tr_file_tty_messagesp,
(undefined) tr_float_can_branch_complex,
(undefined) tr_function_call_default,
\end{verbatim}
(undefined) tr_numer,
(undefined) tr_optimize_max_loop,
(undefined) tr_state_vars,
(undefined) tr_warnings_get,
tr_warn_bad_function_calls
(undefined) tr_warn_fexpr,
(undefined) tr_warn_meval,
(undefined) tr_warn_mode,
(undefined) tr_warn_undeclared,
and (undefined) tr_warn_undefined_variable.

transrun [Option variable]
Default value: true

When transrun is false will cause the interpreted version of all functions to be run (provided they are still around) rather than the translated version.

tr_array_as_ref [Option variable]
Default value: true

If translate_fast_arrays is false, array references in Lisp code emitted by translate_file are affected by tr_array_as_ref. When tr_array_as_ref is true, array names are evaluated, otherwise array names appear as literal symbols in translated code.

tr_array_as_ref has no effect if translate_fast_arrays is true.

tr_bound_function_applyp [Option variable]
Default value: true

When tr_bound_function_applyp is true and tr_function_call_default is general, if a bound variable (such as a function argument) is found being used as a function then Maxima will rewrite that function call using apply and print a warning message.

For example, if \( g \) is defined by \( g(f,x) := f(x+1) \) then translating \( g \) will cause Maxima to print a warning and rewrite \( f(x+1) \) as \( \text{apply}(f,[x+1]) \).

\[
\begin{align*}
(\%i1) & \quad f(x) := x^2$ \\
(\%i2) & \quad g(f) := f(3)$ \\
(\%i3) & \quad \text{tr_bound_function_applyp : true}$ \\
(\%i4) & \quad \text{translate (g)}$ \\
\text{warning: f is a bound variable in f(3), but it is used as a function.} \\
\text{note: instead I'll translate it as: apply(f,[3])}$ \\
(\%i5) & \quad g (\text{lambda} ([x], x)); \\
(\%o5) & \quad 3 \\
(\%i6) & \quad \text{tr_bound_function_applyp : false}$ \\
(\%i7) & \quad \text{translate (g)}$ \\
(\%i8) & \quad g (\text{lambda} ([x], x)); \\
(\%o8) & \quad 9
\end{align*}
\]

tr_file_tty_messagesp [Option variable]
Default value: false
When \texttt{tr\_file\_tty\_messagesp} is \texttt{true}, messages generated by \texttt{translate\_file} during translation of a file are displayed on the console and inserted into the UNLISP file. When \texttt{false}, messages about translation of the file are only inserted into the UNLISP file.

\texttt{tr\_float\_can\_branch\_complex} \hspace{1cm} \textbf{[Option variable]}
\begin{quote}
Default value: \texttt{true}
\end{quote}
Tells the Maxima-to-Lisp translator to assume that the functions \texttt{acos}, \texttt{asin}, \texttt{asec}, \texttt{acsc}, \texttt{acosh}, \texttt{asech}, \texttt{atanh}, \texttt{acoth}, \texttt{log} and \texttt{sqrt} can return complex results.
When it is \texttt{true} then \texttt{acos(x)} is of mode \texttt{any} even if \texttt{x} is of mode \texttt{float} (as set by \texttt{mode\_declare}). When \texttt{false} then \texttt{acos(x)} is of mode \texttt{float} if and only if \texttt{x} is of mode \texttt{float}.

\texttt{tr\_function\_call\_default} \hspace{1cm} \textbf{[Option variable]}
\begin{quote}
Default value: \texttt{general}
\end{quote}
\texttt{false} means give up and call \texttt{meval}, \texttt{expr} means assume Lisp fixed arg function. \texttt{general}, the default gives code good for \texttt{mexprs} and \texttt{mlexprs} but not \texttt{macros}. \texttt{general} assures variable bindings are correct in compiled code. In \texttt{general} mode, when translating \texttt{F(X)}, if \texttt{F} is a bound variable, then it assumes that \texttt{apply (f, [x])} is meant, and translates a such, with appropriate warning. There is no need to turn this off. With the default settings, no warning messages implies full compatibility of translated and compiled code with the Maxima interpreter.

\texttt{tr\_numer} \hspace{1cm} \textbf{[Option variable]}
\begin{quote}
Default value: \texttt{false}
\end{quote}
When \texttt{tr\_numer} is \texttt{true}, \texttt{numer} properties are used for atoms which have them, e.g. \texttt{\%pi}.

\texttt{tr\_optimize\_max\_loop} \hspace{1cm} \textbf{[Option variable]}
\begin{quote}
Default value: \texttt{100}
\end{quote}
\texttt{tr\_optimize\_max\_loop} is the maximum number of times the macro-expansion and optimization pass of the translator will loop in considering a form. This is to catch macro expansion errors, and non-terminating optimization properties.

\texttt{tr\_state\_vars} \hspace{1cm} \textbf{[System variable]}
\begin{quote}
Default value:

\[
[\texttt{translate\_fast\_arrays, tr\_function\_call\_default, tr\_bound\_function\_applyp,}
\texttt{tr\_array\_as\_ref, tr\_numer, tr\_float\_can\_branch\_complex, define\_variable}]
\]
\end{quote}
The list of the switches that affect the form of the translated output. This information is useful to system people when trying to debug the translator. By comparing the translated product to what should have been produced for a given state, it is possible to track down bugs.

\texttt{tr\_warnings\_get ()} \hspace{1cm} \textbf{[Function]}
Prints a list of warnings which have been given by the translator during the current translation.
**tr_warn_bad_function_calls**

Default value: `true`
- Gives a warning when function calls are being made which may not be correct due to improper declarations that were made at translate time.

**tr_warn_fexpr**

Default value: `compfile`
- Gives a warning if any FEXPRs are encountered. FEXPRs should not normally be output in translated code, all legitimate special program forms are translated.

**tr_warn_meval**

Default value: `compfile`
- Gives a warning if the function `meval` gets called. If `meval` is called that indicates problems in the translation.

**tr_warn_mode**

Default value: `all`
- Gives a warning when variables are assigned values inappropriate for their mode.

**tr_warn_undeclared**

Default value: `compile`
- Determines when to send warnings about undeclared variables to the TTY.

**tr_warn_undefined_variable**

Default value: `all`
- Gives a warning when undefined global variables are seen.

**compile_file**

- `compile_file(filename)`
- `compile_file(filename, compiled_filename)`
- `compile_file(filename, compiled_filename, lisp_filename)`

Translates the Maxima file `filename` into Lisp, and executes the Lisp compiler. The compiled code is not loaded into Maxima.

`compile_file` returns a list of the names of four files: the original Maxima file, the Lisp translation, notes on translation, and the compiled code. If the compilation fails, the fourth item is `false`.

Some declarations and definitions take effect as soon as the Lisp code is compiled (without loading the compiled code). These include functions defined with the `:=` operator, macros define with the `::=` operator, `alias`, `declare`, `define_variable`, `mode_declare`, and `infix`, `matchfix`, `nofix`, `postfix`, `prefix`, and `compfile`.

Assignments and function calls are not evaluated until the compiled code is loaded. In particular, within the Maxima file, assignments to the translation flags (`tr_numer`, etc.) have no effect on the translation.

`filename` may not contain `:lisp` statements.

`compile_file` evaluates its arguments.
**declare_translated** (*f_1, f_2, ...*)  
When translating a file of Maxima code to Lisp, it is important for the translator to know which functions it sees in the file are to be called as translated or compiled functions, and which ones are just Maxima functions or undefined. Putting this declaration at the top of the file, lets it know that although a symbol does which does not yet have a Lisp function value, will have one at call time.  

```lisp
(MFUNCTION-CALL fn arg1 arg2 ...) is generated when the translator does not know fn is going to be a Lisp function.
```
37 Program Flow

37.1 Lisp and Maxima

Maxima is a fairly complete programming language. But since it is written in Lisp, it additionally can provide easy access to Lisp functions and variables from Maxima and vice versa. Lisp and Maxima symbols are distinguished by a naming convention. A Lisp symbol which begins with a dollar sign \$ corresponds to a Maxima symbol without the dollar sign.

A Maxima symbol which begins with a question mark ? corresponds to a Lisp symbol without the question mark. For example, the Maxima symbol foo corresponds to the Lisp symbol $FOO$, while the Maxima symbol ?foo corresponds to the Lisp symbol FOO. Note that ?foo is written without a space between ? and foo; otherwise it might be mistaken for describe ("foo").

Hyphen -, asterisk *, or other special characters in Lisp symbols must be escaped by backslash \ where they appear in Maxima code. For example, the Lisp identifier *foo-bar* is written ?\*foo\-bar\* in Maxima.

Lisp code may be executed from within a Maxima session. A single line of Lisp (containing one or more forms) may be executed by the special command :lisp. For example,

\[
(\%i1) \text{:lisp (foo } x \text{ y)}
\]
calls the Lisp function foo with Maxima variables x and y as arguments. The :lisp construct can appear at the interactive prompt or in a file processed by batch or demo, but not in a file processed by load, batchload, translate_file, or compile_file.

The function to_lisp opens an interactive Lisp session. Entering (to-maxima) closes the Lisp session and returns to Maxima.

Lisp functions and variables which are to be visible in Maxima as functions and variables with ordinary names (no special punctuation) must have Lisp names beginning with the dollar sign \$.

Maxima is case-sensitive, distinguishing between lowercase and uppercase letters in identifiers. There are some rules governing the translation of names between Lisp and Maxima.

1. A Lisp identifier not enclosed in vertical bars corresponds to a Maxima identifier in lowercase. Whether the Lisp identifier is uppercase, lowercase, or mixed case, is ignored. E.g., Lisp $foo$, $FOO$, and $Foo$ all correspond to Maxima foo. But this is because $foo$, $FOO$ and $Foo$ are converted by the Lisp reader by default to the Lisp symbol $FOO$.

2. A Lisp identifier which is all uppercase or all lowercase and enclosed in vertical bars corresponds to a Maxima identifier with case reversed. That is, uppercase is changed to lowercase and lowercase to uppercase. E.g., Lisp |$FOO| and |$foo| correspond to Maxima foo and FOO, respectively.

3. A Lisp identifier which is mixed uppercase and lowercase and enclosed in vertical bars corresponds to a Maxima identifier with the same case. E.g., Lisp |$Foo| corresponds to Maxima Foo.

The #$Lisp macro allows the use of Maxima expressions in Lisp code. #$expr$ expands to a Lisp expression equivalent to the Maxima expression expr.

\[
\text{(msetq } $foo \text{ #$[x, y]}$)
\]
This has the same effect as entering

(%i1) foo: [x, y];

The Lisp function \texttt{displa} prints an expression in Maxima format.

(%i1) :lisp #\$[x, y, z]\$
((MLIST SIMP) $X$ $Y$ $Z$)
(%i1) :lisp (displa '((MLIST SIMP) $X$ $Y$ $Z))$
[x, y, z]
NIL

Functions defined in Maxima are not ordinary Lisp functions. The Lisp function \texttt{mfuncall} calls a Maxima function. For example:

(%i1) foo(x,y) := x*y$
(%i2) :lisp (mfuncall '$foo 'a 'b)
((MTIMES SIMP) A B)

Some Lisp functions are shadowed in the Maxima package, namely the following.

- \texttt{complement}
- \texttt{continue} //
- \texttt{float}
- \texttt{functionp}
- \texttt{array}
- \texttt{exp}
- \texttt{listen}
- \texttt{signum}
- \texttt{atan}
- \texttt{asin}
- \texttt{acos}
- \texttt{asinh}
- \texttt{acosh}
- \texttt{atanh}
- \texttt{tanh}
- \texttt{cosh}
- \texttt{sinh}
- \texttt{tan}
- \texttt{break}
- \texttt{gcd}

### 37.2 Garbage Collection

One of the advantages of using \texttt{lisp} is that it uses “Garbage Collection”. In other words it automatically takes care of freeing memory occupied for example of intermediate results that were used during symbolic computation.

Garbage Collection avoids many errors frequently found in \texttt{C} programs (memory being freed too early, multiple times or not at all).

\texttt{garbage\_collect} ()

\texttt{garbage\_collect} () \hspace{1cm} \textbf{[Function]}

Tries to manually trigger the \texttt{lisp}'s garbage collection. This rarely is necessary as the \texttt{lisp} will employ an excellent algorithm for determining when to start garbage collection.

If \texttt{maxima} knows how to do manually trigger the garbage collection for the current \texttt{lisp} \texttt{garbage\_collect} returns \texttt{true}, else \texttt{false}.

### 37.3 Introduction to Program Flow

Maxima provides a \texttt{do} loop for iteration, as well as more primitive constructs such as \texttt{go}.
37.4 Functions and Variables for Program Flow

backtrace [Function]

backtrace ()
backtrace (n)

Prints the call stack, that is, the list of functions which called the currently active function.

backtrace () prints the entire call stack.
backtrace (n) prints the n most recent functions, including the currently active function.
backtrace can be called from a script, a function, or the interactive prompt (not only in a debugging context).

Examples:
• backtrace () prints the entire call stack.

(%%i1) h(x) := g(x/7)$
(%%i2) g(x) := f(x-11)$
(%%i3) f(x) := e(x^-2)$
(%%i4) e(x) := (backtrace(), 2*x + 13)$
(%%i5) h(10);
#0: e(x=4489/49)
#1: f(x=-67/7)
#2: g(x=10/7)
#3: h(x=10)
         9615
(%%o5)     ----
             49

• backtrace (n) prints the n most recent functions, including the currently active function.

(%%i1) h(x) := (backtrace(1), g(x/7))$
(%%i2) g(x) := (backtrace(1), f(x-11))$
(%%i3) f(x) := (backtrace(1), e(x^-2))$
(%%i4) e(x) := (backtrace(1), 2*x + 13)$
(%%i5) h(10);
#0: h(x=10)
#0: g(x=10/7)
#0: f(x=-67/7)
#0: e(x=4489/49)
        9615
(%%o5)    ----
        49

do [Special operator]
while [Special operator]
unless [Special operator]
for [Special operator]
from [Special operator]
The `do` statement is used for performing iteration. The general form of the `do` statements Maxima supports is:

- for variable: initial\_value step increment thru limit do body
- for variable: initial\_value step increment while condition do body
- for variable: initial\_value step increment unless condition do body
- for variable in list do body

If the loop is expected to generate a list as output the command `makelist` may be the appropriate command to use instead, See Section 5.4.3 [Performance considerations for Lists], page 74.

`initial\_value`, `increment`, `limit`, and `body` can be any expression. `list` is a list. If the increment is 1 then "step 1" may be omitted; As always, if `body` needs to contain more than one command these commands can be specified as a comma-separated list surrounded by parenthesis or as a `block`. Due to its great generality the `do` statement will be described in two parts. The first form of the `do` statement (which is shown in the first three items above) is analogous to that used in several other programming languages (Fortran, Algol, PL/I, etc.); then the other features will be mentioned.

The execution of the `do` statement proceeds by first assigning the `initial\_value` to the variable (henceforth called the control-variable). Then: (1) If the control-variable has exceeded the limit of a `thru` specification, or if the condition of the `unless` is `true`, or if the condition of the `while` is `false` then the `do` terminates. (2) The `body` is evaluated. (3) The increment is added to the control-variable. The process from (1) to (3) is performed repeatedly until the termination condition is satisfied. One may also give several termination conditions in which case the `do` terminates when any of them is satisfied.

In general the `thru` test is satisfied when the control-variable is greater than the `limit` if the `increment` was non-negative, or when the control-variable is less than the `limit` if the `increment` was negative. The `increment` and `limit` may be non-numeric expressions as long as this inequality can be determined. However, unless the `increment` is syntactically negative (e.g. is a negative number) at the time the `do` statement is input, Maxima assumes it will be positive when the `do` is executed. If it is not positive, then the `do` may not terminate properly.

Note that the `limit`, `increment`, and termination condition are evaluated each time through the loop. Thus if any of these involve much computation, and yield a result that does not change during all the executions of the `body`, then it is more efficient to set a variable to their value prior to the `do` and use this variable in the `do` form.

The value normally returned by a `do` statement is the atom `done`. However, the function `return` may be used inside the `body` to exit the `do` prematurely and give it any desired value. Note however that a `return` within a `do` that occurs in a `block` will exit only the `do` and not the `block`. Note also that the `go` function may not be used to exit from a `do` into a surrounding `block`. 
The control-variable is always local to the `do` and thus any variable may be used without affecting the value of a variable with the same name outside of the `do`. The control-variable is unbound after the `do` terminates.

```lisp
(%i1) for a:-3 thru 26 step 7 do display(a)$
   a = - 3
   a = 4
   a = 11
   a = 18
   a = 25

(%i1) s: 0$
(%i2) for i: 1 while i <= 10 do s: s+i;
   done
(%i3) s;
   (%o3) 55
```

Note that the condition `while i <= 10` is equivalent to `unless i > 10` and also `thru 10`.

```lisp
(%i1) series: 1$
(%i2) term: exp (sin (x))$
(%i3) for p: 1 unless p > 7 do
   (term: diff (term, x)/p,
    series: series + subst (x=0, term)*x^p)$
(%i4) series;
   7   6   5   4   2
   x   x   x   x   x
   -- - ---- - --- - -- + -- + x + 1
   90 240 15  8  2
```

which gives 8 terms of the Taylor series for $e^{\sin(x)}$.

```lisp
(%i1) poly: 0$
(%i2) for i: 1 thru 5 do
   for j: i step -1 thru 1 do
      poly: poly + i*x^j$
(%i3) poly;
   5   4   3   2
   5 x + 9 x + 12 x + 14 x + 15 x
(%i4) guess: -3.0$
(%i5) for i: 1 thru 10 do
   (guess: subst (guess, x, 0.5*(x + 10/x)),
    if abs (guess^2 - 10) < 0.00005 then return (guess));
(%o5) - 3.162280701754386
```
This example computes the negative square root of 10 using the Newton-Raphson iteration a maximum of 10 times. Had the convergence criterion not been met the value returned would have been done.

Instead of always adding a quantity to the control-variable one may sometimes wish to change it in some other way for each iteration. In this case one may use next expression instead of step increment. This will cause the control-variable to be set to the result of evaluating expression each time through the loop.

\[
\text{(%i6) for count: 2 next 3*count thru 20 do display (count)}$
\]

\[
\begin{align*}
\text{count} & = 2 \\
\text{count} & = 6 \\
\text{count} & = 18
\end{align*}
\]

As an alternative to for variable: value ...do... the syntax for variable from value ...do... may be used. This permits the from value to be placed after the step or next value or after the termination condition. If from value is omitted then 1 is used as the initial value.

Sometimes one may be interested in performing an iteration where the control-variable is never actually used. It is thus permissible to give only the termination conditions omitting the initialization and updating information as in the following example to compute the square-root of 5 using a poor initial guess.

\[
\text{(%i11) x: 1000}$
\]

\[
\text{(%i12) thru 20 do x: 0.5*(x + 5.0/x)}$
\]

\[
\text{(%i13) x;$(%i14) sqrt(5), numer;}
\]

\[
\begin{align*}
\text{(%o3) 2.23606797749979} \\
\text{(%o4) 2.23606797749979}
\end{align*}
\]

If it is desired one may even omit the termination conditions entirely and just give do body which will continue to evaluate the body indefinitely. In this case the function return should be used to terminate execution of the do.

\[
\text{(%i11) newton (f, x):= ([y, df, dfx], df: diff (f ('x), 'x),}$
\]

\[
\text{do (y: ev(df), x: x - f(x)/y,}$
\]

\[
\text{if abs (f (x)) < 5e-6 then return (x)))}}$
\]

\[
\text{(%i12) sqr (x) := x^2 - 5.0}$
\]

\[
\text{(%i13) newton (sqr, 1000);$(%i14) sqrt(5), numer;}
\]

\[
\begin{align*}
\text{(%o3) 2.236068027062195}
\end{align*}
\]

(Note that return, when executed, causes the current value of x to be returned as the value of the do. The block is exited and this value of the do is returned as the value of the block because the do is the last statement in the block.)

One other form of the do is available in Maxima. The syntax is:

\[
\text{for variable in list end_tests do body}$
\]

The elements of list are any expressions which will successively be assigned to the variable on each iteration of the body. The optional termination tests end_tests can be used to terminate execution of the do; otherwise it will terminate when the
list is exhausted or when a `return` is executed in the body. (In fact, list may be any non-atomic expression, and successive parts are taken.)

```maxima
(%i1) for f in [log, rho, atan] do ldisp(f(1))$
(%t1) 0
(%t2) rho(1) %pi
(%t3) --- 4

(%i4) ev(%t3,numer);
(%o4) 0.78539816
```

**errcatch(expr_1, ..., expr_n)**  [Function]

Evaluates `expr_1`, ..., `expr_n` one by one and returns `[expr_n]` (a list) if no error occurs. If an error occurs in the evaluation of any argument, `errcatch` prevents the error from propagating and returns the empty list `[]` without evaluating any more arguments.

`errcatch` is useful in batch files where one suspects an error might occur which would terminate the batch if the error weren’t caught.

See also `errormsg`.

**error(expr_1, ..., expr_n)**  [Function]

Evaluates and prints `expr_1`, ..., `expr_n`, and then causes an error return to top level Maxima or to the nearest enclosing `errcatch`.

The variable `error` is set to a list describing the error. The first element of `error` is a format string, which merges all the strings among the arguments `expr_1`, ..., `expr_n`, and the remaining elements are the values of any non-string arguments.

`errormsg()` formats and prints `error`. This is effectively reprinting the most recent error message.

**warning(expr_1, ..., expr_n)**  [Function]

Evaluates and prints `expr_1`, ..., `expr_n`, as a warning message that is formatted in a standard way so a maxima front-end may be able to recognize the warning and to format it accordingly.

The function `warning` always returns false.

**error_size**  [Option variable]

Default value: 60

`error_size` modifies error messages according to the size of expressions which appear in them. If the size of an expression (as determined by the Lisp function `ERROR-SIZE`) is greater than `error_size`, the expression is replaced in the message by a symbol, and the symbol is assigned the expression. The symbols are taken from the list `error_syms`.

Otherwise, the expression is smaller than `error_size`, and the expression is displayed in the message.

See also `error` and `error_syms`.
Example:
The size of \( U \), as determined by \texttt{ERROR-SIZE}, is 24.

\begin{verbatim}
(%i1) U: (C^D^E + B + A)/(cos(X-1) + 1)$
(%i2) error_size: 20$
(%i3) error ("Example expression is", U);
Example expression is errexp1
-- an error. Quitting. To debug this try debugmode(true);
(%i4) errexp1;
\end{verbatim}

\begin{verbatim}
E
D
C + B + A
--------------
\cos(X - 1) + 1
\end{verbatim}

\begin{verbatim}
(%o4) E
D
C + B + A
--------------
\cos(X - 1) + 1

(%i5) error_size: 30$
(%i6) error ("Example expression is", U);
Example expression is errexp1
-- an error. Quitting. To debug this try debugmode(true);
\end{verbatim}

\begin{verbatim}
E
D
C + B + A
--------------
\cos(X - 1) + 1
\end{verbatim}

\begin{verbatim}
(%o6) E
D
C + B + A
--------------
\cos(X - 1) + 1
\end{verbatim}

\begin{verbatim}
\end{verbatim}

\textbf{error\_syms} \hspace{1cm} \textbf{[Option variable]}

Default value: \textbf{[errexp1, errexp2, errexp3]}

In error messages, expressions larger than \texttt{error\_size} are replaced by symbols, and the symbols are set to the expressions. The symbols are taken from the list \texttt{error\_syms}. The first too-large expression is replaced by \texttt{error\_syms[1]}, the second by \texttt{error\_syms[2]}, and so on.

If there are more too-large expressions than there are elements of \texttt{error\_syms}, symbols are constructed automatically, with the \( n \)-th symbol equivalent to \texttt{concat (\"errexp, \( n \).}.

See also \texttt{error} and \texttt{error\_size}.

\textbf{errormsg ( )} \hspace{1cm} \textbf{[Function]}

Reprints the most recent error message. The variable \texttt{error} holds the message, and \texttt{errormsg} formats and prints it.

\textbf{errormsg} \hspace{1cm} \textbf{[Option variable]}

Default value: \texttt{true}

When \texttt{false} the output of error messages is suppressed.
The option variable `errormsg` can not be set in a block to a local value. The global value of `errormsg` is always present.

```
(%i1) errormsg;
(%o1) true
(%i2) sin(a,b);
sin: wrong number of arguments.
-- an error. To debug this try: debugmode(true);
(%i3) errormsg: false;
(%o3) false
(%i4) sin(a,b);
-- an error. To debug this try: debugmode(true);
```

The option variable `errormsg` can not be set in a block to a local value.

```
(%i1) f(bool):=block([errormsg:bool],
   print ("value of errormsg is",errormsg))$
(%i2) errormsg:true;
(%o2) true
(%i3) f(false);
value of errormsg is true
(%o3) true
(%i4) errormsg:false;
(%o4) false
(%i5) f(true);
value of errormsg is false
(%o5) false
```

\[ \text{go (tag)} \] [Function]

is used within a block to transfer control to the statement of the block which is tagged with the argument to go. To tag a statement, precede it by an atomic argument as another statement in the block. For example:

```
block ([x], x:1, loop, x+1, ..., go(loop), ...)
```

The argument to go must be the name of a tag appearing in the same block. One cannot use go to transfer to tag in a block other than the one containing the go.

\[ \text{if} \] [Special operator]

Represents conditional evaluation. Various forms of if expressions are recognized.

\[ \text{if cond}_1 \text{ then } \text{expr}_1 \text{ else } \text{expr}_0 \text{ evaluates to } \text{expr}_1 \text{ if } \text{cond}_1 \text{ evaluates to true, otherwise the expression evaluates to } \text{expr}_0. \]

The command \[ \text{if cond}_1 \text{ then } \text{expr}_1 \text{ elseif } \text{cond}_2 \text{ then } \text{expr}_2 \text{ elseif } \ldots \text{ else } \text{expr}_0 \text{ evaluates to } \text{expr}_k \text{ if } \text{cond}_k \text{ is true and all preceding conditions are false.} \]

If none of the conditions are true, the expression evaluates to \text{expr}_0.\]

A trailing else false is assumed if else is missing. That is, the command \[ \text{if cond}_1 \text{ then } \text{expr}_1 \text{ is equivalent to } \text{if cond}_1 \text{ then } \text{expr}_1 \text{ else false,} \]

and the command \[ \text{if cond}_1 \text{ then } \text{expr}_1 \text{ elseif } \ldots \text{ elseif } \text{cond}_n \text{ then } \text{expr}_n \text{ is equivalent to } \text{if cond}_1 \text{ then } \text{expr}_1 \text{ elseif } \ldots \text{ elseif } \text{cond}_n \text{ then } \text{expr}_n \text{ else false.} \]
The alternatives \( expr_0, \ldots, expr_n \) may be any Maxima expressions, including nested \texttt{if} expressions. The alternatives are neither simplified nor evaluated unless the corresponding condition is \texttt{true}.

The conditions \( cond_1, \ldots, cond_n \) are expressions which potentially or actually evaluate to \texttt{true} or \texttt{false}. When a condition does not actually evaluate to \texttt{true} or \texttt{false}, the behavior of \texttt{if} is governed by the global flag \texttt{prederror}. When \texttt{prederror} is \texttt{true}, it is an error if any evaluated condition does not evaluate to \texttt{true} or \texttt{false}. Otherwise, conditions which do not evaluate to \texttt{true} or \texttt{false} are accepted, and the result is a conditional expression.

Among other elements, conditions may comprise relational and logical operators as follows.

<table>
<thead>
<tr>
<th>Operation</th>
<th>Symbol</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>less than</td>
<td>&lt;</td>
<td>relational infix</td>
</tr>
<tr>
<td>less than or equal to</td>
<td>&lt;=</td>
<td>relational infix</td>
</tr>
<tr>
<td>equality (syntactic)</td>
<td>=</td>
<td>relational infix</td>
</tr>
<tr>
<td>equality (value)</td>
<td>equal</td>
<td>relational function</td>
</tr>
<tr>
<td>negation of equal</td>
<td>notequal</td>
<td>relational function</td>
</tr>
<tr>
<td>greater than or equal to</td>
<td>&gt;=</td>
<td>relational infix</td>
</tr>
<tr>
<td>greater than</td>
<td>&gt;</td>
<td>relational infix</td>
</tr>
<tr>
<td>and</td>
<td>and</td>
<td>logical infix</td>
</tr>
<tr>
<td>or</td>
<td>or</td>
<td>logical infix</td>
</tr>
<tr>
<td>not</td>
<td>not</td>
<td>logical infix</td>
</tr>
</tbody>
</table>

\( \texttt{map(f, expr_1, \ldots, expr_n)} \) \hspace{1cm} \text{[Function]}\n
Returns an expression whose leading operator is the same as that of the expressions \( expr_1, \ldots, expr_n \) but whose subparts are the results of applying \( f \) to the corresponding subparts of the expressions. \( f \) is either the name of a function of \( n \) arguments or is a \texttt{lambda} form of \( n \) arguments.

\texttt{maperror} - if \texttt{false} will cause all of the mapping functions to (1) stop when they finish going down the shortest \( expr_i \) if not all of the \( expr_i \) are of the same length and (2) apply \( f \) to \([expr_1, expr_2, \ldots]\) if the \( expr_i \) are not all the same type of object. If \texttt{maperror} is \texttt{true} then an error message will be given in the above two instances.

One of the uses of this function is to \texttt{map} a function (e.g. \texttt{partfrac}) onto each term of a very large expression where it ordinarily wouldn’t be possible to use the function on the entire expression due to an exhaustion of list storage space in the course of the computation.

See also \texttt{scanmap}, \texttt{maplist}, \texttt{outermap}, \texttt{matrixmap} and \texttt{apply}.

\begin{verbatim}
(%i1) map(f,x+a*y+b*z);
(%o1) f(b z) + f(a y) + f(x)
(%i2) map(lambda([u],partfrac(u,x)),x+1/(x^3+4*x^2+5*x+2));
1 1 1
---- - ------- + -------- + x
x + 2 x + 1 2
\end{verbatim}
(%i3) map(ratsimp, x/(x^2+x)+(y^2+y)/y);
 1
(%o3) y + ----- + 1
     x + 1
(%i4) map("=",[a,b],[-0.5,3]);
(%o4) [a = - 0.5, b = 3]

mapatom (expr)

Returns true if and only if expr is treated by the mapping routines as an atom. "Mapatoms" are atoms, numbers (including rational numbers), subcripted variables and structure references.

maperror

Default value: true

When maperror is false, causes all of the mapping functions, for example

map (f, expr_1, expr_2, ...)

to (1) stop when they finish going down the shortest expr_i if not all of the expr_i are of the same length and (2) apply f to [expr_1, expr_2, ...] if the expr_i are not all the same type of object.

If maperror is true then an error message is displayed in the above two instances.

mapprint

Default value: true

When mapprint is true, various information messages from map, maplist, and fullmap are produced in certain situations. These include situations where map would use apply, or map is truncating on the shortest list.

If mapprint is false, these messages are suppressed.

maplist (f, expr_1, ..., expr_n)

Returns a list of the applications of f to the parts of the expressions expr_1, ..., expr_n. f is the name of a function, or a lambda expression.

maplist differs from map(f, expr_1, ..., expr_n) which returns an expression with the same main operator as expr_i has (except for simplifications and the case where map does an apply).

prederror

Default value: false

When prederror is true, an error message is displayed whenever the predicate of an if statement or an is function fails to evaluate to either true or false.

If false, unknown is returned instead in this case. The prederror: false mode is not supported in translated code; however, maybe is supported in translated code.

See also is and maybe.
return (value)  [Function]

May be used to exit explicitly from the current block, while, for or do loop bringing its argument. It therefore can be compared with the return statement found in other programming languages but it yields one difference: In maxima only returns from the current block, not from the entire function it was called in. In this aspect it more closely resembles the break statement from C.

(%i1) for i:1 thru 10 do o:i;
(%o1) done
(%i2) for i:1 thru 10 do if i=3 then return(i);
(%o2) 3
(%i3) for i:1 thru 10 do
   (block([i],
        i:3,
        return(i)
   ),
    return(8)
);
(%o3) 8
(%i4) block([i],
        i:4,
        block([o],
            o:5,
            return(o)
        ),
        return(i),
        return(10)
);  
(%o4) 4
See also for, while, do and block.

scanmap  [Function]

scanmap (f, expr)
scanmap (f, expr, bottomup)

Recursively applies f to expr, in a top down manner. This is most useful when complete factorization is desired, for example:

(%i1) exp:(a^2+2*a+1)*y + x^2$
(%i2) scanmap(factor,exp);  
2  2
   (a + 1) y + x

Note the way in which scanmap applies the given function factor to the constituent subexpressions of expr; if another form of expr is presented to scanmap then the result may be different. Thus, %o2 is not recovered when scanmap is applied to the expanded form of exp:

(%i3) scanmap(factor,expand(exp));  
2  2
Here is another example of the way in which `scanmap` recursively applies a given function to all subexpressions, including exponents:

```
(%i4) expr : u*v^(a*x+b) + c$
(%i5) scanmap('f, expr);
    f(f(f(a) f(x)) + f(b))
(%o5) f(f(f(u) f(f(v) )) + f(c))
```

`scanmap (f, expr, bottomup)` applies `f` to `expr` in a bottom-up manner. E.g., for undefined `f`,

```
scanmap(f,a*x+b) ->
    f(a*x+b) -> f(f(a*x)+f(b)) -> f(f(f(a)*f(x))+f(b))
scanmap(f,a*x+b,bottomup) -> f(a)*f(x)+f(b)
    -> f(f(f(a)*f(x))+f(b))
```

In this case, you get the same answer both ways.

\section*{Function throw (expr)}

Evaluates `expr` and throws the value back to the most recent `catch`. `throw` is used with `catch` as a nonlocal return mechanism.

\section*{Function outermap (f, a_1, \ldots, a_n)}

Applies the function `f` to each one of the elements of the outer product `a_1` cross `a_2` \ldots cross `a_n`.

`f` is the name of a function of `n` arguments or a lambda expression of `n` arguments. Each argument `a_k` may be a list or nested list, or a matrix, or any other kind of expression.

The `outermap` return value is a nested structure. Let `x` be the return value. Then `x` has the same structure as the first list, nested list, or matrix argument, `x[i_1]...[i_m]` has the same structure as the second list, nested list, or matrix argument, `x[i_1]...[i_m][j_1]...[j_n]` has the same structure as the third list, nested list, or matrix argument, and so on, where `m`, `n`, \ldots are the numbers of indices required to access the elements of each argument (one for a list, two for a matrix, one or more for a nested list). Arguments which are not lists or matrices have no effect on the structure of the return value.

Note that the effect of `outermap` is different from that of applying `f` to each one of the elements of the outer product returned by `cartesian_product`. `outermap` preserves the structure of the arguments in the return value, while `cartesian_product` does not.

`outermap` evaluates its arguments.

See also `map`, `maplist`, and `apply`.

Examples:

Elementary examples of `outermap`. To show the argument combinations more clearly, `F` is left undefined.

```
(%i1) outermap (F, [a, b, c], [1, 2, 3]);
(%o11) outermap (F, [a, b, c], [1, 2, 3]);
```

```
(\%o1) [[F(a, 1), F(a, 2), F(a, 3)], [F(b, 1), F(b, 2), F(b, 3)],
       [F(c, 1), F(c, 2), F(c, 3)]]
```
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(%i2) outermap (F, matrix ([a, b], [c, d]), matrix ([1, 2], [3, 4]));
   [ [ F(a, 1) F(a, 2) ] [ F(b, 1) F(b, 2) ] ]
   [ [ F(a, 3) F(a, 4) ] [ F(b, 3) F(b, 4) ] ]
   [ [ F(c, 1) F(c, 2) ] [ F(d, 1) F(d, 2) ] ]
   [ [ F(c, 3) F(c, 4) ] [ F(d, 3) F(d, 4) ] ]
(%o2)

(%i3) outermap (F, [a, b], x, matrix ([1, 2], [3, 4]));
   [ F(a, x, 1) F(a, x, 2) ] [ F(b, x, 1) F(b, x, 2) ]
   [ [ ] [ ] ]
   [ F(a, x, 3) F(a, x, 4) ] [ F(b, x, 3) F(b, x, 4) ]
(%o3)

(%i4) outermap (F, [a, b], matrix ([1, 2]), matrix ([x], [y]));
   [ [ F(a, 1, x) ] [ F(a, 2, x) ] ]
   [ [ F(a, 1, y) ] [ F(a, 2, y) ] ]
   [ [ F(b, 1, x) ] [ F(b, 2, x) ] ]
   [ [ F(b, 1, y) ] [ F(b, 2, y) ] ]
(%o4)

A closer examination of the outermap return value. The first, second, and third arguments are a matrix, a list, and a matrix, respectively. The return value is a matrix. Each element of that matrix is a list, and each element of each list is a matrix.

(%i11) arg_1 : matrix ([a, b], [c, d]);
   [ a b ]
   [ ]
   [ c d ]
(%o11)

(%i12) arg_2 : [11, 22];
(%o12)

(%i13) arg_3 : matrix ([xx, yy]);
(%o13)

(%i14) xx_0 : outermap (lambda ([x, y, z], x / y + z), arg_1,
     arg_2, arg_3);
   [ [ a a ] [ a a ] ]
   [ [ xx + -- yy + -- ], [ xx + -- yy + -- ]]]
   [ [ 11 11 ] [ 22 22 ] ]
(%o14) Col 1 =

   [ [ c c ] [ c c ] ]
   [ [ xx + -- yy + -- ], [ xx + -- yy + -- ]]]
   [ [ 11 11 ] [ 22 22 ] ]
   [ [ b b ] [ b b ] ]
   [ [ xx + -- yy + -- ], [ xx + -- yy + -- ]]]
   [ [ 11 11 ] [ 22 22 ] ]
(\%i5) \texttt{xx}_{1} : \texttt{xx}_{0} \{1\}[1];
\begin{verbatim}
[ a a ]
\end{verbatim}

(\%o5) \begin{verbatim}
\[ xx + -- yy + -- \]
\end{verbatim}
\begin{verbatim}
[ 11 11 ]
\end{verbatim}

(\%i6) \texttt{xx}_{2} : \texttt{xx}_{0} \{1\}[1][1];
\begin{verbatim}
[ a a ]
\end{verbatim}

(\%o6) \begin{verbatim}
\[ xx + -- yy + -- \]
\end{verbatim}
\begin{verbatim}
[ 11 11 ]
\end{verbatim}

(\%i7) \texttt{xx}_{3} : \texttt{xx}_{0} \{1\}[1][1][1];
\begin{verbatim}
a
\end{verbatim}

(\%o7) \begin{verbatim}
\texttt{xx} + --
\end{verbatim}
\begin{verbatim}
11
\end{verbatim}

outermap preserves the structure of the arguments in the return value, while \texttt{cartesian_product} does not.

(\%i11) outermap (F, \{a, b, c\}, \{1, 2, 3\});
(\%o11) \begin{verbatim}
[[F(a, 1), F(a, 2), F(a, 3)], [F(b, 1), F(b, 2), F(b, 3)], [F(c, 1), F(c, 2), F(c, 3)]
\end{verbatim}

(\%i12) setify (flatten (\%));
(\%o12) \{F(a, 1), F(a, 2), F(a, 3), F(b, 1), F(b, 2), F(b, 3), F(c, 1), F(c, 2), F(c, 3)\}

(\%i13) map (lambda ([L], apply (F, L)),
\begin{verbatim}
cartesian_product ({a, b, c}, \{1, 2, 3\});
\end{verbatim}
(\%o13) \{F(a, 1), F(a, 2), F(a, 3), F(b, 1), F(b, 2), F(b, 3), F(c, 1), F(c, 2), F(c, 3)\}

(\%i14) is (equal (%, \%th (2)));
(\%o14) \texttt{true}
38 Debugging

38.1 Source Level Debugging

Maxima has a built-in source level debugger. The user can set a breakpoint at a function, and then step line by line from there. The call stack may be examined, together with the variables bound at that level.

The command :help or :h shows the list of debugger commands. (In general, commands may be abbreviated if the abbreviation is unique. If not unique, the alternatives will be listed.) Within the debugger, the user can also use any ordinary Maxima functions to examine, define, and manipulate variables and expressions.

A breakpoint is set by the :br command at the Maxima prompt. Within the debugger, the user can advance one line at a time using the :n (“next”) command. The :bt (“backtrace”) command shows a list of stack frames. The :r (“resume”) command exits the debugger and continues with execution. These commands are demonstrated in the example below.

```
(%i1) load ("/tmp/foobar.mac");

(%o1) /tmp/foobar.mac

(%i2) :br foo
Turning on debugging debugmode(true)
Bkpt 0 for foo (in /tmp/foobar.mac line 1)

(%i2) bar (2,3);
Bkpt 0:(foobar.mac 1)
/tmp/foobar.mac:1::

(dbm:1) :bt <-- :bt typed here gives a backtrace
#0: foo(y=5)(foobar.mac line 1)
#1: bar(x=2,y=3)(foobar.mac line 9)

(dbm:1) :n <-- Here type :n to advance line
(dbm:1) u; <-- Investigate value of u
28

(dbm:1) u: 33; <-- Change u to be 33
33
```
The file /tmp/foobar.mac is the following:

foo(y) := block ([u:y^2],
              u: u+3,
              u: u^2,
              u);

bar(x,y) := (x: x+2,
             y: y+2,
             x: foo(y),
             x+y);

USE OF THE DEBUGGER THROUGH EMACS

If the user is running the code under GNU emacs in a shell window (dbl shell), or is running the graphical interface version, Xmaxima, then if he stops at a break point, he will see his current position in the source file which will be displayed in the other half of the window, either highlighted in red, or with a little arrow pointing at the right line. He can advance single lines at a time by typing M-n (Alt-n).

Under Emacs you should run in a dbl shell, which requires the dbl.el file in the elisp directory. Make sure you install the elisp files or add the Maxima elisp directory to your path: e.g., add the following to your .emacs file or the site-init.el

(setq load-path (cons "/usr/share/maxima/5.9.1/emacs" load-path))
(autoload 'dbl "dbl")

then in emacs

M-x dbl

should start a shell window in which you can run programs, for example Maxima, gcl, gdb etc. This shell window also knows about source level debugging, and display of source code in the other window.

The user may set a break point at a certain line of the file by typing C-x space. This figures out which function the cursor is in, and then it sees which line of that function the cursor is on. If the cursor is on, say, line 2 of foo, then it will insert in the other window the command, “:br foo 2”, to break foo at its second line. To have this enabled, the user must have maxima-mode.el turned on in the window in which the file foobar.mac is visiting. There are additional commands available in that file window, such as evaluating the function into the Maxima, by typing Alt-Control-x.

38.2 Keyword Commands

Keyword commands are special keywords which are not interpreted as Maxima expressions. A keyword command can be entered at the Maxima prompt or the debugger prompt, although not at the break prompt. Keyword commands start with a colon, ’:’. For example, to evaluate a Lisp form you may type :lisp followed by the form to be evaluated.

(\%i1) :lisp (+ 2 3)
5

The number of arguments taken depends on the particular command. Also, you need not type the whole command, just enough to be unique among the break keywords. Thus :br would suffice for :break.

The keyword commands are listed below.

**:break F n**
Set a breakpoint in function F at line offset n from the beginning of the function. If F is given as a string, then it is assumed to be a file, and n is the offset from the beginning of the file. The offset is optional. If not given, it is assumed to be zero (first line of the function or file).

**:bt**
Print a backtrace of the stack frames

**:continue**
Continue the computation

**:delete**
Delete the specified breakpoints, or all if none are specified

**:disable**
Disable the specified breakpoints, or all if none are specified

**:enable**
Enable the specified breakpoints, or all if none are specified

**:frame n**
Print stack frame n, or the current frame if none is specified

**:help**
Print help on a debugger command, or all commands if none is specified

**:info**
Print information about item

**:lisp some-form**
Evaluate some-form as a Lisp form

**:lisp-quiet some-form**
Evaluate Lisp form some-form without any output

**:next**
Like :step, except :next steps over function calls

**:quit**
Quit the current debugger level without completing the computation

**:resume**
Continue the computation

**:step**
Continue the computation until it reaches a new source line

**:top**
Return to the Maxima prompt (from any debugger level) without completing the computation

### 38.3 Functions and Variables for Debugging

debugmode [Option variable]

Default value: false

When debugmode is true, Maxima will start the Maxima debugger when a Maxima error occurs. At this point the user may enter commands to examine the call stack, set breakpoints, step through Maxima code, and so on. See debugging for a list of Maxima debugger commands.

When debugmode is lisp, Maxima will start the Lisp debugger when a Maxima error occurs.

In either case, enabling debugmode will not catch Lisp errors.
**refcheck**  
[Option variable]  
Default value: *false*  
When *refcheck* is *true*, Maxima prints a message each time a bound variable is used for the first time in a computation.

**setcheck**  
[Option variable]  
Default value: *false*  
If *setcheck* is set to a list of variables (which can be subscripted), Maxima prints a message whenever the variables, or subscripted occurrences of them, are bound with the ordinary assignment operator : , the :: assignment operator, or function argument binding, but not the function assignment ::= nor the macro assignment :::= operators. The message comprises the name of the variable and the value it is bound to.  
*setcheck* may be set to *all* or *true* thereby including all variables.  
Each new assignment of *setcheck* establishes a new list of variables to check, and any variables previously assigned to *setcheck* are forgotten.  
The names assigned to *setcheck* must be quoted if they would otherwise evaluate to something other than themselves. For example, if *x*, *y*, and *z* are already bound, then enter  
```  
setcheck: ['x, 'y, 'z]$
```

to put them on the list of variables to check.  
No printout is generated when a variable on the *setcheck* list is assigned to itself, e.g., *X*: 'X.

**setcheckbreak**  
[Option variable]  
Default value: *false*  
When *setcheckbreak* is *true*, Maxima will present a break prompt whenever a variable on the *setcheck* list is assigned a new value. The break occurs before the assignment is carried out. At this point, *setval* holds the value to which the variable is about to be assigned. Hence, one may assign a different value by assigning to *setval*.  
See also *setcheck* and *setval*.

**setval**  
[System variable]  
Holds the value to which a variable is about to be set when a *setcheckbreak* occurs. Hence, one may assign a different value by assigning to *setval*.  
See also *setcheck* and *setcheckbreak*.

**timer**  
[Function]  
```  
timer (f_1, . . ., f_n)
```

Given functions *f_1*, . . ., *f_n*, *timer* puts each one on the list of functions for which timing statistics are collected. *timer(f)*$ timer(g)$ puts *f* and then *g* onto the list; the list accumulates from one call to the next.  
*timer(all)* puts all user-defined functions (as named by the global variable functions) on the list of timed functions.
With no arguments, \texttt{timer} returns the list of timed functions. Maxima records how much time is spent executing each function on the list of timed functions. \texttt{timer\_info} returns the timing statistics, including the average time elapsed per function call, the number of calls, and the total time elapsed. \texttt{untimer} removes functions from the list of timed functions.

\texttt{timer} quotes its arguments. \texttt{f(x) := x^2$ g:f$ timer(g)$} does not put \texttt{f} on the timer list.

If \texttt{trace(f)} is in effect, then \texttt{timer(f)} has no effect; \texttt{trace} and \texttt{timer} cannot both be in effect at the same time.

See also \texttt{timer\_devalue}.

\textbf{untimer} \((f_1, \ldots, f_n)\)

\texttt{untimer()}\[\text{[Function]}\]

Given functions \texttt{f_1, \ldots, f_n}, \texttt{untimer} removes each function from the timer list.

With no arguments, \texttt{untimer} removes all functions currently on the timer list.

After \texttt{untimer} \((f)\) is executed, \texttt{timer\_info} \((f)\) still returns previously collected timing statistics, although \texttt{timer\_info()} (with no arguments) does not return information about any function not currently on the timer list. \texttt{timer} \((f)\) resets all timing statistics to zero and puts \texttt{f} on the timer list again.

\textbf{timer\_devalue}

\texttt{[Option variable]}\[\text{Default value: false}\]

When \texttt{timer\_devalue} is \texttt{true}, Maxima subtracts from each timed function the time spent in other timed functions. Otherwise, the time reported for each function includes the time spent in other functions. Note that time spent in untimed functions is not subtracted from the total time.

See also \texttt{timer} and \texttt{timer\_info}.

\textbf{timer\_info} \((f_1, ..., f_n)\)

\texttt{timer\_info()}\[\text{[Function]}\]

Given functions \texttt{f_1, ..., f_n}, \texttt{timer\_info} returns a matrix containing timing information for each function. With no arguments, \texttt{timer\_info} returns timing information for all functions currently on the timer list.

The matrix returned by \texttt{timer\_info} contains the function name, time per function call, number of function calls, total time, and \texttt{gctime}, which meant "garbage collection time" in the original Macsyma but is now always zero.

The data from which \texttt{timer\_info} constructs its return value can also be obtained by the \texttt{get} function:

\texttt{get(f, 'calls); get(f, 'runtime); get(f, 'gctime);}

See also \texttt{timer}.

\textbf{trace} \((f_1, \ldots, f_n)\)

\texttt{trace(all)}\[\text{[Function]}\]

\texttt{trace()}\[\text{[Function]}\]

Given functions \texttt{f_1, \ldots, f_n}, \texttt{trace} instructs Maxima to print out debugging information whenever those functions are called. \texttt{trace(f)$ trace(g)$} puts \texttt{f} and then \texttt{g} onto the list of functions to be traced; the list accumulates from one call to the next.
trace(all) puts all user-defined functions (as named by the global variable functions) on the list of functions to be traced.

With no arguments, trace returns a list of all the functions currently being traced.

The untrace function disables tracing. See also trace_options.

trace quotes its arguments. Thus, \( f(x) := x^2 \) \( g: f \) trace(g)\$ does not put \( f \) on the trace list.

When a function is redefined, it is removed from the timer list. Thus after timer(f)\$ \( f(x) := x^2 \), function \( f \) is no longer on the timer list.

If timer (f) is in effect, then trace (f) has no effect; trace and timer can’t both be in effect for the same function.

trace_options (f, option_1, ..., option_n) [Function]

trace_options (f)
Sets the trace options for function \( f \). Any previous options are superseded. trace_options (f, ...) has no effect unless trace (f) is also called (either before or after trace_options).

trace_options (f) resets all options to their default values.

The option keywords are:

- **noprint** Do not print a message at function entry and exit.
- **break** Put a breakpoint before the function is entered, and after the function is exited. See break.
- **lisp_print** Display arguments and return values as Lisp objects.
- **info** Print \( \rightarrow \) true at function entry and exit.
- **errorcatch** Catch errors, giving the option to signal an error, retry the function call, or specify a return value.

Trace options are specified in two forms. The presence of the option keyword alone puts the option into effect unconditionally. (Note that option foo is not put into effect by specifying foo: true or a similar form; note also that keywords need not be quoted.) Specifying the option keyword with a predicate function makes the option conditional on the predicate.

The argument list to the predicate function is always [level, direction, function, item] where level is the recursion level for the function, direction is either enter or exit, function is the name of the function, and item is the argument list (on entering) or the return value (on exiting).

Here is an example of unconditional trace options:

\[
\begin{align*}
  \text{(\%i1)} \ f(n) & := \text{if equal}(n, 0) \text{ then } 1 \text{ else } n \ast f(n - 1) \\
  \text{(\%i2)} \ \text{trace}(f) & \\
  \text{(\%i3)} \ \text{trace_options}(f, \text{lisp_print}, \text{break}) & \\
  \text{(\%i4)} \ f(3); & 
\end{align*}
\]
Here is the same function, with the break option conditional on a predicate:

(%i5) trace_options (ff, break(pp))$

(%i6) pp (level, direction, function, item) := block (print (item),
    return (function = 'ff and level = 3 and direction = exit))$

(%i7) ff(6);

untrace  
  untrace (f_1, . . ., f_n)
  untrace ()

Given functions f_1, . . ., f_n, untrace disables tracing enabled by the trace function. With no arguments, untrace disables tracing for all functions. untrace returns a list of the functions for which it disabled tracing.
39 alt-display

39.1 Introduction to alt-display

The *alt-display* package provides a means to change the way that Maxima displays its output. The *alt-display1d* and *alt-display2d* Lisp hooks were introduced to Maxima in 2002, but were not easily accessible from the Maxima REPL until the introduction of this package.

The package provides a general purpose function to define alternative display functions, and a separate function to set the display function. The package also provides customized display functions to produce output in \TeX, Texinfo, XML and all three output formats within Texinfo.

Here is a sample session:

```maxima
(%i1) load("alt-display.mac")$
(%i2) set_alt_display(2,tex_display)$

(%i3) x/(x^2+y^2) = 1;
\mbox{\tt\red({\it \%o_3}) \black}$$\frac{x}{y^2+x^2}=1$$

(%i4) set_alt_display(2,mathml_display)$

(%i5) x/(x^2+y^2) = 1;
<math xmlns="http://www.w3.org/1998/Math/MathML"> <mi>mlabel</mi> <mfenced separators=""> <msub><mi>%o</mi> <mn>5</mn> </msub> <mo>,</mo> <mfrac><mrow><mi>x</mi> </mrow> <mrow><msup><mrow><mi>y</mi> </mrow> <mn>2</mn> </msup> <mo>+</mo> <msup><mrow><mi>x</mi> </mrow> <mn>2</mn> </msup> </mrow></mfrac> <mo>=</mo> <mn>1</mn> </mfenced> </math>

(%i6) set_alt_display(2,multi_display_for_texinfo)$

(%i7) x/(x^2+y^2) = 1;
@iftex
@tex
\mbox{\tt\red({\it \%o_7}) \black}$$\frac{x}{y^2+x^2}=1$$
@end tex
@end iftex
@ifhtml
@html
<math xmlns="http://www.w3.org/1998/Math/MathML"> <mi>mlabel</mi> <mfenced separators=""> <msub><mi>%o</mi> <mn>7</mn> </msub> <mo>,</mo> <mfrac><mrow><mi>x</mi> </mrow> <mrow><msup><mrow><mi>y</mi> </mrow> <mn>2</mn> </msup> <mo>+</mo> <msup><mrow><mi>x</mi> </mrow> <mn>2</mn> </msup> </mrow></mfrac> <mo>=</mo> <mn>1</mn> </mfenced> </math>
@end ifhtml
```
If the alternative display function causes an error, the error is trapped and the display function is reset to the default display. In the following example, the error function is set to display the output. This throws an error, which is handled by resetting the 2d-display to the default.

(%i8) set_alt_display(2,?error)$

(%i9) x;

Error in *alt-display2d*.
Message: Condition designator ((MLABEL) $%O9 $X) is not of type
(OR SYMBOL STRING FUNCTION).
*alt-display2d* reset to nil.
-- an error. To debug this try: debugmode(true);

(%i10) x;
(%o10) x

39.2 Functions and Variables for alt-display

alt_display_output_type (form) [Function]
Determine the type of output to be printed. Form must be a lisp form suitable for printing via Maxima's built-in displa function. At present, this function returns one of three values: text, label or unknown.

An example where alt_display_output_type is used. In my_display, a text form is printed between a pair of tags TEXT;>> and <<TEXT;~% while a label form is printed between a pair tags OUT;>> and <<OUT;~%; in addition to the usual output label.

The function set_prompt also ensures that input labels are printed between matching PROMPT;>> and <<PROMPT; tags.

Thanks to Eric Stemmler.

(%i11) (load("mactex-utilities"), load("alt-display.mac"))$

(%i2) define_alt_display(my_display(form),
block([type,txttmplt,labtmplt],
 txttmplt:"~%TEXT;>>"~%a~"<<TEXT;"~%",
 labtmplt:"~%OUT;>>"% (~a) "~a~a~%"<<OUT;"~%",
 type:alt_display_output_type(form),
 if type='text then
printf(true,txttmplt,first(form))
else if type='label then
  printf(true,labtmplt,first(form),"$$",tex1(second(form)),"$$")
else
  block([alt_display1d:false, alt_display2d:false], displa(form))) $

(%i3) (set_prompt('prefix, "PROMPT;>>",'suffix, "<<PROMPT;"),
    set_alt_display(1,my_display)) $

PROMPT;>>(%i4) <<PROMPT;integrate(x^n,x);

Is n equal to -1?

n;

OUT;>>

(%o4) $$\frac{x^{n+1}}{n+1}$$

<<OUT;

define_alt_display \([\text{function(input), expr}]\) [Function]

This function is similar to define: it evaluates its arguments and expands into a
function definition. The function is a function of a single input input. For convenience,
a substitution is applied to expr after evaluation, to provide easy access to Lisp
variable names.

Set a time-stamp on each prompt:

(%i1) load("alt-display.mac")$

(%i2) display2d: false$

(%i3) define_alt_display(time_stamp(x),
    block([alt_display1d:false,alt_display2d:false],
        prompt_prefix:printf(false,"\textcolor{black}{~a~%}\textcolor{black}{time_date()}",
        displa(x)));

(%o3) time_stamp(x):=block(
    [\*alt\-display1d\*:false,
     \*alt\-display2d\*:false,
     \*prompt\-prefix\*\textcolor{black}{:printf(false,\textcolor{black}{\textcolor{black}{"\textcolor{black}{~a~}%}\textcolor{black}{time_date()}}},
     displa(x))

(%i4) set_alt_display(1,time_stamp);

(%o4) done

2017-11-27 16:15:58-06:00

(%i5)
The input line %i3 defines `time_stamp` using `define_alt_display`. The output line %o3 shows that the Maxima variable names `alt_display1d`, `alt_display2d` and `prompt_prefix` have been replaced by their Lisp translations, as has `displa` been replaced by `?displa` (the display function).

The display variables `alt_display1d` and `alt_display2d` are both bound to `false` in the body of `time_stamp` to prevent an infinite recursion in `displa`.

info_display (form)  [Function]

This is an alias for the default 1-d display function. It may be used as an alternative 1-d or 2-d display function.

(%i1) load("alt-display.mac")$
(%i2) set_alt_display(2, info_display);
(%o2) done
(%i3) x/y;
(%o3) x/y

mathml_display (form)  [Function]

Produces MathML output.

(%i1) load("alt-display.mac")$
(%i2) set_alt_display(2, mathml_display);
<math xmlns="http://www.w3.org/1998/Math/MathML"> <mi>mlabel</mi> <mfenced separators=""> <msub><mi>%o</mi> <mn>2</mn> </msub> <mo>,</mo> <mi>done</mi> </mfenced> </math>

tex_display (form)  [Function]

Produces TeX output.

(%i2) set_alt_display(2, tex_display);
\mbox{\tt\red{{\it \%o_2}) \black}} \mathbf{\{x\over{y^2+x^2}\}}$
(%i3) x/(x^2+y^2);
(%o3) \mbox{\tt\red{{\it \%o_3}) \black}})$

multi_display_for_texinfo (form)  [Function]

Produces Texinfo output using all three display functions.

(%i2) set_alt_display(2, multi_display_for_texinfo)$
(%i3) x/(x^2+y^2);
@if TeX
@end TeX
@if Html
@end Html

reset_displays()  \[\text{Functions}\]
Resets the prompt prefix and suffix to the empty string, and sets both 1-d and 2-d display functions to the default.

set_alt_display(num, display-function)  \[\text{Function}\]
The input num is the display to set; it may be either 1 or 2. The second input display-function is the display function to use. The display function may be either a Maxima function or a \texttt{lambda} expression.

Here is an example where the display function is a \texttt{lambda} expression; it just displays the result as \TeX.

\begin{verbatim}
(%i1) load("alt-display.mac")$

(%i2) set_alt_display(2, lambda([form], tex(?caddr(form))))$

(%i3) integrate(exp(-t^2), t, 0, inf);
$$\frac{\sqrt{\pi}}{2}$$
\end{verbatim}

A user-defined display function should take care that it prints its output. A display function that returns a string will appear to display nothing, nor cause any errors.

set_prompt(fix, expr)  \[\text{Function}\]
Set the prompt prefix or suffix to expr. The input fix must evaluate to one of \texttt{prefix}, \texttt{suffix}, \texttt{general}, \texttt{prolog} or \texttt{epilog}. The input expr must evaluate to either a string or \texttt{false}; if \texttt{false}, the fix is reset to the default value.

\begin{verbatim}
(%i1) load("alt-display.mac")$

(%i2) set_prompt('prefix,printf(false,"It is now: \text{"a"\%},timedate()))$

It is now: 2014-01-07 15:23:23-05:00
(%i3)
\end{verbatim}

The following example shows the effect of each option, except \texttt{prolog}. Note that the \texttt{epilog} prompt is printed as Maxima closes down. The \texttt{general} is printed between the end of input and the output, unless the input line ends in $.$
Here is an example to show where the prompt strings are placed.

(\%i1) load("alt-display.mac")$

(\%i2) set_prompt(prefix, "<<prefix>> ", suffix, "<<suffix>> ", 
general, printf(false,"<<general>>~%"), 
epilog, printf(false,"<<epilog>>~%"));

(\%o2) done
<<prefix>> (\%i3) <<suffix>> x/y;
<<general>>

(\%o3)
\x
-
\y
<<prefix>> (\%i4) <<suffix>> quit();
<<general>>
<<epilog>>

Here is an example that shows how to colorize the input and output when Maxima is running in a terminal or terminal emulator like Emacs\(^1\).

Each prompt string starts with the ASCII escape character (27) followed by an open square bracket (91); each string ends with a lower-case m (109). The webpages 
https://misc.flogisoft.com/bash/tip_colors_and_formatting and 
https://www.tldp.org/HOWTO/Bash-Prompt-HOWTO/x329.html provide information on how to use control strings to set the terminal colors.

\(^1\) Readers using the info reader in Emacs will see the actual prompt strings; other readers will see the colorized output
40 asympa

40.1 Introduction to asympa

asympa

asympa is a package for asymptotic analysis. The package contains simplification functions for asymptotic analysis, including the “big O” and “little o” functions that are widely used in complexity analysis and numerical analysis.

load ("asympa") loads this package.

40.2 Functions and variables for asympa
41 augmented_lagrangian

41.1 Functions and Variables for augmented_lagrangian

augmented_lagrangian_method

Returns an approximate minimum of the expression \( FOM \) with respect to the variables \( xx \), holding the constraints \( C \) equal to zero. \( yy \) is a list of initial guesses for \( xx \). The method employed is the augmented Lagrangian method (see Refs [1] and [2]).

\( grad \), if present, is the gradient of \( FOM \) with respect to \( xx \), represented as a list of expressions, one for each variable in \( xx \). If not present, the gradient is constructed automatically.

\( FOM \) and each element of \( grad \), if present, must be ordinary expressions, not names of functions or lambda expressions.

\( \text{optional_args} \) represents additional arguments, specified as \( \text{symbol} = \text{value} \). The optional arguments recognized are:

- \text{niter} Number of iterations of the augmented Lagrangian algorithm
- \text{lbfgs_tolerance} Tolerance supplied to LBFGS
- \text{iprint} IPRINT parameter (a list of two integers which controls verbosity) supplied to LBFGS
- \%\text{lambda} Initial value of \%\text{lambda} to be used for calculating the augmented Lagrangian

This implementation minimizes the augmented Lagrangian by applying the limited-memory BFGS (LBFGS) algorithm, which is a quasi-Newton algorithm.

\text{load("augmented_lagrangian")} loads this function.

See also Chapter 69 [lbfgs-pkg], page 1055.

References:


Examples:

\( (%i1) \text{load ("lbfgs");} \)
\( (%o1) /home/gunter/src/maxima-code/share/lbfgs/lbfgs.mac \)
\( (%i2) \text{load("augmented_lagrangian");} \)
\( (%o2) /home/gunter/src/maxima-code/share/contrib/augmented_lagrangian.mac \)
(%i3) FOM: x^2 + 2*y^2;
   2 2
(%o3)            2 y + x
(%i4) xx: [x, y];
   [x, y]
(%o4)                
(%i5) C: [x + y - 1];
   [y + x - 1]
(%o5)                
(%i6) yy: [1, 1];
   [1, 1]
(%o6)                
(%i7) augmented_lagrangian_method(FOM, xx, C, yy, iprint=[-1,0]);
(%o7) [[[x = 0.666659841080023, y = 0.333340272455448],
        %lambda = [- 1.333337940892518]]

Same example as before, but this time the gradient is supplied as an argument.
(%i11) load ("1bfgs")$
(%i12) load ("augmented_lagrangian")$
(%i13) FOM: x^2 + 2*y^2;
   2 2
(%o13)           2 y + x
(%i14) xx: [x, y];
   [x, y]
(%o14)                
(%i15) grad : [2*x, 4*y];
   [2 x, 4 y]
(%o15)                
(%i16) C: [x + y - 1];
   [y + x - 1]
(%o16)                
(%i17) yy: [1, 1];
   [1, 1]
(%o17)                
(%i18) augmented_lagrangian_method ([FOM, grad], xx, C, yy,
     iprint = [-1, 0]);
(%o18) [[[x = 0.6666598410800247, y = 0.3333402724554464],
        %lambda = [- 1.333337940892525]]
42 Bernstein

42.1 Functions and Variables for Bernstein

bernstein_poly (k, n, x) [Function]
Provided k is not a negative integer, the Bernstein polynomials are defined by
bernstein_poly(k,n,x) = binomial(n,k) x^k (1-x)^(n-k); for a negative integer
k, the Bernstein polynomial bernstein_poly(k,n,x) vanishes. When either k or n
are non integers, the option variable bernstein_explicit controls the expansion
of the Bernstein polynomials into its explicit form; example:
(\%i1) load("bernstein")$

(\%i2) bernstein_poly(k,n,x);
(\%o2) bernstein_poly(k, n, x)
(\%i3) bernstein_poly(k,n,x), bernstein_explicit : true;
\n  _n - k_ k
  \binomial(n, k) (1 - x) x

The Bernstein polynomials have both a gradef property and an integrate property:
(\%i4) diff(bernstein_poly(k,n,x),x);
(\%o4) (bernstein_poly(k - 1, n - 1, x)
  - bernstein_poly(k, n - 1, x)) n
(\%i5) integrate(bernstein_poly(k,n,x),x);
(\%o5)
\n  \frac{k + 1}{\hypergeometric([k + 1, k - n], [k + 2], x) \binomial(n, k) x}
  \______________________________
  k + 1

For numeric inputs, both real and complex, the Bernstein polynomials evaluate to a
numeric result:
(\%i6) bernstein_poly(5,9, 1/2 + \%i);
39375 \%i 39375
(\%o6)
\n  ------- + ------
  128 256
(\%i7) bernstein_poly(5,9, 0.5b0 + \%i);
3.076171875b2 \%i + 1.5380859375b2
(\%o7)

To use bernstein_poly, first load("bernstein").

bernstein_explicit [Variable]
Default value: false

When either k or n are non integers, the option variable bernstein_explicit controls
the expansion of bernstein(k,n,x) into its explicit form; example:
(\%i1) bernstein_poly(k,n,x);
(\%o1) bernstein_poly(k, n, x)
(\%i2) bernstein_poly(k,n,x), bernstein_explicit : true;
\[ \text{binomial}(n, k) \quad (1 - x)^k x \]  

When both \( k \) and \( n \) are explicitly integers, \( \text{bernstein}(k, n, x) \) always expands to its explicit form.

**multibernstein_poly** ([\( k_1, k_2, \ldots, k_p \]), [\( n_1, n_2, \ldots, n_p \]), [\( x_1, x_2, \ldots, x_p \)])

The multibernstein polynomial \( \text{multibernstein_poly} \) is the product of bernstein polynomials \( \text{bernstein_poly}(k_1, n_1, x_1) \text{bernstein_poly}(k_2, n_2, x_2) \ldots \text{bernstein_poly}(k_p, n_p, x_p) \).

To use \( \text{multibernstein_poly} \), first \text{load}("bernstein")

**bernstein_approx** (\( f \), [\( x_1, x_2, \ldots, x_n \)], \( n \))

Return the \( n \)-th order uniform Bernstein polynomial approximation for the function \( (x_1, x_2, \ldots, x_n) \mapsto f \). Examples

\[ f(1) x + 2 f(-) (1 - x) x + f(0) (1 - x) \]

\[ f(1, 1) x y + 2 f(-, 1) (1 - x) x y + f(0, 1) (1 - x) y + 2 f(-, 0) (1 - x) (1 - y) \]

To use \( \text{bernstein_approx} \), first \text{load}("bernstein")

**bernstein_expand** (\( e \), [\( x_1, x_2, \ldots, x_n \)])

Express the \( polynomial \) \( e \) exactly as a linear combination of multi-variable Bernstein polynomials.

\[ x y + 1 \]

Maxima signals an error when the first argument isn’t a polynomial.

To use \( \text{bernstein_expand} \), first \text{load}("bernstein").
43 bitwise

The package bitwise provides functions that allow to manipulate bits of integer constants. As always maxima attempts to simplify the result of the operation if the actual value of a constant isn't known considering attributes that might be known for the variables, see the declare mechanism.

43.1 Functions and Variables for bitwise

bit_not (int)

Inverts all bits of a signed integer. The result of this action reads \(-\text{int} - 1\).

```
(%i1) load("bitwise")
(%i2) bit_not(i);
(%o2) bit_not(i)
(%i3) bit_not(bit_not(i));
(%o3) i
(%i4) bit_not(3);
(%o4) - 4
(%i5) bit_not(100);
(%o5) - 101
(%i6) bit_not(-101);
(%o6) 100
```

bit_and (int1, ...)

This function calculates a bitwise and of two or more signed integers.

```
(%i1) load("bitwise")
(%i2) bit_and(i,i);
(%o2) i
(%i3) bit_and(i,i,i);
(%o3) i
(%i4) bit_and(1,3);
(%o4) 1
(%i5) bit_and(-7,7);
(%o5) 1
```

If it is known if one of the parameters to bit_and is even this information is taken into consideration by the function.

```
(%i1) load("bitwise")
(%i2) declare(e,even,o,odd);
(%o2) done
(%i3) bit_and(1,e);
(%o3) 0
(%i4) bit_and(1,o);
(%o4) 1
```

bit_or (int1, ...)

This function calculates a bitwise or of two or more signed integers.

```
(%i1) load("bitwise")
```
(\%i2) bit_or(i,i);  \hspace{1cm} i
(\%o2) i
(\%i3) bit_or(i,i,i);  \hspace{1cm} i
(\%o3) i
(\%i4) bit_or(1,3);  \hspace{1cm} 3
(\%o4) 3
(\%i5) bit_or(-7,7);  \hspace{1cm} -1
(\%o5) -1

If it is known if one of the parameters to \texttt{bit\_or} is even this information is taken into consideration by the function.

(\%i1) load("bitwise")$
(\%i2) declare(e,even,o,odd);  \hspace{1cm} done
(\%i3) bit_or(1,e);  \hspace{1cm} e + 1
(\%o3) e + 1
(\%i4) bit_or(1,o);  \hspace{1cm} o
(\%o4) o

\textbf{bit\_xor} (int1, ...)

\texttt{bit\_xor} calculates a bitwise or of two or more signed integers.

(\%i1) load("bitwise")$
(\%i2) bit_xor(i,i);  \hspace{1cm} 0
(\%o2) 0
(\%i3) bit_xor(i,i,i);  \hspace{1cm} i
(\%o3) i
(\%i4) bit_xor(1,3);  \hspace{1cm} 2
(\%o4) 2
(\%i5) bit_xor(-7,7);  \hspace{1cm} -2
(\%o5) -2

If it is known if one of the parameters to \texttt{bit\_xor} is even this information is taken into consideration by the function.

(\%i1) load("bitwise")$
(\%i2) declare(e,even,o,odd);  \hspace{1cm} done
(\%i3) bit_xor(1,e);  \hspace{1cm} e + 1
(\%o3) e + 1
(\%i4) bit_xor(1,o);  \hspace{1cm} o - 1
(\%o4) o - 1

\textbf{bit\_lsh} (int, n\texttt{Bits})

\texttt{bit\_lsh} shifts all bits of the signed integer \texttt{int} to the left by \texttt{n\texttt{Bits}} bits. The width of the integer is extended by \texttt{n\texttt{Bits}} for this process. The result of \texttt{bit\_lsh} therefore is \texttt{int} * 2.

(\%i1) load("bitwise")$
(\%i2) bit_lsh(0,1);  \hspace{1cm} 0
(\%o2) 0
Chapter 43: bitwise

(bit_lsh(1,0); %i3) 1
(bit_lsh(1,1); %o4) 2
(bit_lsh(1,i); %i5) bit_lsh(1, i)
(bit_lsh(-3,1); %o6) -6
(bit_lsh(-2,1); %o7) -4

(bit_lsh (int, nBits))
This function shifts all bits of the signed integer int to the right by nBits bits. The width of the integer is reduced by nBits for this process.

(bit_rsh(0,1); %i1) 0
(bit_rsh(2,0); %o2) 2
(bit_rsh(2,1); %o3) 1
(bit_rsh(2,2); %o4) 0
(bit_rsh(-3,1); %i5) -2
(bit_rsh(-2,1); %o6) -1
(bit_rsh(-2,2); %o7) -1

(bit_length (int))
determines how many bits a variable needs to be long in order to store the number int. This function only operates on positive numbers.

(bit_length(0); %i1) 0
(bit_length(1); %o2) 1
(bit_length(7); %o3) 3
(bit_length(8); %o4) 4

(bit_onep (int, nBit))
determines if bits nBit is set in the signed integer int.

(bit_onep(85,0); %i1) true
(%i3) bit_onep(85,1);                         false
(%o3) bit_onep(85,2);                         true
(%o4) bit_onep(85,3);                         false
(%o5) bit_onep(85,100);                       false
(%o6) bit_onep(i,100);                       bit_onep(i, 100)

For signed numbers the sign bit is interpreted to be more than nBit to the left of the
leftmost bit of int that reads 1.

(%i1) load("bitwise")$
(%i2) bit_onep(-2,0);                        false
(%o2) bit_onep(-2,1);                        true
(%o3) bit_onep(-2,2);                        true
(%o4) bit_onep(-2,3);                        true
(%o5) bit_onep(-2,4);                        true
(%o6) bit_onep(-2,5);                        true

If it is known if the number to be tested is even this information is taken into consid-
eration by the function.

(%i1) load("bitwise")$
(%i2) declare(e,even,o,odd);
(%o2) done
(%i3) bit_onep(e,0);                         false
(%o3) bit_onep(o,0);                         true
44 bode

44.1 Functions and Variables for bode

\texttt{bode\_gain (H, range, ...plot\_opts...)}

[Function]

Function to draw Bode gain plots.

Examples (1 through 7 from http://www.swarthmore.edu/NatSci/echeeve1/Ref/Bode/BodeHow.html, 8 from Ron Crummett):

(\%i1) load("bode")$

(\%i2) H1 (s) := 100 * (1 + s) / ((s + 10) * (s + 100))$

(\%i3) bode\_gain (H1 (s), [w, 1/1000, 1000])$

(\%i4) H2 (s) := 1 / (1 + s/omega0)$

(\%i5) bode\_gain (H2 (s), [w, 1/1000, 1000]), omega0 = 10$

(\%i6) H3 (s) := 1 / (1 + s/omega0)^2$

(\%i7) bode\_gain (H3 (s), [w, 1/1000, 1000]), omega0 = 10$

(\%i8) H4 (s) := 1 + s/omega0$

(\%i9) bode\_gain (H4 (s), [w, 1/1000, 1000]), omega0 = 10$

(\%i10) H5 (s) := 1/s$

(\%i11) bode\_gain (H5 (s), [w, 1/1000, 1000])$

(\%i12) H6 (s) := 1/((s/omega0)^2 + 2 * zeta * (s/omega0) + 1)$

(\%i13) bode\_gain (H6 (s), [w, 1/1000, 1000]), omega0 = 10, zeta = 1/10$

(\%i14) H7 (s) := (s/omega0)^2 + 2 * zeta * (s/omega0) + 1$

(\%i15) bode\_gain (H7 (s), [w, 1/1000, 1000]), omega0 = 10, zeta = 1/10$

(\%i16) H8 (s) := 0.5 / (0.0001 * s^3 + 0.002 * s^2 + 0.01 * s)$

(\%i17) bode\_gain (H8 (s), [w, 1/1000, 1000])$

To use this function write first load("bode"). See also \texttt{bode\_phase}.
Function to draw Bode phase plots.

Examples (1 through 7 from http://www.swarthmore.edu/NatSci/eccheeve1/Ref/Bode/BodeHow.html, 8 from Ron Crummett):

```maxima
(%i1) load("bode")$
(%i2) H1 (s) := 100 * (1 + s) / ((s + 10) * (s + 100))$
(%i3) bode_phase (H1 (s), [w, 1/1000, 1000])$
(%i4) H2 (s) := 1 / (1 + s/omega0)$
(%i5) bode_phase (H2 (s), [w, 1/1000, 1000]), omega0 = 10$
(%i6) H3 (s) := 1 / (1 + s/omega0)^2$
(%i7) bode_phase (H3 (s), [w, 1/1000, 1000]), omega0 = 10$
(%i8) H4 (s) := 1 + s/omega0$
(%i9) bode_phase (H4 (s), [w, 1/1000, 1000]), omega0 = 10$
(%i10) H5 (s) := 1/s$
(%i11) bode_phase (H5 (s), [w, 1/1000, 1000])$
(%i12) H6 (s) := 1/((s/omega0)^2 + 2 * zeta * (s/omega0) + 1)$
(%i13) bode_phase (H6 (s), [w, 1/1000, 1000]),
   omega0 = 10, zeta = 1/10$
(%i14) H7 (s) := (s/omega0)^2 + 2 * zeta * (s/omega0) + 1$
(%i15) bode_phase (H7 (s), [w, 1/1000, 1000]),
   omega0 = 10, zeta = 1/10$
(%i16) H8 (s) := 0.5 / (0.0001 * s^3 + 0.002 * s^2 + 0.01 * s)$
(%i17) bode_phase (H8 (s), [w, 1/1000, 1000])$
(%i18) block ([bode_phase_unwrap : false], bode_phase (H8 (s), [w, 1/1000, 1000]));
(%i19) block ([bode_phase_unwrap : true], bode_phase (H8 (s), [w, 1/1000, 1000]));
```
To use this function write first \texttt{load("bode"). See also \texttt{bode\_gain}.}
45 celine

45.1 Introduction to celine

Maxima implementation of Sister Celine’s method. Barton Willis wrote this code. It is released under the Creative Commons CC0 license.


Let \( f = F(n,k) \). The function celine returns a set of recursion relations for \( F \) of the form

\[
p_0(n) * f(n,k) + p_1(n) * f(n+1,k) + \ldots + p_p(n) * f(n+p,k+q),
\]

where \( p_0 \) through \( p_p \) are polynomials. If Maxima is unable to determine that

\[
\sum(\sum(a(i,j) * F(n+i,k+j),i,0,p),j,0,q) / F(n,k)
\]

is a rational function of \( n \) and \( k \), celine returns the empty set. When \( f \) involves parameters (variables other than \( n \) or \( k \)), celine might make assumptions about these parameters. Using 'put' with a key of 'proviso,' Maxima saves these assumptions on the input label.

To use this function, first load the package integer_sequence, opsubst, and to_poly_solve.

Examples:

```
(%i1) load("integer_sequence")$
(%i2) load("opsubst")$
(%i3) load("to_poly_solve")$
(%i4) load("celine")$
(%i5) celine(n!,n,k,1,0);
(%o5) \{fff(n + 1, k) - n fff(n, k) - fff(n, k)\}
```

Verification that this result is correct:

```
(%i1) load("integer_sequence")$
(%i2) load("opsubst")$
(%i3) load("to_poly_solve")$
(%i4) load("celine")$
(%i5) g1:{fff(n+1,k)-n*fff(n,k)-fff(n,k)};
(%o5) \{fff(n + 1, k) - n fff(n, k) - fff(n, k)\}
(%i6) ratsimp(minfactorial(first(g1))),fff(n,k) := n!;
(%o6) 0
```

An example with parameters including the test that the result of the example is correct:

```
(%i1) load("integer_sequence")$
(%i2) load("opsubst")$
(%i3) load("to_poly_solve")$
(%i4) load("celine")$
(%i5) e : pochhammer(a,k) * pochhammer(-k,n) / (pochhammer(b,k));
       (a)  (- k)
          k     n
       ---------
        (b)     k
(%o5)
```

```
```
(%i6) recur : celine(e,n,k,2,1);
(%o6) \{fff(n + 2, k + 1) - fff(n + 2, k) - b fff(n + 1, k + 1)
+ n ((- fff(n + 1, k + 1)) + 2 fff(n + 1, k) - a fff(n, k)
- fff(n, k)) + a (fff(n + 1, k) - fff(n, k)) + 2 fff(n + 1, k)
- n fff(n, k)\}
(%i7) /* Test this result for correctness */
(%i8) first(%), fff(n,k) := ''(e)$
(%i9) makefact(makegamma(%))$
(%o9) 0
(%i10) minfactorial(factor(minfactorial(factor(%))))

The proviso data suggests that setting a = b may result in a lower order recursion which
is shown by the following example:

(%i11) load("integer_sequence")$
(%i12) load("opsubst")$
(%i13) load("to_poly_solve")$
(%i14) load("celine")$
(%i15) e : pochhammer(a,k) * pochhammer(-k,n) / (pochhammer(b,k));
  (a) (- k)
     k n
   -----------
(b)             k
(%o5)

(%i6) recur : celine(e,n,k,2,1);
(%o6) \{fff(n + 2, k + 1) - fff(n + 2, k) - b fff(n + 1, k + 1)
+ n ((- fff(n + 1, k + 1)) + 2 fff(n + 1, k) - a fff(n, k)
- fff(n, k)) + a (fff(n + 1, k) - fff(n, k)) + 2 fff(n + 1, k)
- n fff(n, k)\}
(%i7) get(%, 'proviso);
(%o7) false
(%i8) celine(subst(b=a,e),n,k,1,1);
(%o8) \{fff(n + 1, k + 1) - fff(n + 1, k) + n fff(n, k)
+ fff(n, k)\}
46 clebsch_gordan

46.1 Functions and Variables for clebsch_gordan

clebsch_gordan (j1, j2, m1, m2, j, m)  
Compute the Clebsch-Gordan coefficient \langle j1, j2, m1, m2 | j, m \rangle.

racah_v (a, b, c, a1, b1, c1)  
Compute Racah’s V coefficient (computed in terms of a related Clebsch-Gordan coefficient).

racah_w (j1, j2, j5, j4, j3, j6)  
Compute Racah’s W coefficient (computed in terms of a Wigner 6j symbol)

wigner_3j (j1, j2, j3, m1, m2, m3)  
Compute Wigner’s 3j symbol (computed in terms of a related Clebsch-Gordan coefficient).

wigner_6j (j1, j2, j3, j4, j5, j6)  
Compute Wigner’s 6j symbol.

wigner_9j (a, b, c, d, e, f, g, h, i, j)  
Compute Wigner’s 9j symbol.
47 cobyla

47.1 Introduction to cobyla

**fmin_cobyla** is a Common Lisp translation (via `f2cl`) of the Fortran constrained optimization routine COBYLA by Powell [1][2][3].

COBYLA minimizes an objective function $F(X)$ subject to $M$ inequality constraints of the form $g(X) \geq 0$ on $X$, where $X$ is a vector of variables that has $N$ components.

Equality constraints $g(X) = 0$ can often be implemented by a pair of inequality constraints $g(X) \geq 0$ and $-g(X) \geq 0$. Maxima’s interface to COBYLA allows equality constraints and internally converts the equality constraints to a pair of inequality constraints.

The algorithm employs linear approximations to the objective and constraint functions, the approximations being formed by linear interpolation at $N + 1$ points in the space of the variables. The interpolation points are regarded as vertices of a simplex. The parameter $RHO$ controls the size of the simplex and it is reduced automatically from $RHOBEG$ to $RHOEND$. For each $RHO$ the subroutine tries to achieve a good vector of variables for the current size, and then $RHO$ is reduced until the value $RHOEND$ is reached. Therefore, $RHOBEG$ and $RHOEND$ should be set to reasonable initial changes to and the required accuracy in the variables respectively, but this accuracy should be viewed as a subject for experimentation because it is not guaranteed. The routine treats each constraint individually when calculating a change to the variables, rather than lumping the constraints together into a single penalty function. The name of the subroutine is derived from the phrase Constrained Optimization BY Linear Approximations.

References:

[1] Fortran Code is from [http://plato.asu.edu/sub/nlores.html#general](http://plato.asu.edu/sub/nlores.html#general)


47.2 Functions and Variables for cobyla

**fmin_cobyla**

```lisp
fmin_cobyla (F, X, Y)
fmin_cobyla (F, X, Y, optional_args)
```

Returns an approximate minimum of the expression $F$ with respect to the variables $X$, subject to an optional set of constraints. $Y$ is a list of initial guesses for $X$.

$F$ must be ordinary expressions, not names of functions or lambda expressions.

`optional_args` represents additional arguments, specified as `symbol = value`. The optional arguments recognized are:
constraints
List of inequality and equality constraints that must be satisfied by \( X \).
The inequality constraints must be actual inequalities of the form \( g(X) \geq h(X) \) or \( g(X) \leq h(X) \). The equality constraints must be of the form \( g(X) = h(X) \).

\textbf{rhobeg}  
Initial value of the internal \texttt{RHO} variable which controls the size of simplex. (Defaults to 1.0)

\textbf{rhoend}  
The desired final value \( \rho \) parameter. It is approximately the accuracy in the variables. (Defaults to 1d-6.)

\textbf{iprint}  
Verbose output level. (Defaults to 0)
- 0 - No output
- 1 - Summary at the end of the calculation
- 2 - Each new value of \( \texttt{RHO} \) and \( \texttt{SIGMA} \) is printed, including the vector of variables, some function information when \( \texttt{RHO} \) is reduced.
- 3 - Like 2, but information is printed when \( F(X) \) is computed.

\textbf{maxfun}  
The maximum number of function evaluations. (Defaults to 1000).

On return, a vector is given:

1. The value of the variables giving the minimum. This is a list of elements of the form \( \texttt{var} = \texttt{value} \) for each of the variables listed in \( X \).
2. The minimized function value
3. The number of function evaluations.
4. Return code with the following meanings
   - 0 - No errors.
   - 1 - Limit on maximum number of function evaluations reached.
   - 2 - Rounding errors inhibiting progress.
   - -1 - \texttt{MAXCV} value exceeds \texttt{RHOEND}. This indicates that the constraints were probably not satisfied. User should investigate the value of the constraints.

\texttt{MAXCV} stands for “\texttt{MAXimum Constraint Violation}” and is the value of \( \max(0.0, -c1(x), -c2(x), ... - cm(x)) \) where \( ck(x) \) denotes the \( k \)'th constraint function. (Note that \texttt{maxima} allows constraints of the form \( f(x) = g(x) \), which are internally converted to \( f(x) - g(x) \geq 0 \) and \( g(x) - f(x) \geq 0 \) which is required by COBYLA).

\texttt{load("fmin_cobyla")} loads this function.

\textbf{bf_fmin_cobyla}  
This function is identical to \texttt{fmin_cobyla}, except that bigfloat operations are used, and the default value for \texttt{rhoend} is \( 10^{-(\text{fpprec}/2)} \).

See \texttt{fmin_cobyla} for more information.

\texttt{load("bf_fmin_cobyla")} loads this function.
47.3 Examples for cobyla

Minimize $x_1 x_2$ with $1 - x_1^2 - x_2^2 \geq 0$.

The theoretical solution is

$$x_1 = \frac{1}{\sqrt{2}}$$

$$x_2 = -\frac{1}{\sqrt{2}}$$

(%i1) load("fmin_cobyla")$
(%i2) fmin_cobyla(x1*x2, [x1, x2], [1,1],
constraints = [x1^2+x2^2<=1], iprint=1);

Normal return from subroutine COBYLA

NFVALS = 66  F = -5.000000E-01  MAXCV = 1.999845E-12
X = 7.071058E-01 -7.071077E-01
(%o2) [[x1 = 0.70710584934848, x2 = -0.7071077130248],
    -0.49999999999926, [[-1.99995575659757e-12]], [], 66]

Here is the same example but the constraint is $x_1^2 + x_2^2 \leq -1$ which is impossible over the reals.

(%i1) fmin_cobyla(x1*x2, [x1, x2], [1,1],
    constraints = [x1^2+x2^2 <= -1], iprint=1);

Normal return from subroutine COBYLA

NFVALS = 65  F = 3.016417E-13  MAXCV = 1.000000E+00
X = -3.375179E-07 -8.937057E-07
(%o1) [[x1 = -3.375178983064622e-7, x2 = -8.937056510780022e-7],
    3.016416530564557e-13, 65, -1]
(%o2) subst(%o1[2], [x1^2+x2^2 <= -1]);

We see the return code (%o1[4]) is -1 indicating that the constraints may not be satisfied. Substituting the solution into the constraint equation as shown in %o2 shows that the constraint is, of course, violated.

There are additional examples in the share/cobyla/ex directory and in share/cobyla/rtest_cobyla.mac.
48 combinatorics

48.1 Package combinatorics

The `combinatorics` package provides several functions to work with permutations and to permute elements of a list. The permutations of degree \( n \) are all the \( n! \) possible orderings of the first \( n \) positive integers, 1, 2, \ldots, \( n \). The functions in this package expect a permutation to be represented by a list of those integers.

Cycles are represented as a list of two or more integers \( i_1, i_2, \ldots, i_m \), all different. Such a list represents a permutation where the integer \( i_2 \) appears in the \( i_1 \)th position, the integer \( i_3 \) appears in the \( i_2 \)th position and so on, until the integer \( i_m \), which appears in the \( i_m \)th position.

For instance, \([4, 2, 1, 3]\) is one of the 24 permutations of degree four, which can also be represented by the cycle \([1, 4, 3]\). The functions where cycles are used to represent permutations also require the order of the permutation to avoid ambiguity. For instance, the same cycle \([1, 4, 3]\) could refer to the permutation of order 6: \([4, 2, 1, 3, 5, 6]\). A product of cycles must be represented by a list of cycles; the cycles at the end of the list are applied first. For example, \([[2, 4], [1, 3, 6, 5]]\) is equivalent to the permutation \([3, 4, 6, 2, 1, 5]\).

A cycle can be written in several ways. For instance, \([1, 3, 6, 5]\), \([3, 6, 5, 1]\) and \([6, 5, 1, 3]\) are all equivalent. The canonical form used in the package is the one that places the lowest index in the first place. A cycle with only two indices is also called a transposition and if the two indices are consecutive, it is called an adjacent transposition.

To run an interactive tutorial, use the command `demo (combinatorics)`. Since this is an additional package, it must be loaded with the command `load("combinatorics")`.

48.2 Functions and Variables for Combinatorics

**apply_cycles (cl,l)**

Permutes the list or set \( l \) applying to it the list of cycles \( cl \). The cycles at the end of the list are applied first and the first cycle in the list \( cl \) is the last one to be applied.

See also `permute`.

Example:

```plaintext
(%i1) load("combinatorics")$
(%i2) lis1: [a,b*c^2,4,z,x/y,1/2,ff23(x),0];
2 2
2 x 1
(%o2) [a, b c , 4, z, -, -, ff23(x), 0]

y 2
(%i3) apply_cycles ([[1, 6], [2, 6, 5, 7]], lis1);
2
2 x 1
(%o3) [a, b c , 4, z, -, -, ff23(x), 0]

y 2
```

**cyclep (c, n)**

Returns true if \( c \) is a valid cycle of order \( n \) namely, a list of non-repeated positive integers less or equal to \( n \). Otherwise, it returns false.
See also `perm`.  
Examples:

```
(%i1) load("combinatorics")$
(%i2) cyclep([-2,3,4], 5);
(%o2) false
(%i3) cyclep([2,3,4,2], 5);
(%o3) false
(%i4) cyclep([6,3,4], 5);
(%o4) false
(%i5) cyclep([6,3,4], 6);
(%o5) true
```

perm_cycles (p)  
Returns permutation p as a product of cycles. The cycles are written in a canonical form, in which the lowest index in the cycle is placed in the first position.  
See also `perm_decomp`.  
Example:

```
(%i1) load("combinatorics")$
(%i2) perm_cycles([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2) [[1, 4], [2, 6, 5, 7]]
```

perm_decomp (p)  
Returns the minimum set of adjacent transpositions whose product equals the given permutation p.  
See also `perm_cycles`.  
Example:

```
(%i1) load("combinatorics")$
(%i2) perm_decomp([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2) [[6, 7], [5, 6], [6, 7], [3, 4], [4, 5], [2, 3], [3, 4],
[4, 5], [5, 6], [1, 2], [2, 3], [3, 4]]
```

perm_inverse (p)  
Returns the inverse of a permutation of p, namely, a permutation q such that the products pq and qp are equal to the identity permutation: [1, 2, 3, ..., n], where n is the length of p.  
See also `permult`.  
Example:

```
(%i1) load("combinatorics")$
(%i2) perm_inverse([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2) [4, 7, 3, 1, 6, 2, 5, 8]
```

perm_length (p)  
Determines the minimum number of adjacent transpositions necessary to write permutation p as a product of adjacent transpositions. An adjacent transposition is a cycle with only two numbers, which are consecutive integers.  
See also `perm_decomp`.  

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_length ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2) 12
```

**perm_lex_next** \(p\)  
Returns the permutation that comes after the given permutation \(p\), in the sequence of permutations in lexicographic order.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_lex_next ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2) [4, 6, 3, 1, 7, 5, 8, 2]
```

**perm_lex_rank** \(p\)  
Finds the position of permutation \(p\), an integer from 1 to the degree \(n\) of the permutation, in the sequence of permutations in lexicographic order.

See also **perm_lex_unrank** and **perms_lex**.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_lex_rank ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2) 18255
```

**perm_lex_unrank** \((n, i)\)  
Returns the \(n\)-degree permutation at position \(i\) (from 1 to \(n!\)) in the lexicographic ordering of permutations.

See also **perm_lex_rank** and **perms_lex**.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_lex_unrank (8, 18255);
(%o2) [4, 6, 3, 1, 7, 5, 2, 8]
```

**perm_next** \(p\)  
Returns the permutation that comes after the given permutation \(p\), in the sequence of permutations in Trotter-Johnson order.

See also **perms**.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_next ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2) [4, 6, 3, 1, 7, 5, 8, 2]
```

**perm_parity** \(p\)  
Finds the parity of permutation \(p\): 0 if the minimum number of adjacent transpositions necessary to write permutation \(p\) as a product of adjacent transpositions is even, or 1 if that number is odd.

See also **perm_decomp**.
Example:

(%i1) load("combinatorics")$
(%i2) perm_parity ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2) 0

perm_rank (p)  [Function]
Finds the position of permutation p, an integer from 1 to the degree n of the permutation, in the sequence of permutations in Trotter-Johnson order.

See also perm_unrank and perms.

Example:

(%i1) load("combinatorics")$
(%i2) perm_rank ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2) 19729

perm_undecomp (cl, n)  [Function]
Converts the list of cycles cl of degree n into an n degree permutation, equal to their product.

See also perm_decomp.

Example:

(%i1) load("combinatorics")$
(%i2) perm_undecomp ([[1,6],[2,6,5,7]], 8);
(%o2) [5, 6, 3, 4, 7, 1, 2, 8]

perm_unrank (n, i)  [Function]
Returns the n-degree permutation at position i (from 1 to n!) in the Trotter-Johnson ordering of permutations.

See also perm_rank and perms.

Example:

(%i1) load("combinatorics")$
(%i2) perm_unrank (8, 19729);
(%o2) [4, 6, 3, 1, 7, 5, 2, 8]

permp (p)  [Function]
Returns true if p is a valid permutation namely, a list of length n, whose elements are all the positive integers from 1 to n, without repetitions. Otherwise, it returns false.

Examples:

(%i1) load("combinatorics")$
(%i2) permp ([2,0,3,1]);
(%o2) false
(%i3) permp ([2,1,4,3]);
(%o3) true
perms

perms (n)
perms (n, i)
perms (n, i, j)

perms(n) returns a list of all n-degree permutations in the so-called Trotter-Johnson order.
perms(n, i) returns the n-degree permutation which is at the ith position (from 1 to n!) in the Trotter-Johnson ordering of the permutations.
perms(n, i, j) returns a list of the n-degree permutations between positions i and j in the Trotter-Johnson ordering of the permutations.

The sequence of permutations in Trotter-Johnson order starts with the identity permutation and each consecutive permutation can be obtained from the previous one by single adjacent transposition.

See also perm_next, perm_rank and perm_unrank.

Examples:

(%i1) load("combinatorics")$
(%i2) perms (4);
(%o2) [\[1, 2, 3, 4\], [1, 2, 4, 3], [1, 4, 2, 3], [4, 1, 2, 3],
     [4, 1, 3, 2], [1, 4, 3, 2], [1, 3, 4, 2], [1, 3, 2, 4],
     [3, 1, 2, 4], [3, 1, 4, 2], [3, 4, 1, 2], [4, 3, 1, 2],
     [4, 3, 2, 1], [3, 4, 2, 1], [3, 2, 4, 1], [3, 2, 1, 4],
     [2, 3, 1, 4], [2, 4, 3, 1], [4, 2, 3, 1], [4, 2, 1, 3],
     [2, 4, 1, 3], [2, 1, 4, 3], [2, 1, 3, 4], [2, 1, 3, 4]]
(%i3) perms (4, 12);
(%o3) [\[4, 3, 1, 2\]]
(%i4) perms (4, 12, 14);
(%o4) [\[4, 3, 1, 2\], [4, 3, 2, 1], [3, 4, 2, 1]]

perms_lex

perms_lex (n)
perms_lex (n, i)
perms_lex (n, i, j)

perms_lex(n) returns a list of all n-degree permutations in the so-called lexicographic order.
perms_lex(n, i) returns the n-degree permutation which is at the ith position (from 1 to n!) in the lexicographic ordering of the permutations.
perms_lex(n, i, j) returns a list of the n-degree permutations between positions i and j in the lexicographic ordering of the permutations.

The sequence of permutations in lexicographic order starts with all the permutations with the lowest index, 1, followed by all permutations starting with the following index, 2, and so on. The permutations starting by an index i are the permutations of the first n integers different from i and they are also placed in lexicographic order, where the permutations with the lowest of those integers are placed first and so on.

See also perm_lex_next, perm_lex_rank and perm_lex_unrank.
Examples:

```
(%i1) load("combinatorics")$
(%i2) perms_lex (4);
(%o2) [[1, 2, 3, 4], [1, 2, 4, 3], [1, 3, 2, 4], [1, 3, 4, 2],
     [1, 4, 2, 3], [1, 4, 3, 2], [2, 1, 3, 4], [2, 1, 4, 3],
     [2, 3, 1, 4], [2, 3, 4, 1], [2, 4, 1, 3], [2, 4, 3, 1],
     [3, 1, 2, 4], [3, 1, 4, 2], [3, 2, 1, 4], [3, 2, 4, 1],
     [3, 4, 1, 2], [3, 4, 2, 1], [4, 1, 2, 3], [4, 1, 3, 2],
     [4, 2, 1, 3], [4, 2, 3, 1], [4, 3, 1, 2], [4, 3, 2, 1]]
(%i3) perms_lex (4, 12);
(%o3) [[2, 4, 3, 1]]
(%i4) perms_lex (4, 12, 14);
(%o4) [[2, 4, 3, 1], [3, 1, 2, 4], [3, 1, 4, 2]]
```

**permult (p_1, ..., p_m)**

Returns the product of two or more permutations \( p_1, \ldots, p_m \).

Example:

```
(%i1) load("combinatorics")$
(%i2) permult ([2,3,1], [3,1,2], [2,1,3]);
(%o2) [2, 1, 3]
```

**permute (p, l)**

Applies the permutation \( p \) to the elements of the list (or set) \( l \).

Example:

```
(%i1) load("combinatorics")$
(%i2) lis1: [a,b*c^2,4,z,x/y,1/2,ff23(x),0];
2 x 1
(%o2) [a, b c , 4, z, -, -, ff23(x), 0]
     y 2
(%i3) permute ([4, 6, 3, 1, 7, 5, 2, 8], lis1);
1 x 2
(%o3) [z, -, 4, a, ff23(x), -, b c , 0]
     y 2
```

**random_perm (n)**

Returns a random permutation of degree \( n \).

See also random_permutation.

Example:

```
(%i1) load("combinatorics")$
(%i2) random_perm (7);
(%o2) [6, 3, 4, 7, 5, 1, 2]
```
49 contrib_ode

49.1 Introduction to contrib_ode

Maxima’s ordinary differential equation (ODE) solver \texttt{ode2} solves elementary linear ODEs of first and second order. The function \texttt{contrib_ode} extends \texttt{ode2} with additional methods for linear and non-linear first order ODEs and linear homogeneous second order ODEs. The code is still under development and the calling sequence may change in future releases. Once the code has stabilized it may be moved from the contrib directory and integrated into Maxima.

This package must be loaded with the command \texttt{load("contrib_ode")} before use.

The calling convention for \texttt{contrib_ode} is identical to \texttt{ode2}. It takes three arguments: an ODE (only the left hand side need be given if the right hand side is 0), the dependent variable, and the independent variable. When successful, it returns a list of solutions.

The form of the solution differs from \texttt{ode2}. As non-linear equations can have multiple solutions, \texttt{contrib_ode} returns a list of solutions. Each solution can have a number of forms:

- an explicit solution for the dependent variable,
- an implicit solution for the dependent variable,
- a parametric solution in terms of variable \%t, or
- a transformation into another ODE in variable \%u.

\texttt{contrib_ode} uses the global variables \%c, \%k1, \%k2, \texttt{method} and \texttt{yp} similarly to \texttt{ode2}.

If \texttt{contrib_ode} cannot obtain a solution for whatever reason, it returns \texttt{false}, after perhaps printing out an error message.

It is necessary to return a list of solutions, as even first order non-linear ODEs can have multiple solutions. For example:

\begin{verbatim}
(%i1) load("contrib_ode")$
(%i2) eqn:x*'diff(y,x)^2-(1+x*y)*'diff(y,x)+y=0;
   dy 2          dy
(%o2) x (--) - (1 + x y) -- + y = 0
   dx          dx
(%i3) contrib_ode(eqn,y,x);
   dy 2          dy
(%t3) x (--) - (1 + x y) -- + y = 0
   dx          dx

first order equation not linear in y'

   x
(%o3) [y = log(x) + %c, y = %c %e ]
(%i4) method;
(%o4) factor
\end{verbatim}
Nonlinear ODEs can have singular solutions without constants of integration, as in the second solution of the following example:

\begin{verbatim}
(%i1) load("contrib_ode")$
(%i2) eqn:'diff(y,x)^2+x*'diff(y,x)-y=0;
     dy 2    dy
   (--) + x -- - y = 0
     dx    dx
(%i3) contrib_ode(eqn,y,x);
     dy 2    dy
   (--) + x -- - y = 0
     dx    dx
(%t3) first order equation not linear in y'

(%o3) [y = %c x + %c , y = - --]
        4
(%i4) method;
(%o4) clairaut
\end{verbatim}

The following ODE has two parametric solutions in terms of the dummy variable \( \%t \). In this case the parametric solutions can be manipulated to give explicit solutions.

\begin{verbatim}
(%i1) load("contrib_ode")$
(%i2) eqn:'diff(y,x)=(x+y)^2;
   dy 2
   -- = (x + y)
   dx
(%i3) contrib_ode(eqn,y,x);
   dy 2
   (--) + x -- - y = 0
   dx    dx
(%t3) [x = %c - atan(sqrt(%t)), y = (- x) - sqrt(%t)],
    [x = atan(sqrt(%t)) + %c, y = sqrt(%t) - x]

(%i4) method;
(%o4) lagrange
\end{verbatim}

The following example (Kamke 1.112) demonstrates an implicit solution.

\begin{verbatim}
(%i1) load("contrib_ode")$
(%i2) assume(x>0,y>0);
    [x > 0, y > 0]
(%i3) eqn:x*'diff(y,x)-x*sqrt(y^2+x^2)-y;
   dy 2 2
   x -- - x sqrt(y + x) - y
   dx
(%i4) contrib_ode(eqn,y,x);
   y
   [x - asinh(--) = %c]
   x
(%i5) method;
(%o5) lie
\end{verbatim}
The following Riccati equation is transformed into a linear second order ODE in the variable \( \%u \). Maxima is unable to solve the new ODE, so it is returned unevaluated.

\[
%i1) \text{load("contrib_ode")}$
%i2) \text{eqn:x^2*'diff(y,x)=a+b*x^n+c*x^2*y^2;}
\]
\[
%o2) x \frac{dy}{dx} = c x y + b x + a
\]
\[
%i3) \text{contrib_ode(eqn,y,x);}
\]
\[
\frac{d%u}{dx} = \frac{2 a}{c} - \frac{2 d %u}{x c 2 2} \]
\[
%o3) \{y = - \frac{2}{d %u}, \%u c (\frac{-2}{x} + b x) + \frac{2}{d %u} c = 0]\]
\[
%i4) \text{method;}
\]
\[
%o4) \text{riccati}
\]

For first order ODEs \textit{contrib_ode} calls \textit{ode2}. It then tries the following methods: factorization, Clairaut, Lagrange, Riccati, Abel and Lie symmetry methods. The Lie method is not attempted on Abel equations if the Abel method fails, but it is tried if the Riccati method returns an unsolved second order ODE.

For second order ODEs \textit{contrib_ode} calls \textit{ode2} then \textit{odelin}.

Extensive debugging traces and messages are displayed if the command \texttt{put('contrib_ode,'true,'verbose)} is executed.

### 49.2 Functions and Variables for contrib_ode

\textbf{contrib_ode (eqn, y, x)}

Returns a list of solutions of the ODE eqn with independent variable x and dependent variable y.

\textbf{odelin (eqn, y, x)}

\textit{odelin} solves linear homogeneous ODEs of first and second order with independent variable x and dependent variable y. It returns a fundamental solution set of the ODE.

For second order ODEs, \textit{odelin} uses a method, due to Bronstein and Lafaillle, that searches for solutions in terms of given special functions.

\[
%i1) \text{load("contrib_ode")}$
%i2) \text{odelin(x*(x+1)*'diff(y,x,2)+(x+5)*'diff(y,x,1)+(-4)*y,y,x);}
\]
\[
\text{gauss_a(-6, -2, -3, -x) gauss_b(-6, -2, -3, -x)}
\]
\[
%o2) \{---------------------------, ---------------------------\}
\]
\[
4 4
\]
\[
x x
\]

\textbf{ode_check (eqn, soln)}

Returns the value of ODE eqn after substituting a possible solution soln. The value is equivalent to zero if soln is a solution of eqn.

\[
%i1) \text{load("contrib_ode")}$
\]
(%i2) eqn: 'diff(y, x, 2) + (a*x + b)*y;
    2
   d y
---- + (b + a*x) y
   dx
(%o2)

(%i3) ans: [y = bessel_y(1/3, 2*(a*x + b)^(3/2)/(3*a))*%k2*sqrt(a*x + b) +
         bessel_j(1/3, 2*(a*x + b)^(3/2)/(3*a))*%k1*sqrt(a*x + b)];
       1  2  3/2
   1  2 (b + a x)
      3  3 a
   3/2
%k2 sqrt(a x + b)
(%o3)
       1  2  3/2
   1  2 (b + a x)
      3  3 a
   3/2
+ bessel_j(1/3, 2*(a*x + b)^(3/2)/(3*a))*%k1*sqrt(a*x + b)]

(%i4) ode_check(eqn, ans[1]);
(%o4) 0

gauss_a (a, b, c, x)

gauss_a(a,b,c,x) and gauss_b(a,b,c,x) are 2F1 hypergeometric functions. They
represent any two independent solutions of the hypergeometric differential equation
x*(1-x) diff(y, x, 2) + [c-(a+b+1)x] diff(y, x) - a*b*y = 0 (A&S 15.5.1).

The only use of these functions is in solutions of ODEs returned by odelin and
contrib_ode. The definition and use of these functions may change in future releases
of Maxima.

See also gauss_b, dgauss_a and gauss_b.

gauss_b (a, b, c, x)

See gauss_a.

dgauss_a (a, b, c, x)

The derivative with respect to x of gauss_a(a, b, c, x).

dgauss_b (a, b, c, x)

The derivative with respect to x of gauss_b(a, b, c, x).

kummer_m (a, b, x)

Kummer’s M function, as defined in Abramowitz and Stegun, Handbook of Mathemati-
cal Functions, Section 13.1.2.

The only use of this function is in solutions of ODEs returned by odelin and contrib_ode. The definition and use of this function may change in future releases of Maxima.

See also kummer_u, dkummer_m, and dkummer_u.

kummer_u (a, b, x)

Kummer’s U function, as defined in Abramowitz and Stegun, Handbook of Mathemati-
cal Functions, Section 13.1.3.

See kummer_m.
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\texttt{dkummer}_m (a, b, x) \quad \text{[Function]}

The derivative with respect to \( x \) of \texttt{kummer}_m(a, b, x).

\texttt{dkummer}_u (a, b, x) \quad \text{[Function]}

The derivative with respect to \( x \) of \texttt{kummer}_u(a, b, x).

\texttt{bessel_simplify} (expr) \quad \text{[Function]}

Simplifies expressions containing Bessel functions \texttt{bessel}_j, \texttt{bessel}_y, \texttt{bessel}_i, \texttt{bessel}_k, \texttt{hankel}_1, \texttt{hankel}_2, \texttt{struve}_h and \texttt{struve}_l. Recurrence relations (DLMF §10.6(i))(A&S 9.1.27) are used to replace functions of highest order \( n \) by functions of order \( n-1 \) and \( n-2 \).

This process is repeated until all the orders differ by less than 2.

\begin{verbatim}
(%i1) load("contrib_ode")$
(%i2) bessel_simplify(4*bessel_j(n,x^2)*(x^2-n^2/x^2)
   +x*((bessel_j(n-2,x^2)-bessel_j(n,x^2))*x
   -(bessel_j(n,x^2)-bessel_j(n+2,x^2))*x)
   -2*bessel_j(n+1,x^2)+2*bessel_j(n-1,x^2));
(%o2) 0

(%i3) bessel_simplify(-2*bessel_j(1,z)*z^3 - 10*bessel_j(2,z)*z^2
   + 15*%pi*bessel_j(1,z)*struve_h(3,z)*z - 15*%pi*struve_h(1,z)
   *bessel_j(3,z)*z - 15*%pi*bessel_j(0,z)*struve_h(2,z)*z
   + 15*%pi*struve_h(0,z)*bessel_j(2,z)*z - 30*%pi*bessel_j(1,z)
   *struve_h(2,z) + 30*%pi*struve_h(1,z)*bessel_j(2,z));
(%o3) 0
\end{verbatim}

\texttt{expintegral_e_simplify} (expr) \quad \text{[Function]}

Simplify expressions containing exponential integral \texttt{expintegral}_e using the recurrence (A&S 5.1.14).

\texttt{expintegral}_e(n+1,z) = (1/n) * (exp(-z)*z^neexpintegral}_e(n,z)) \ n = 1,2,3 ....

49.3 Possible improvements to contrib_ode

These routines are work in progress. I still need to:

- Extend the FACTOR method \texttt{ode1_factor} to work for multiple roots.
- Extend the FACTOR method \texttt{ode1_factor} to attempt to solve higher order factors. At present it only attempts to solve linear factors.
- Fix the LAGRANGE routine \texttt{ode1_lagrange} to prefer real roots over complex roots.
- Add additional methods for Riccati equations.
- Improve the detection of Abel equations of second kind. The existing pattern matching is weak.
- Work on the Lie symmetry group routine \texttt{ode1_lie}. There are quite a few problems with it: some parts are unimplemented; some test cases seem to run forever; other test cases crash; yet others return very complex "solutions". I wonder if it really ready for release yet.
- Add more test cases.
49.4 Test cases for contrib_ode

The routines have been tested on a approximately one thousand test cases from Murphy, Kamke, Zwillinger and elsewhere. These are included in the tests subdirectory.

- The Clairaut routine `ode1_clairaut` finds all known solutions, including singular solutions, of the Clairaut equations in Murphy and Kamke.
- The other routines often return a single solution when multiple solutions exist.
- Some of the "solutions" from `ode1_lie` are overly complex and impossible to check.
- There are some crashes.

49.5 References for contrib_ode

2. G. M. Murphy, Ordinary Differential Equations and Their Solutions, Van Nostrand, New York, 1960
50 descriptive

50.1 Introduction to descriptive

Package **descriptive** contains a set of functions for making descriptive statistical computations and graphing. Together with the source code there are three data sets in your Maxima tree: *pidigits.data, wind.data* and *biomed.data*.

Any statistics manual can be used as a reference to the functions in package **descriptive**.

For comments, bugs or suggestions, please contact me at 'riotorto AT yahoo DOT com'.

Here is a simple example on how the descriptive functions in **descriptive** do they work, depending on the nature of their arguments, lists or matrices,

```maxima
(%i1) load ("descriptive")$
(%i2) /* univariate sample */ mean ([a, b, c]);
   c + b + a
-----
  3
(%o2)
(%i3) matrix ([a, b], [c, d], [e, f]);
   [ a b ]
   [ c d ]
   [ e f ]
(%o3)
(%i4) /* multivariate sample */ mean (%);
   e + c + a f + d + b
[---------, ---------]
  3  3
(%o4)
```

Note that in multivariate samples the mean is calculated for each column.

In case of several samples with possible different sizes, the Maxima function **map** can be used to get the desired results for each sample,

```maxima
(%i1) load ("descriptive")$
(%i2) map (mean, [[a, b, c], [d, e]]);
   c + b + a e + d
[---------, -----
  3  2
(%o2)
```

In this case, two samples of sizes 3 and 2 were stored into a list.

Univariate samples must be stored in lists like

```maxima
(%i1) s1 : [3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5];
(%o1) [3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5]
```

and multivariate samples in matrices as in
In this case, the number of columns equals the random variable dimension and the number of rows is the sample size.

Data can be introduced by hand, but big samples are usually stored in plain text files. For example, file `pidigits.data` contains the first 100 digits of number $\pi$:

3
1
4
1
5
9
2
5
3 ...

In order to load these digits in Maxima,

(%i1) s1 : read_list (file_search ("pidigits.data"))$
(%i2) length (s1);
(%o2) 100

On the other hand, file `wind.data` contains daily average wind speeds at 5 meteorological stations in the Republic of Ireland (This is part of a data set taken at 12 meteorological stations. The original file is freely downloadable from the StatLib Data Repository and its analysis is discussed in Haslett, J., Raftery, A. E. (1989) Space-time Modelling with Long-memory Dependence: Assessing Ireland’s Wind Power Resource, with Discussion. Applied Statistics 38, 1-50). This loads the data:

(%i1) s2 : read_matrix (file_search ("wind.data"))$
(%i2) length (s2);
(%o2) 100
(%i3) s2 [%, /* last record */
(%o3) [3.58, 6.0, 4.58, 7.62, 11.25]

Some samples contain non numeric data. As an example, file `biomed.data` (which is part of another bigger one downloaded from the StatLib Data Repository) contains four blood measures taken from two groups of patients, A and B, of different ages,
The first individual belongs to group A, is 30 years old and his/her blood measures were 167.0, 89.0, 25.6 and 364.

One must take care when working with categorical data. In the next example, symbol a is assigned a value in some previous moment and then a sample with categorical value a is taken,

```lisp
(%i1) a : 1
(%i2) matrix ([a, 3], [b, 5]);
      [ 1 3 ]
      [      ]
      [ b 5 ]
```

50.2 Functions and Variables for data manipulation

`build_sample` [Function]

`build_sample(list)`
`build_sample(matrix)`

Builds a sample from a table of absolute frequencies. The input table can be a matrix or a list of lists, all of them of equal size. The number of columns or the length of the lists must be greater than 1. The last element of each row or list is interpreted as the absolute frequency. The output is always a sample in matrix form.

Examples:

Univariate frequency table.

```lisp
(%i1) load ("descriptive")
(%i2) sam1: build_sample([[6,1], [j,2], [2,1]]);
      [ 6 ]
      [    ]
      [ j ]
      [    ]
      [ 2 ]
(%o2)
(%i3) mean(sam1);
   2 j + 8
(%o3) [----------]
    4
(%i4) barsplot(sam1) $
```

Multivariate frequency table.

```lisp
(%i1) load ("descriptive")
(%i2) sam2: build_sample([[6,3,1], [5,6,2], [u,2,1],[6,8,2]]); 
      [ 6 3 ]
```
continuous_freq

continuous_freq (data)
continuous_freq (data, m)

The first argument of continuous_freq must be a list or 1-dimensional array (as created by make_array) of numbers. Divides the range in intervals and counts how many values are inside them. The second argument is optional and either equals the number of classes we want, 10 by default, or equals a list containing the class limits and the number of classes we want, or a list containing only the limits.

If sample values are all equal, this function returns only one class of amplitude 2.

Examples:

Optional argument indicates the number of classes we want. The first list in the output contains the interval limits, and the second the corresponding counts: there are 16 digits inside the interval [0, 1.8], 24 digits in (1.8, 3.6], and so on.

(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) continuous_freq (s1, 5); 
[0, 1.8, 3.6, 5.4, 7.2, 9.0], [16, 24, 18, 17, 25]]

Optional argument indicates we want 7 classes with limits -2 and 12:

(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) continuous_freq (s1, [-2,12,7]); 
[[- 2, 0, 2, 4, 6, 8, 10, 12], [8, 20, 22, 17, 20, 13, 0]]

Optional argument indicates we want the default number of classes with limits -2 and 12:

(%i1) load ("descriptive")$
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(%i2) s1 : read_list (file_search ("pidigits.data"));
(%i3) continuous_freq (s1, [-2,12]);
   3  4  11  18  32  39  46  53
(%o3)  [[-2, -, -, -, -, 5, -, -, -, 12],
       [0, 8, 20, 12, 18, 9, 8, 25, 0, 0]]

The first argument may be an array.

(%i1) load ("descriptive");
(%i2) s1 : read_list (file_search ("pidigits.data"));
(%i3) a1 : make_array (fixnum, length (s1)) $
(%i4) fillarray (a1, s1);
   {Lisp Array:
    #(3 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 8 4 6 2 6 4 3 8 3 2 7 9 \
     5 0 2 8 8 4 1 9 7 1 6 9 3 9 9 3 7 5 1 0 5 8 2 0 9 7 4 9 4 4 5 9 \
     2 3 0 7 8 1 6 4 0 6 2 8 6 2 0 8 9 9 8 6 2 8 0 3 4 8 2 5 3 4 2 \
     1 1 7 0 6 7})
(%i5) continuous_freq (a1);
   9  9  27  18  9  27  63  36  81
(%o5)  [[0, -, -, -, -, -, -, -, 9],
       [8, 8, 12, 12, 10, 8, 9, 8, 12, 13]]

discrete_freq (data) [Function]
Counts absolute frequencies in discrete samples, both numeric and categorical. Its unique argument is a list, or 1-dimensional array (as created by make_array).

(%i1) load ("descriptive");
(%i2) s1 : read_list (file_search ("pidigits.data"));
(%i3) discrete_freq (s1);
   [0, 1, 2, 3, 4, 5, 6, 7, 8, 9],
   [8, 8, 12, 12, 10, 8, 9, 8, 12, 13]

The first list gives the sample values and the second their absolute frequencies. Commands ? col and ? transpose should help you to understand the last input.

The argument may be an array.

(%i1) load ("descriptive");
(%i2) s1 : read_list (file_search ("pidigits.data"));
(%i3) a1 : make_array (fixnum, length (s1)) $
(%i4) fillarray (a1, s1);
   {Lisp Array:
    #(3 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 8 4 6 2 6 4 3 8 3 2 7 9 \
     5 0 2 8 8 4 1 9 7 1 6 9 3 9 9 3 7 5 1 0 5 8 2 0 9 7 4 9 4 4 5 9 \
     2 3 0 7 8 1 6 4 0 6 2 8 6 2 0 8 9 9 8 6 2 8 0 3 4 8 2 5 3 4 2 \
     1 1 7 0 6 7})
(%i5) discrete_freq (a1);
   [0, 1, 2, 3, 4, 5, 6, 7, 8, 9],
   [8, 8, 12, 12, 10, 8, 9, 8, 12, 13]
**standardize**

standardize (list)
standardize (matrix)

Subtracts to each element of the list the sample mean and divides the result by the standard deviation. When the input is a matrix, `standardize` subtracts to each row the multivariate mean, and then divides each component by the corresponding standard deviation.

**subsample**

subsample (data_matrix, predicate_function)
subsample (data_matrix, predicate_function, col_num1, col_num2, ...)

This is a sort of variant of the Maxima `submatrix` function. The first argument is the data matrix, the second is a predicate function and optional additional arguments are the numbers of the columns to be taken. Its behaviour is better understood with examples.

These are multivariate records in which the wind speed in the first meteorological station were greater than 18. See that in the lambda expression the i-th component is referred to as v[i].

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"));
(%i3) subsample (s2, lambda([v], v[1] > 18));
    [ 19.38  15.37  15.12  23.09  25.25 ]
    [        ]
    [ 18.29  18.66  19.08  26.08  27.63 ]
    [        ]
    [ 20.25  21.46  19.95  27.71  23.38 ]
    [        ]
    [ 18.97  18.96  14.46  26.38  21.84 ]
```

In the following example, we request only the first, second and fifth components of those records with wind speeds greater or equal than 16 in station number 1 and less than 25 knots in station number 4. The sample contains only data from stations 1, 2 and 5. In this case, the predicate function is defined as an ordinary Maxima function.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"));
(%i3) g(x):= x[1] >= 16 and x[4] < 25$
(%i4) subsample (s2, g, 1, 2, 5);  
    [ 19.38  15.37  25.25 ]
    [        ]
    [ 17.33  14.67  19.58 ]
    [        ]
    [ 16.92  13.21  21.21 ]
    [        ]
    [ 17.25  18.46  23.87 ]
```

Here is an example with the categorical variables of `biomed.data`. We want the records corresponding to those patients in group B who are older than 38 years.
(\%i1) load ("descriptive")$
(\%i2) s3 : read_matrix (file_search ("biomed.data"))$
(\%i3) \(h(u) := u[1] = B \text{ and } u[2] > 38\)$
(\%i4) subsample (s3, h);

\[
\begin{bmatrix}
B & 39 & 28.0 & 102.3 & 17.1 & 146 \\
B & 39 & 21.0 & 92.4 & 10.3 & 197 \\
B & 39 & 23.0 & 111.5 & 10.0 & 133 \\
B & 39 & 26.0 & 92.6 & 12.3 & 196 \\
B & 39 & 25.0 & 98.7 & 10.0 & 174 \\
B & 39 & 21.0 & 93.2 & 5.9 & 181 \\
B & 39 & 18.0 & 95.0 & 11.3 & 66 \\
B & 39 & 39.0 & 88.5 & 7.6 & 168 \\
\end{bmatrix}
\]

Probably, the statistical analysis will involve only the blood measures,

(\%i1) load ("descriptive")$
(\%i2) s3 : read_matrix (file_search ("biomed.data"))$
(\%i3) subsample (s3, lambda([v], v[1] = B and v[2] > 38),
  3, 4, 5, 6);

\[
\begin{bmatrix}
28.0 & 102.3 & 17.1 & 146 \\
21.0 & 92.4 & 10.3 & 197 \\
23.0 & 111.5 & 10.0 & 133 \\
26.0 & 92.6 & 12.3 & 196 \\
25.0 & 98.7 & 10.0 & 174 \\
21.0 & 93.2 & 5.9 & 181 \\
18.0 & 95.0 & 11.3 & 66 \\
39.0 & 88.5 & 7.6 & 168 \\
\end{bmatrix}
\]

This is the multivariate mean of \(s3\),

(\%i1) load ("descriptive")$
(\%i2) s3 : read_matrix (file_search ("biomed.data"))$
Here, the first component is meaningless, since A and B are categorical, the second component is the mean age of individuals in rational form, and the fourth and last values exhibit some strange behaviour. This is because symbol NA is used here to indicate non available data, and the two means are nonsense. A possible solution would be to take out from the matrix those rows with NA symbols, although this deserves some loss of information.

\begin{verbatim}
(%i1) load ("descriptive")$
(%i2) s3 : read_matrix (file_search ("biomed.data"))$
(%i4) mean (subsample (s3, g, 3, 4, 5, 6));
(%o4) [79.4923076923077, 86.2032967032967, 16.93186813186813, 2514
     ----
     13
     [Function]

\end{verbatim}

**transform_sample** (matrix, varlist, exprlist)

Transforms the sample matrix, where each column is called according to varlist, following expressions in exprlist.

Examples:

The second argument assigns names to the three columns. With these names, a list of expressions define the transformation of the sample.

\begin{verbatim}
(%i1) load ("descriptive")$
(%i2) data: matrix([3,2,7],[3,7,2],[8,2,4],[5,2,4]) $
(%i3) transform_sample(data, [a,b,c], [c, a*b, log(a)]);
(%o3)
   [ 7  6  log(3) ]
   [            ]
   [ 2  21 log(3) ]
   [            ]
   [ 4  16 log(8) ]
   [            ]
   [ 4  10 log(5) ]

Add a constant column and remove the third variable.

\begin{verbatim}
(%i1) load ("descriptive")$
(%i2) data: matrix([3,2,7],[3,7,2],[8,2,4],[5,2,4]) $
(%i3) transform_sample(data, [a,b,c], [makelist(1,k,length(data)),a,b]);
\end{verbatim}
Chapter 50: descriptive statistics

50.3 Functions and Variables for descriptive statistics

mean

mean (list)
mean (matrix)

This is the sample mean, defined as

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

Example:

(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) mean (s1);
(%o3) 471

(%i4) %, numer;
(%o4) 4.71

(%i5) s2 : read_matrix (file_search ("wind.data"))$
(%i6) mean (s2);
(%o6) [9.9485, 10.1607, 10.8685, 15.7166, 14.8441]

var

var (list)
var (matrix)

This is the sample variance, defined as

$$\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

Example:

(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) var (s1), numer;
(%o3) 8.425899999999999

See also function var1.
var1

\[ \text{var1}(\text{list}) \]
\[ \text{var1}(\text{matrix}) \]

This is the sample variance, defined as
\[ \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \]

Example:
\begin{verbatim}
(%i1) load ("descriptive")
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) var1 (s1), numer;
(%o3) 8.5110101010101
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) var1 (s2);
(%o5) [17.39586540404041, 15.13912778787879, 15.63204924242424,
     32.50152569696971, 24.66977392929294]
\end{verbatim}

See also function \texttt{var}.

std

\[ \text{std}(\text{list}) \]
\[ \text{std}(\text{matrix}) \]

This is the square root of the function \texttt{var}, the variance with denominator \( n \).

Example:
\begin{verbatim}
(%i1) load ("descriptive")
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) std (s1), numer;
(%o3) 2.902740084816414
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) std (s2);
(%o5) [4.149928523480858, 3.871399812729241, 3.933920277534866,
     5.672434260526957, 4.941970881136392]
\end{verbatim}

See also functions \texttt{var} and \texttt{std1}.

std1

\[ \text{std1}(\text{list}) \]
\[ \text{std1}(\text{matrix}) \]

This is the square root of the function \texttt{var1}, the variance with denominator \( n - 1 \).

Example:
\begin{verbatim}
(%i1) load ("descriptive")
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) std1 (s1), numer;
(%o3) 2.917363553109228
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) std1 (s2);
(%o5) [4.170835096721089, 3.89090320978032, 3.953738641137555,
     5.701010936401517, 4.966867617451963]
\end{verbatim}

See also functions \texttt{var1} and \texttt{std}.
noncentral_moment

noncentral_moment (list, k)
noncentral_moment (matrix, k)

The noncentral moment of order \(k\), defined as

\[
\frac{1}{n} \sum_{i=1}^{n} x_i^k
\]

Example:

\[
\begin{align*}
&\text{(i1) load ("descriptive")$} \\
&\text{(i2) s1 : read_list (file_search ("pidigits.data"))}$ \\
&\text{(i3) noncentral_moment (s1, 1), numer; /* the mean */}$ \\
&\text{(o3) 4.71}$ \\
&\text{(i5) s2 : read_matrix (file_search ("wind.data"))}$ \\
&\text{(i6) noncentral_moment (s2, 5);}$ \\
&\text{(o6) [319793.8724761505, 320532.1923892463,}$ \\
&\text{391249.5621381556, 2502278.205988911, 1691881.797742255]}$
\end{align*}
\]

See also function central_moment.

central_moment

central_moment (list, k)
central_moment (matrix, k)

The central moment of order \(k\), defined as

\[
\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^k
\]

Example:

\[
\begin{align*}
&\text{(i1) load ("descriptive")$} \\
&\text{(i2) s1 : read_list (file_search ("pidigits.data"))}$ \\
&\text{(i3) central_moment (s1, 2), numer; /* the variance */}$ \\
&\text{(o3) 8.425899999999999}$ \\
&\text{(i5) s2 : read_matrix (file_search ("wind.data"))}$ \\
&\text{(i6) central_moment (s2, 3);}$ \\
&\text{(o6) [11.29584771375004, 16.97988248298583, 5.626661952750102,}$ \\
&\text{37.5986572057918, 25.85981904394192]}$
\end{align*}
\]

See also functions central_moment and mean.

cv

cv (list)
cv (matrix)

The variation coefficient is the quotient between the sample standard deviation \((std)\) and the mean.

\[
\text{cv}(x) = \frac{\text{std}(x)}{\text{mean}(x)}
\]

Example:

\[
\begin{align*}
&\text{(i1) load ("descriptive")$} \\
&\text{(i2) s1 : read_list (file_search ("pidigits.data"))}$ \\
&\text{(i3) cv (s1), numer;}$ \\
&\text{(o3) .6193977819764815}$
\end{align*}
\]
(i4) s2 : read_matrix (file_search ("wind.data"));
(%i5) cv (s2);
(%o5) [.4192426091090204, .3829365309260502, 0.363779605385983, 
.3627381836021478, .3346021393989506]

See also functions std and mean.

smin

smin (list)
smin (matrix)
This is the minimum value of the sample list. When the argument is a matrix, smin
returns a list containing the minimum values of the columns, which are associated to
statistical variables.

(i1) load ("descriptive");
(i2) s1 : read_list (file_search ("pidigits.data"));
(i3) smin (s1);
(%o3) 0
(i4) s2 : read_matrix (file_search ("wind.data"));
(i5) smin (s2);
(%o5) [0.58, 0.5, 2.67, 5.25, 5.17]

See also function smax.

smax

smax (list)
smax (matrix)
This is the maximum value of the sample list. When the argument is a matrix, smax
returns a list containing the maximum values of the columns, which are associated to
statistical variables.

(i1) load ("descriptive");
(i2) s1 : read_list (file_search ("pidigits.data"));
(i3) smax (s1);
(%o3) 9
(i4) s2 : read_matrix (file_search ("wind.data"));
(i5) smax (s2);
(%o5) [20.25, 21.46, 20.04, 29.63, 27.63]

See also function smin.

range

range (list)
rngae (matrix)
The range is the difference between the extreme values.

Example:

(i1) load ("descriptive");
(i2) s1 : read_list (file_search ("pidigits.data"));
(i3) range (s1);
(%o3) 9
(i4) s2 : read_matrix (file_search ("wind.data"));
quantile

quantile(list, p)
quantile(matrix, p)

This is the p-quantile, with p a number in [0, 1], of the sample list. Although there are several definitions for the sample quantile (Hyndman, R. J., Fan, Y. (1996) Sample quantiles in statistical packages. American Statistician, 50, 361-365), the one based on linear interpolation is implemented in package Chapter 50 [descriptive-pkg], page 725.

Example:

```plaintext
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) /* 1st and 3rd quartiles */
   [quantile (s1, 1/4), quantile (s1, 3/4)], numer;
(%o3) [2.0, 7.25]

(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) quantile (s2, 1/4);
(%o5) [7.2575, 7.477500000000001, 7.82, 11.28, 11.48]
```

median

median(list)
median(matrix)

Once the sample is ordered, if the sample size is odd the median is the central value, otherwise it is the mean of the two central values.

Example:

```plaintext
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) median (s1); 9
(%o3) 9

(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) median (s2); 2
(%o5) [10.06, 9.855, 10.73, 15.48, 14.105]
```

The median is the 1/2-quantile.

See also function quantile.

qrange

qrange(list)
qrange(matrix)

The interquartile range is the difference between the third and first quartiles, quantile(list,3/4) - quantile(list,1/4),

```plaintext
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
```
(%i3) qrange (s1);
    21
(%o3) --
    4

(%i4) s2 : read_matrix (file_search ("wind.data"));
(%i5) qrange (s2);
(%o5) [5.385, 5.572499999999998, 6.022500000000001, 8.729999999999999, 6.649999999999999]

See also function quantile.

mean_deviation [Function]

mean_deviation (list)
mean_deviation (matrix)

The mean deviation, defined as
\[ \frac{1}{n} \sum_{i=1}^{n} |x_i - \bar{x}| \]

Example:

(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"));
(%i3) mean_deviation (s1);
    51
(%o3) --
    20

(%i4) s2 : read_matrix (file_search ("wind.data"));
(%i5) mean_deviation (s2);
(%o5) [3.287959999999999, 3.075342, 3.23907, 4.715664000000001, 4.028546000000002]

See also function mean.

median_deviation [Function]

median_deviation (list)
median_deviation (matrix)

The median deviation, defined as
\[ \frac{1}{n} \sum_{i=1}^{n} |x_i - med| \]

where med is the median of list.

Example:

(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"));
(%i3) median_deviation (s1);
    5
(%o3) -
    2
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(\%i4) s2 : read_matrix (file_search ("wind.data"))$
(\%i5) median_deviation (s2);
(\%o5) [2.75, 2.755, 3.08, 4.315, 3.31]

See also function mean.

**harmonic_mean**

harmonic_mean (list)
harmonic_mean (matrix)

The harmonic mean, defined as

\[
\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}
\]

Example:

(\%i1) load ("descriptive")$
(\%i2) y : [5, 7, 2, 5, 9, 5, 6, 4, 9, 2, 4, 2, 5]$
(\%i3) harmonic_mean (y), numer;
(\%o3) 3.901858027632205
(\%i4) s2 : read_matrix (file_search ("wind.data"))$
(\%i5) harmonic_mean (s2);
(\%o5) [6.948015590052786, 7.391967752360356, 9.056658197151745, 13.4459028193692, 13.01439145898509]

See also functions mean and geometric_mean.

**geometric_mean**

geometric_mean (list)
geometric_mean (matrix)

The geometric mean, defined as

\[
\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}
\]

Example:

(\%i1) load ("descriptive")$
(\%i2) y : [5, 7, 2, 5, 9, 5, 6, 4, 9, 2, 4, 2, 5]$
(\%i3) geometric_mean (y), numer;
(\%o3) 4.454845412337012
(\%i4) s2 : read_matrix (file_search ("wind.data"))$
(\%i5) geometric_mean (s2);
(\%o5) [8.82476274347979, 9.22652604739361, 10.0442675714889, 14.61274126349021, 13.96184163444275]

See also functions mean and harmonic_mean.

**kurtosis**

kurtosis (list)
kurtosis (matrix)

The kurtosis coefficient, defined as

\[
\frac{1}{ns^4} \sum_{i=1}^{n} (x_i - \bar{x})^4 - 3
\]
Example:

(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) kurtosis (s1), numer;
(%o3) - 1.273247946514421
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) kurtosis (s2);
(%o5) [- .2715445622195385, 0.119998784429451,
               - .4275233490482861, - .6405361979019522, - .4952382132352935]

See also functions mean, var and skewness.

skewness  [Function]

  skewness (list)
  skewness (matrix)

  The skewness coefficient, defined as

  \[ \frac{1}{n s^3} \sum_{i=1}^{n} (x_i - \bar{x})^3 \]

Example:

(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) skewness (s1), numer;
(%o3) .009196180476450424
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) skewness (s2);
(%o5) [.1580509020000978, .2926379232061854, .09242174416107717,
               .2059984348148687, .2142520248890831]

See also functions mean, var and kurtosis.

pearson_skewness  [Function]

  pearson_skewness (list)
  pearson_skewness (matrix)

  Pearson’s skewness coefficient, defined as

  \[ \frac{3 (\bar{x} - med)}{s} \]

where med is the median of list.

Example:

(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) pearson_skewness (s1), numer;
(%o3) .2159484029093895
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) pearson_skewness (s2);
(%o5) [- .08019976629211892, .2357036272952649,
               .1050904062491204, .1245042340592368, .4464181795804519]

See also functions mean, var and median.
quartile_skewness [Function]

quartile_skewness (list)
quartile_skewness (matrix)

The quartile skewness coefficient, defined as
\[ \frac{c_3 - 2c_2 + c_1}{c_2 - c_1} \]

where \( c_p \) is the \( p \)-quantile of sample \( list \).

Example:

\[
(%i1) \text{load ("descriptive")}$
(%i2) s1 : \text{read_list (file_search ("pidigits.data"))}$
(%i3) \text{quartile_skewness (s1), numer;}
(%o3) 0.04761904761904762
(%i4) s2 : \text{read_matrix (file_search ("wind.data"))}$
(%i5) \text{quartile_skewness (s2);}
(%o5) [- 0.0408542246982353, 0.1467025572005382,
     0.0336239103362392, 0.03780068728522298,
     0.2105263157894735]

See also function quantile.

km [Function]

km (list, option ...)
km (matrix, option ...)

Kaplan Meier estimator of the survival, or reliability, function \( S(x) = 1 - F(x) \).

Data can be introduced as a list of pairs, or as a two column matrix. The first component is the observed time, and the second component a censoring index (1 = non censored, 0 = right censored).

The optional argument is the name of the variable in the returned expression, which is \( x \) by default.

Examples:

Sample as a list of pairs.

\[
(%i1) \text{load ("descriptive")}$
(%i2) S: \text{km([[2,1], [3,1], [5,0], [8,1]]);}
    \text{charfun((3 <= x) and (x < 8))}
(%o2) \text{charfun(x < 0) + } \frac{2}{3} \text{charfun((2 <= x) and (x < 3))}
    + \frac{4}{3} \text{charfun((0 <= x) and (x < 2))}

(%i3) \text{load ("draw")}$
(%i4) \text{draw2d(line_width = 3, grid = true,}
     \text{explicit(S, x, -0.1, 10))}$

Estimate survival probabilities.

\[
(%i11) \text{load ("descriptive")}$
\]
\begin{verbatim}
(%i2) S(t):= 'km([[2,1], [3,1], [5,0], [8,1]], t)) $
(%i3) S(6);
1
(%o3) 1
2

cdf_empirical [Function]

cdf_empirical (list, option ...)
cdf_empirical (matrix, option ...)

Empirical distribution function $F(x)$.

Data can be introduced as a list of numbers, or as an one column matrix.

The optional argument is the name of the variable in the returned expression, which is $x$ by default.

Example:

Empirical distribution function.

(%i1) load ("descriptive")$
(%i2) F(x):= ' (cdf_empirical([1,3,3,5,7,7,7,8,9]));
(%o2) F(x) := (charfun(x >= 9) + charfun(x >= 8)
 + 3 charfun(x >= 7) + charfun(x >= 5)
 + 2 charfun(x >= 3) + charfun(x >= 1))/9
(%i3) F(6);
4
(%o3) 4
9

(%i4) load("draw")$
(%i5) draw2d(
   line_width = 3,
   grid = true,
   explicit(F(z), z, -2, 12)) $

cov (matrix) [Function]

The covariance matrix of the multivariate sample, defined as

\[
S = \frac{1}{n} \sum_{j=1}^{n} (X_j - \bar{X}) (X_j - \bar{X})'
\]

where $X_j$ is the $j$-th row of the sample matrix.

Example:

(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"));
(%i3) fpprintprec : 7$ /* change precision for pretty output */
\end{verbatim}
cov1 (matrix)  

The covariance matrix of the multivariate sample, defined as

\[
\frac{1}{n - 1} \sum_{j=1}^{n} (X_j - \bar{X}) (X_j - \bar{X})'
\]

where \(X_j\) is the \(j\)-th row of the sample matrix.

Example:

```
(%i1) load ("descriptive")
(%i2) s2 : read_matrix (file_search ("wind.data"))
(%i3) fpprintprec : 7$ /* change precision for pretty output */
(%i4) cov1 (s2);
(%o4) [ 17.39587 13.75567 14.51734 19.59216 15.5774 ]
[               ]
[ 13.75567 15.13913 13.43887 15.31145 15.12232 ]
[               ]
[ 19.59216 15.31145 17.50044 32.50153 20.65338 ]
[               ]
[ 15.5774 15.12232 16.34516 20.65338 24.66977 ]
```

See also function cov1.

global_variances  

Function `global_variances` returns a list of global variance measures:

- **total variance**: \(\text{trace}(S_1)\),
- **mean variance**: \(\text{trace}(S_1)/p\),
- **generalized variance**: \(\text{determinant}(S_1)\),
- **generalized standard deviation**: \(\sqrt{\text{determinant}(S_1)}\),
- **effective standard deviation**: \(\text{determinant}(S_1)^{(1/(2*p))}\).
where \( p \) is the dimension of the multivariate random variable and \( S \) is the covariance matrix returned by \texttt{cov1}.

Option:
- \texttt{'data}, default \texttt{'true}, indicates whether the input matrix contains the sample data, in which case the covariance matrix \texttt{cov1} must be calculated, or not, and then the covariance matrix (symmetric) must be given, instead of the data.

Example:

\begin{verbatim}
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) global_variances (s2);
(%o3) [105.338342060606, 21.06766841212119, 12874.34690469686, 113.4651792608501, 6.636590811800795, 2.576158149609762]
\end{verbatim}

Calculate the \texttt{global_variances} from the covariance matrix.

\begin{verbatim}
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) s : cov1 (s2)$
(%i4) global_variances (s, data=false);
(%o4) [105.338342060606, 21.06766841212119, 12874.34690469686, 113.4651792608501, 6.636590811800795, 2.576158149609762]
\end{verbatim}

See also \texttt{cov} and \texttt{cov1}.

cor

\texttt{cor (matrix)}

\texttt{cor (matrix, logical_value)}

The correlation matrix of the multivariate sample.

Option:
- \texttt{'data}, default \texttt{'true}, indicates whether the input matrix contains the sample data, in which case the covariance matrix \texttt{cov1} must be calculated, or not, and then the covariance matrix (symmetric) must be given, instead of the data.

Example:

\begin{verbatim}
(%i1) load ("descriptive")$
(%i2) fpprintprec : 7$
(%i3) s2 : read_matrix (file_search ("wind.data"))$
(%i4) cor (s2);
(%o4) [
  [ 1.0 0.8476339 0.8803515 0.8239624 0.7519506 ]
  [ 0.8476339 1.0 0.8735834 0.6902622 0.782502 ]
  [ 0.8803515 0.8735834 1.0 0.7764065 0.8323358 ]
  [ 0.8239624 0.6902622 0.7764065 1.0 0.7293848 ]
  [ 0.7519506 0.782502 0.8323358 0.7293848 1.0 ]
\end{verbatim}
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Calculate the correlation matrix from the covariance matrix.

(%i1) load ("descriptive")
(%i2) fpprintprec : 7$
(%i3) s2 : read_matrix (file_search ("wind.data"))$
(%i4) s : cov1 (s2)$
(%i5) cor (s, data=false); /* this is faster */
    [ 1.0  .8476339  .8803515  .8239624  .7519506 ]
    [ ]
    [ .8476339  1.0  .8735834  .6902622  .782502  ]
    [ ]
    [ ]
(%o5) [ .8803515  .8735834  1.0  .7764065  .8323358 ]
    [ ]
    [ .8239624  .6902622  .7764065  1.0  .7293848 ]
    [ ]
    [ .7519506  .782502  .8323358  .7293848  1.0  ]

See also cov and cov1.

list_correlations

list_correlations (matrix)
list_correlations (matrix, options ...)

Function list_correlations returns a list of correlation measures:

- **precision matrix**: the inverse of the covariance matrix \( S_1 \),
  \[
  S_1^{-1} = (s^{ij})_{i,j=1,2,...,p}
  \]

- **multiple correlation vector**: \( (R^2_1, R^2_2, ..., R^2_p) \), with
  \[
  R^2_i = 1 - \frac{1}{s^{ii} s_{ii}}
  \]
  being an indicator of the goodness of fit of the linear multivariate regression model on \( X_i \) when the rest of variables are used as regressors.

- **partial correlation matrix**: with element \((i,j)\) being
  \[
  r_{ij,rest} = -\frac{s^{ij}}{\sqrt{s^{ii} s^{jj}}}
  \]

Option:

- `'data`, default `'true`, indicates whether the input matrix contains the sample data, in which case the covariance matrix `cov1` must be calculated, or not, and then the covariance matrix (symmetric) must be given, instead of the data.

Example:

(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) z : list_correlations (s2)$
(%i4) fpprintprec : 5$ /* for pretty output */
(%i5) z[1]; /* precision matrix */
[ .38486  - .13856  - .15626  - .10239  .031179 ]
[ ]
[ - .13856  .34107  - .15233  .038447  - .052842 ]
[ ]
(%o5) [ - .15626  - .15233  .47296  - .024816  - .10054 ]
[ ]
[ - .10239  .038447  - .024816  .10937  - .034033 ]
[ ]
[ .031179  - .052842  - .10054  - .034033  .14834 ]
(%i6) z[2]; /* multiple correlation vector */
(%o6) [.85063, .80634, .86474, .71867, .72675]
(%i7) z[3]; /* partial correlation matrix */
[ - 1.0  .38244  .36627  .49908  - .13049 ]
[ ]
[ .38244  - 1.0  .37927  - .19907  .23492 ]
[ ]
(%o7) [ .36627  .37927  - 1.0  .10911  .37956 ]
[ ]
[ .49908  - .19907  .10911  - 1.0  .26719 ]
[ ]
[ - .13049  .23492  .37956  .26719  - 1.0 ]

See also `cov` and `cov1`.

principal_components

`principal_components (matrix)`
`principal_components (matrix, options ...)`
Calculates the principal components of a multivariate sample. Principal components are used in multivariate statistical analysis to reduce the dimensionality of the sample. 

Option:
- `data`, default `true`, indicates whether the input matrix contains the sample data, in which case the covariance matrix `cov1` must be calculated, or not, and then the covariance matrix (symmetric) must be given, instead of the data.

The output of function `principal_components` is a list with the following results:
- variances of the principal components,
- percentage of total variance explained by each principal component,
- rotation matrix.

Examples:
In this sample, the first component explains 83.13 per cent of total variance.

(%i11) load ("descriptive")$
(%i12) s2 : read_matrix (file_search ("wind.data"))$
(%i13) fpprintprec:4$
(%i14) res: principal_components(s2); 
0 errors, 0 warnings
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(%o4) [[87.57, 8.753, 5.515, 1.889, 1.613],
[83.13, 8.31, 5.235, 1.793, 1.531],
[.4149, .03379, -.4757, -.581, -.5126],
[0.369, -.3657, -.4298, .7237, -.1469],
[.3959, -.2178, -.2181, -.2749, .8201],
[.5548, .7744, .1857, .2319, .06498],
[.4765, -.4669, 0.712, -.09605, -.1969]]
(%i5) /* accumulated percentages */
block([ap: copy(res[2])],
  for k:2 thru length(ap) do ap[k]: ap[k]+ap[k-1],
ap);
(%o5) [83.13, 91.44, 96.68, 98.47, 100.0]
(%i6) /* sample dimension */
p: length(first(res));
(%o6) 5
(%i7) /* plot percentages to select number of
  principal components for further work */
draw2d(
  fill_density = 0.2,
  apply(bars, makelist([k, res[2][k], 1/2], k, p)),
  pointsJoined = true,
  point_type = filled_circle,
  point_size = 3,
  points(makelist([k, res[2][k]], k, p)),
  xlabel = "Variances",
  ylabel = "Percentages",
  xtics = setify(makelist([concat("PC",k),k], k, p)))$ In case de covariance matrix is known, it can be passed to the function, but option
data=false must be used.

(%i1) load("descriptive")$
(%i2) S: matrix([[1,-2,0],[2,5,0],[0,0,2]]);
(%o2) [[1, -2, 0],
[-2, 5, 0],
[0, 0, 2]]
(%i3) fpprintprec:4$
(%i4) /* the argument is a covariance matrix */
res: principal_components(S, data=false);
0 errors, 0 warnings
  [ - .3827, 0.0, .9239 ]
  [ ]
(%o4) \[
[[5.828, 2.0, .1716], [72.86, 25.0, 2.145], [.9239, 0.0, .3827]]
[ ]
[ 0.0 1.0 0.0 ]
(%i5) /* transformation to get the principal components
from original records */
    matrix([a1,b2,c3],[a2,b2,c2]).last(res);
[ .9239 b2 - .3827 a1 1.0 c3 .3827 b2 + .9239 a1 ]
(%o5) [ ]
[ .9239 b2 - .3827 a2 1.0 c2 .3827 b2 + .9239 a2 ]

50.4 Functions and Variables for statistical graphs

barsplot (data1, data2, ..., option_1, option_2, ...) [Function]
Plots bars diagrams for discrete statistical variables, both for one or multiple samples.
data can be a list of outcomes representing one sample, or a matrix of m rows and n
columns, representing n samples of size m each.
Available options are:
- box_width (default, 3/4): relative width of rectangles. This value must be in the
  range [0,1].
- grouping (default, clustered): indicates how multiple samples are shown. Valid
  values are: clustered and stacked.
- groups_gap (default, 1): a positive integer number representing the gap between
  two consecutive groups of bars.
- bars_colors (default, []): a list of colors for multiple samples. When there are
  more samples than specified colors, the extra necessary colors are chosen at ran-
dom. See color to learn more about them.
- frequency (default, absolute): indicates the scale of the ordinates. Possible
  values are: absolute, relative, and percent.
- ordering (default, orderlessp): possible values are orderlessp or ordergreatp,
  indicating how statistical outcomes should be ordered on the x-axis.
- sample_keys (default, []): a list with the strings to be used in the legend. When
  the list length is other than 0 or the number of samples, an error message is
  returned.
- start_at (default, 0): indicates where the plot begins to be plotted on the x axis.
- All global draw options, except xtics, which is internally assigned by barsplot.
  If you want to set your own values for this option or want to build complex
  scenes, make use of barsplot_description. See example below.
- The following local Chapter 53 [draw-pkg], page 815, options: key, color_draw,
  fill_color, fill_density and line_width. See also barsplot.
There is also a function wxbarsplot for creating embedded histograms in interfaces
wxMaxima and iMaxima. barsplot in a multiplot context.
Examples:
Univariate sample in matrix form. Absolute frequencies.
(%i11) load ("descriptive")$
Two samples of different sizes, with relative frequencies and user declared colors.

Four non numeric samples of equal size.

Stacked bars.

For bars diagrams related options, see barsplot of package Chapter 53 [draw-pkg], page 815. See also functions histogram and piechart.
**barsplot_description (**)  

Function **barsplot_description** creates a graphic object suitable for creating complex scenes, together with other graphic objects.

Example: **barsplot** in a multiplot context.

```
(%i1) load ("descriptive")$
(%i2) l1:makelist(random(10),k,1,50)$
(%i3) l2:makelist(random(10),k,1,100)$
(%i4) bp1 : barsplot_description(
   l1,
   box_width = 1,
   fill_density = 0.5,
   bars_colors = [blue],
   frequency = relative)$
(%i5) bp2 : barsplot_description(
   l2,
   box_width = 1,
   fill_density = 0.5,
   bars_colors = [red],
   frequency = relative)$
(%i6) draw(gr2d(bp1), gr2d(bp2))$
```

**boxplot (**)  

**boxplot** is a function for drawing box-and-whisker diagrams. Argument **data** can be a list, which is not of great interest, since these diagrams are mainly used for comparing different samples, or a matrix, so it is possible to compare two or more components of a multivariate statistical variable. But it is also allowed **data** to be a list of samples with possible different sample sizes, in fact this is the only function in package **descriptive** that admits this type of data structure.

The box is plotted from the first quartile to the third, with an horizontal segment situated at the second quartile or median. By default, lower and upper whiskers are plotted at the minimum and maximum values, respectively. Option **range** can be used to indicate that values greater than \( \text{quantile}(x, 3/4) + \text{range} \times (\text{quantile}(x, 3/4) - \text{quantile}(x, 1/4)) \) or less than \( \text{quantile}(x, 1/4) - \text{range} \times (\text{quantile}(x, 3/4) - \text{quantile}(x, 1/4)) \) must be considered as outliers, in which case they are plotted as isolated points, and the whiskers are located at the extremes of the rest of the sample.

Available options are:

- **box_width** (default, 3/4): relative width of boxes. This value must be in the range \([0, 1]\).
- **box_orientation** (default, vertical): possible values: vertical and horizontal.
- **range** (default, inf): positive coefficient of the interquartilic range to set outliers boundaries.
- **outliers_size** (default, 1): circle size for isolated outliers.
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- All `draw` options, except `points_joined`, `point_size`, `point_type`, `xtics`, `ytics`, `xrange`, and `yrange`, which are internally assigned by `boxplot`. If you want to set your own values for this options or want to build complex scenes, make use of `boxplot_description`.
- The following local `draw` options: `key`, `color`, and `line_width`.

There is also a function `wxboxplot` for creating embedded histograms in interfaces `wxMaxima` and `iMaxima`.

Examples:

Box-and-whisker diagram from a multivariate sample.

```maxima
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix(file_search("wind.data"))$
(%i3) boxplot(s2,
              box_width = 0.2,
              title = "Windspeed in knots",
              xlabel = "Stations",
              color = red,
              line_width = 2)$
```

Box-and-whisker diagram from three samples of different sizes.

```maxima
(%i1) load ("descriptive")$
(%i2) A : 
   [[6, 4, 6, 2, 4, 8, 6, 4, 6, 4, 3, 2],
    [8, 10, 7, 9, 12, 8, 10],
    [16, 13, 17, 12, 11, 18, 13, 18, 14, 12]]$
(%i3) boxplot (A, box_orientation = horizontal)$
```

Option `range` can be used to handle outliers.

```maxima
(%i1) load ("descriptive")$
(%i2) B : [[7, 15, 5, 8, 6, 5, 7, 3, 1],
    [10, 8, 12, 8, 11, 9, 20],
    [23, 17, 19, 7, 22, 19]]$
(%i3) boxplot (B, range=1)$
(%i4) boxplot (B, range=1.5, box_orientation = horizontal)$
(%i5) draw2d(
    boxplot_description (B,
    range = 1.5,
    line_width = 3,
    outliers_size = 2,
    color = red,
    background_color = light_gray),
    xtics = {{"Low",1},{"Medium",2},{"High",3}})$
```

`boxplot_description (...)` [Function]

Function `boxplot_description` creates a graphic object suitable for creating complex scenes, together with other graphic objects.
histogram

histogram (list)
histogram (list, option_1, option_2, ...)
histogram (one_column_matrix)
histogram (one_column_matrix, option_1, option_2, ...)
histogram (one_row_matrix)
histogram (one_row_matrix, option_1, option_2, ...)

This function plots an histogram from a continuous sample. Sample data must be stored in a list of numbers or an one dimensional matrix.

Available options are:

- **nclasses** (default, 10): number of classes of the histogram, or a list indicating the limits of the classes and the number of them, or only the limits. This option also accepts bounds for varying bin widths, or a symbol with the name of one of the three optimal algorithms available for the number of classes: 'fd (Freedman, D. and Diaconis, P. (1981) On the histogram as a density estimator: L_2 theory. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 57, 453-476.), 'scott (Scott, D. W. (1979) On optimal and data-based histograms. Biometrika 66, 605-610.), and 'sturges (Sturges, H. A. (1926) The choice of a class interval. Journal of the American Statistical Association 21, 65-66).

- **frequency** (default, absolute): indicates the scale of the ordinates. Possible values are: absolute, relative, percent, and density. With density, the histogram area has a total area of one.

- **htics** (default, auto): format of the histogram tics. Possible values are: auto, endpoints, intervals, or a list of labels.

- All global draw options, except xrange, yrange, and xtics, which are internally assigned by histogram. If you want to set your own values for these options, make use of histogram_description. See examples below.

- The following local Chapter 53 [draw-pkg], page 815, options: key, color, fill_color, fill_density and line_width. See also barsplot.

There is also a function wxhistogram for creating embedded histograms in interfaces wxMaxima and iMaxima.

Examples:

A simple with eight classes:

```plaintext
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) histogram (s1,
   nclasses = 8,
   title = "pi digits",
   xlabel = "digits",
   ylabel = "Absolute frequency",
   fill_color = grey,
   fill_density = 0.6)$
```

Setting the limits of the histogram to -2 and 12, with 3 classes. Also, we introduce predefined tics:
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(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) histogram (s1,
    nclasses = [-2,12,3],
    htics = ["A", "B", "C"],
    terminal = png,
    fill_color = "#23afa0",
    fill_density = 0.6)$

Bounds for varying bin widths.

(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) histogram (s1, nclasses = {0,3,6,7,11})$

Freedman - Diaconis robust method for optimal search of the number of classes.

(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) histogram (s1, nclasses = fd) $

Function \texttt{histogram\_description} creates a graphic object suitable for creating complex scenes, together with other graphic objects. We make use of \texttt{histogram\_description} for setting the \texttt{xrange} and adding an explicit curve into the scene:

(%i1) load ("descriptive")$
(%i2) ( load("distrib"),
    m: 14, s: 2,
    s2: random_normal(m, s, 1000) )$
(%i3) draw2d(
    grid = true,
    xrange = [5, 25],
    histogram\_description (s2,
        nclasses = 9,
        frequency = density,
        fill\_density = 0.5),
    explicit(pdf_normal(x,m,s), x, m - 3*s, m + 3* s))$

\texttt{piechart} \hfill [Function]

\texttt{piechart (list)}
\texttt{piechart (list, option\_1, option\_2, \ldots)}
\texttt{piechart (one\_column\_matrix)}
\texttt{piechart (one\_column\_matrix, option\_1, option\_2, \ldots)}
\texttt{piechart (one\_row\_matrix)}
\texttt{piechart (one\_row\_matrix, option\_1, option\_2, \ldots)}

Similar to \texttt{barsplot}, but plots sectors instead of rectangles.

Available options are:
• **sector_colors** (default, []): a list of colors for sectors. When there are more sectors than specified colors, the extra necessary colors are chosen at random. See **color** to learn more about them.

• **pie_center** (default, [0,0]): diagram’s center.

• **pie_radius** (default, 1): diagram’s radius.

• All global draw options, except **key**, which is internally assigned by piechart. If you want to set your own values for this option or want to build complex scenes, make use of **piechart_description**.

• The following local draw options: **key**, **color**, **fill_density** and **line_width**. See also **ellipse**.

There is also a function **wxpiechart** for creating embedded histograms in interfaces wxMaxima and iMaxima.

Example:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) piechart(
   s1,
   xrange  = [-1.1, 1.3],
   yrange  = [-1.1, 1.1],
   title   = "Digit frequencies in pi")$
```

See also function **barsplot**.

---

### Function piechart_description (...)

Function **piechart_description** creates a graphic object suitable for creating complex scenes, together with other graphic objects.

### Function scatterplot

plots scatter diagrams both for univariate (list) and multivariate (matrix) samples.

Available options are the same admitted by **histogram**.

There is also a function **wxscatterplot** for creating embedded histograms in interfaces wxMaxima and iMaxima.

Examples:

Univariate scatter diagram from a simulated Gaussian sample.

```
(%i1) load ("descriptive")$
(%i2) load ("distrib")$
(%i3) scatterplot(
   random_normal(0,1,200),
   xaxis    = true,
   point_size = 2,
   dimensions = [600,150])$
```
Two dimensional scatter plot.

(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) scatterplot(
    submatrix(s2, 1,2,3),
    title = "Data from stations #4 and #5",
    point_type = diamant,
    point_size = 2,
    color = blue)$

Three dimensional scatter plot.

(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) scatterplot(submatrix (s2, 1,2), nclasses=4)$

Five dimensional scatter plot, with five classes histograms.

(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) scatterplot(
    s2,
    nclasses = 5,
    frequency = relative,
    fill_color = blue,
    fill_density = 0.3,
    xtics = 5)$

For plotting isolated or line-joined points in two and three dimensions, see points. See also histogram.

scatterplot_description (...) [Function]

Function scatterplot_description creates a graphic object suitable for creating complex scenes, together with other graphic objects.

starplot (data1, data2, ..., option_1, option_2, ...)[Function]
Plots star diagrams for discrete statistical variables, both for one or multiple samples.

data can be a list of outcomes representing one sample, or a matrix of m rows and n columns, representing n samples of size m each.

Available options are:

- **stars_colors** (default, []): a list of colors for multiple samples. When there are more samples than specified colors, the extra necessary colors are chosen at random. See color to learn more about them.

- **frequency** (default, absolute): indicates the scale of the radii. Possible values are: absolute and relative.

- **ordering** (default, orderlessp): possible values are orderlessp or ordergreatp, indicating how statistical outcomes should be ordered.

- **sample_keys** (default, []): a list with the strings to be used in the legend. When the list length is other than 0 or the number of samples, an error message is returned.
• **star_center** (default, \([0,0]\)) : diagram’s center.

• **star_radius** (default, 1) : diagram’s radius.

• All global **draw** options, except **pointsJoined**, **point_type**, and **key**, which are internally assigned by **starplot**. If you want to set your own values for this options or want to build complex scenes, make use of **starplot_description**.

• The following local **draw** option: **line_width**.

There is also a function **wxstarplot** for creating embedded histograms in interfaces wxMaxima and iMaxima.

Example:

Plot based on absolute frequencies. Location and radius defined by the user.

```maxima
(%i1) load ("descriptive")$
(%i2) l1: makelist(random(10),k,1,50)$
(%i3) l2: makelist(random(10),k,1,200)$
(%i4) starplot(
   l1, l2,
   stars_colors = [blue,red],
   sample_keys = ["1st sample", "2nd sample"],
   star_center = [1,2],
   star_radius = 4,
   proportional_axes = xy,
   line_width = 2 )$
```

**starplot_description** (...)  
Function **starplot_description** creates a graphic object suitable for creating complex scenes, together with other graphic objects.

**stemplot**  

```maxima
stemplot (data)
stemplot (data, option)
```

Plots stem and leaf diagrams.

Unique available option is:

• **leaf_unit** (default, 1) : indicates the unit of the leaves; must be a power of 10.

Example:

```maxima
(%i1) load ("descriptive")$
(%i2) load("distrib")$
```
(%%i3) stemplot(
    random_normal(15, 6, 100),
    leaf_unit = 0.1);
-5|4
0|37
1|7
3|6
4|4
5|4
6|57
7|0149
8|3
9|1334588
10|07888
11|0114467789
12|1256889
13|24778
14|047
15|223458
16|4
17|11557
18|000247
19|4467799
20|00
21|1
22|2335
23|01457
24|12356
25|455
27|79
key: 6|3 = 6.3
(%%o3) done
51 diag

51.1 Functions and Variables for diag

diag (lm) [Function]
Constructs a matrix that is the block sum of the elements of lm. The elements of lm are assumed to be matrices; if an element is scalar, it treated as a 1 by 1 matrix.
The resulting matrix will be square if each of the elements of lm is square.
Example:
(\%i1) load("diag")$
(\%i2) a1:matrix([1,2,3],[0,4,5],[0,0,6])$
(\%i3) a2:matrix([1,1],[1,0])$
(\%i4) diag([a1,x,a2]);

\[
\begin{bmatrix}
1 & 2 & 3 & 0 & 0 & 0 \\
0 & 4 & 5 & 0 & 0 & 0 \\
0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
(\%o4)

(\%i5) diag ([matrix([1,2]), 3]);

\[
\begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 3 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
(\%o5)

To use this function write first load("diag").

JF (lambda,n) [Function]
Returns the Jordan cell of order n with eigenvalue lambda.
Example:
(\%i1) load("diag")$
(\%i2) JF(2,5);

\[
\begin{bmatrix}
2 & 1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 \\
\end{bmatrix}
\]
(\%o2)
To use this function write first \texttt{load("diag")}.

\textbf{jordan (mat)}

Returns the Jordan form of matrix \texttt{mat}, encoded as a list in a particular format. To get the corresponding matrix, call the function \texttt{dispJordan} using the output of \texttt{jordan} as the argument.

The elements of the returned list are themselves lists. The first element of each is an eigenvalue of \texttt{mat}. The remaining elements are positive integers which are the lengths of the Jordan blocks for this eigenvalue. These integers are listed in decreasing order. Eigenvalues are not repeated.

The functions \texttt{dispJordan}, \texttt{minimalPoly} and \texttt{ModeMatrix} expect the output of a call to \texttt{jordan} as an argument. If you construct this argument by hand, rather than by calling \texttt{jordan}, you must ensure that each eigenvalue only appears once and that the block sizes are listed in decreasing order, otherwise the functions might give incorrect answers.

Example:

\begin{verbatim}
(%i1) load("diag")$
(%i2) A: matrix([2,0,0,0,0,0,0,0],
               [1,2,0,0,0,0,0,0],
               [-4,1,2,0,0,0,0,0],
               [2,0,0,2,0,0,0,0],
               [-7,2,0,0,2,0,0,0],
               [9,0,-2,0,1,2,0,0],
               [-34,7,1,-2,-1,1,2,0],
               [145,-17,-16,3,9,-2,0,3])$
(%i3) jordan (A);
(%o3) \[
[2, 3, 3, 1], [3, 1]
\]
(%i4) dispJordan (%);
\[
[2 1 0 0 0 0 0 0 ]
[ ]
[0 2 1 0 0 0 0 0 ]
[ ]
[0 0 2 0 0 0 0 0 ]
[ ]
[0 0 0 2 1 0 0 0 ]
[ ]
[0 0 0 0 2 1 0 0 ]
[ ]
[0 0 0 0 0 2 0 0 ]
\]
\end{verbatim}
To use this function write first `load("diag")`. See also `dispJordan` and `minimalPoly`.

dispJordan (l)  [Function]
Returns a matrix in Jordan canonical form (JCF) corresponding to the list of eigenvalues and multiplicities given by `l`. This list should be in the format given by the `jordan` function. See `jordan` for details of this format.
Example:
```lisp
(%i1) load("diag")$
(%i2) b1:matrix([0,0,1,1,1],
    [0,0,0,1,1],
    [0,0,0,0,1],
    [0,0,0,0,0],
    [0,0,0,0,0])$
(%i3) jordan(b1);
(%o3) [[0, 3, 2]]
(%i4) dispJordan(%);
[ 0 1 0 0 0 ]
[ ]
[ 0 0 1 0 0 ]
[ ]
[ 0 0 0 0 0 ]
[ ]
[ 0 0 0 0 1 ]
[ ]
[ 0 0 0 0 0 ]

To use this function write first `load("diag")`. See also `jordan` and `minimalPoly`.

minimalPoly (l)  [Function]
Returns the minimal polynomial of the matrix whose Jordan form is described by the list `l`. This list should be in the format given by the `jordan` function. See `jordan` for details of this format.
Example:
```lisp
(%i1) load("diag")$
(%i2) a:matrix([2,1,2,0],
    [-2,2,1,2],
    [-2,-1,-1,1],
    [3,1,2,-1])$
(%i3) jordan(a);
(\%i3) minimalPoly(%);
(\%o3) \([[- 1, 1], [1, 3]]\)

(\%i4) (x - 1) (x + 1)
(\%o4) \((x - 1) (x + 1)\)

To use this function write first load("diag"). See also \texttt{jordan} and \texttt{dispJordan}.

\textbf{ModeMatrix} \((A, [\text{jordan\_info}])\)

Returns an invertible matrix \(M\) such that \((Mm_{-1})A.M\) is the Jordan form of \(A\).

To calculate this, Maxima must find the Jordan form of \(A\), which might be quite computationally expensive. If that has already been calculated by a previous call to \texttt{jordan}, pass it as a second argument, \texttt{jordan\_info}. See \texttt{jordan} for details of the required format.

Example:

(\%i1) load("diag")$
(\%i2) A: matrix([2,1,0,1], [-2,2,1,2], [-2,-1,-1,1], [3,1,2,-1])$
(\%i3) M: ModeMatrix (A);
(\%o3) \[
\begin{bmatrix}
1 & -1 & 1 & 1 \\
\emptyset & \emptyset \\
- & - & -1 & 0 & 0 \\
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\
\end{bmatrix}
\]
(\%i4) is ((M^^-1) . A . M = dispJordan (jordan (A)));
(\%o4) true

Note that, in this example, the Jordan form of \(A\) is computed twice. To avoid this, we could have stored the output of \texttt{jordan(A)} in a variable and passed that to both \texttt{ModeMatrix} and \texttt{dispJordan}.

To use this function write first load("diag"). See also \texttt{jordan} and \texttt{dispJordan}.

\textbf{mat\_function} \((f,A)\)

Returns \(f(A)\), where \(f\) is an analytic function and \(A\) a matrix. This computation is based on the Taylor expansion of \(f\). It is not efficient for numerical evaluation, but can give symbolic answers for small matrices.

Example 1:

The exponential of a matrix. We only give the first row of the answer, since the output is rather large.

(\%i1) load("diag")$
(\%i2) A: matrix ([0,1,0], [0,0,1], [-1,-3,-3])$
Chapter 51: diag

Example 2:

Comparison with the Taylor series for the exponential and also comparing $\exp(\%i*A)$ with sine and cosine.

(%i1) load("diag")$
(%i2) A: matrix ([0,1,1,1],
       [0,0,0,1],
       [0,0,0,1],
       [0,0,0,0])$
(%i3) ratsimp (mat_function (exp, t*A));
(%o3) [--------------------, (t + t) %e , --------]
      2 2  

(%i4) minimalPoly (jordan (A));
(%o4) 3

(%i5) ratsimp (ident(4) + t*A + 1/2*(t^2)*A^^2);
(%o5) [ 2 ]
      [ 1 t t t + t ]
      [ ]

(%i6) ratsimp (mat_function (exp, %i*t*A));
(%o6) [ 2 ]
      [ 1 %i t %i t %i t - t ]
      [ ]

(%i7)
Example 3:

Power operations.

(%i1) load("diag")$
(%i2) A: matrix([1,2,0], [0,1,0], [1,0,1])$
(%i3) integer_pow(x) := block ([k], declare (k, integer), x^k)$
(%i4) mat_function (integer_pow, A);

(%i7) ratsimp (mat_function (cos, t*A) + %i*mat_function (sin, t*A));
(%o7)

Example 3:

Power operations.

(%i1) load("diag")$
(%i2) A: matrix([1,2,0], [0,1,0], [1,0,1])$
(%i3) integer_pow(x) := block ([k], declare (k, integer), x^k)$
(%i4) mat_function (integer_pow, A);

(%i7) ratsimp (mat_function (cos, t*A) + %i*mat_function (sin, t*A));
(%o7)

To use this function write first load("diag").
52 distrib

52.1 Introduction to distrib

Package distrib contains a set of functions for making probability computations on both discrete and continuous univariate models.

What follows is a short reminder of basic probabilistic related definitions.

Let $f(x)$ be the density function of an absolute continuous random variable $X$. The distribution function is defined as

$$F(x) = \int_{-\infty}^{x} f(u) \, du$$

which equals the probability $\Pr(X \leq x)$.

The mean value is a localization parameter and is defined as

$$E[X] = \int_{-\infty}^{\infty} x f(x) \, dx$$

The variance is a measure of variation,

$$V[X] = \int_{-\infty}^{\infty} f(x) (x - E[X])^2 \, dx$$

which is a positive real number. The square root of the variance is the standard deviation, $D[X] = \sqrt{V[X]}$, and it is another measure of variation.

The skewness coefficient is a measure of non-symmetry,

$$SK[X] = \frac{\int_{-\infty}^{\infty} f(x) (x - E[X])^3 \, dx}{D[X]^3}$$

And the kurtosis coefficient measures the peakedness of the distribution,

$$KU[X] = \frac{\int_{-\infty}^{\infty} f(x) (x - E[X])^4 \, dx}{D[X]^4} - 3$$

If $X$ is gaussian, $KU[X] = 0$. In fact, both skewness and kurtosis are shape parameters used to measure the non-gaussianity of a distribution.

If the random variable $X$ is discrete, the density, or probability, function $f(x)$ takes positive values within certain countable set of numbers $x_i$, and zero elsewhere. In this case, the distribution function is

$$F(x) = \sum_{x_i \leq x} f(x_i)$$
The mean, variance, standard deviation, skewness coefficient and kurtosis coefficient take the form

\[
E[X] = \sum_i x_i f(x_i),
\]

\[
V[X] = \sum_i f(x_i) (x_i - E[X])^2,
\]

\[
D[X] = \sqrt{V[X]},
\]

\[
SK[X] = \sum_i f(x) \frac{(x - E[X])^3}{D[X]^3} dx,
\]

\[
KU[X] = \sum_i f(x) \frac{(x - E[X])^4}{D[X]^4} dx - 3,
\]

respectively.

There is a naming convention in package \texttt{distrib}. Every function name has two parts, the first one makes reference to the function or parameter we want to calculate,

Functions:

- Density function \texttt{(pdf_*)}
- Distribution function \texttt{(cdf_*)}
- Quantile \texttt{(quantile_*)}
- Mean \texttt{(mean_*)}
- Variance \texttt{(var_*)}
- Standard deviation \texttt{(std_*)}
- Skewness coefficient \texttt{(skewness_*)}
- Kurtosis coefficient \texttt{(kurtosis_*)}
- Random variate \texttt{(random_*)}

The second part is an explicit reference to the probabilistic model,

Continuous distributions:

- Normal \texttt{(*normal)}
- Student \texttt{(*student_t)}
- Chi^2 \texttt{(*chi2)}
- Noncentral Chi^2 \texttt{(*noncentral_chi2)}
- F \texttt{(*f)}
- Exponential \texttt{(*exp)}
- Lognormal \texttt{(*lognormal)}
- Gamma \texttt{(*gamma)}
- Beta \texttt{(*beta)}
- Continuous uniform \texttt{(*continuous_uniform)}
- Logistic \texttt{(*logistic)}
- Pareto \texttt{(*pareto)}
- Weibull \texttt{(*weibull)}
- Rayleigh \texttt{(*rayleigh)}
- Laplace \texttt{(*laplace)}
- Cauchy \texttt{(*cauchy)}
- Gumbel \texttt{(*gumbel)}
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Discrete distributions:
- Binomial (*binomial)
- Poisson (*poisson)
- Bernoulli (*bernoulli)
- Geometric (*geometric)
- Discrete uniform (*discrete_uniform)
- Hypergeometric (*hypergeometric)
- Negative binomial (*negative_binomial)
- Finite discrete (*general_finite_discrete)

For example, pdf_student_t(x,n) is the density function of the Student distribution with n degrees of freedom, std_pareto(a,b) is the standard deviation of the Pareto distribution with parameters a and b and kurtosis_poisson(m) is the kurtosis coefficient of the Poisson distribution with mean m.

In order to make use of package distrib you need first to load it by typing

```
(%i1) load("distrib")$
```

For comments, bugs or suggestions, please contact the author at 'riotorto AT yahoo DOT com'.

### 52.2 Functions and Variables for continuous distributions

Maxima knows the following kinds of continuous distributions.

#### 52.2.1 Normal Random Variable

Normal random variables (also called Gaussian) is denoted by $\text{Normal}(m, s)$ where m is the mean and $s > 0$ is the standard deviation.

**pdf_normal (x,m,s)**

[Function]

Returns the value at $x$ of the density function of a $\text{Normal}(m, s)$ random variable, with $s > 0$. To make use of this function, write first load("distrib").

The pdf is

$$f(x; m, s) = \frac{1}{s\sqrt{2\pi}}e^{-\frac{(x - m)^2}{2s^2}}$$

**cdf_normal (x,m,s)**

[Function]

Returns the value at $x$ of the distribution function of a $\text{Normal}(m, s)$ random variable, with $s > 0$. This function is defined in terms of Maxima’s built-in error function erf.

The cdf can be written analytically:

$$F(x; m, s) = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{x - m}{s\sqrt{2}}\right)$$

```
(%i1) load("distrib")$
(%i2) cdf_normal(x,m,s);
```

```
\left(-\frac{x - m}{\sqrt{2}s}\right) + \frac{1}{2}
\right) + \frac{1}{2}
```

See also \texttt{erf}.

\texttt{quantile_normal (q,m,s)}  \hfill \text{[Function]}

Returns the $q$-quantile of a $\text{Normal}(m, s)$ random variable, with $s > 0$; in other words, this is the inverse of \texttt{cdf_normal}. Argument $q$ must be an element of $[0, 1]$. To make use of this function, write first \texttt{load("distrib")}.

(%i1) load ("distrib")$

(%i2) quantile_normal(95/100,0,1);

\texttt{9}

(%o2) \texttt{sqrt(2) \text{ inverse\_erf}(-)}

\texttt{10}

(%i3) float(%);

(%o3) 1.644853626951472

\texttt{mean_normal (m,s)}  \hfill \text{[Function]}

Returns the mean of a $\text{Normal}(m, s)$ random variable, with $s > 0$. To make use of this function, write first \texttt{load("distrib")}.

The mean is

$$E[X] = m$$

\texttt{var_normal (m,s)}  \hfill \text{[Function]}

Returns the variance of a $\text{Normal}(m, s)$ random variable, with $s > 0$. To make use of this function, write first \texttt{load("distrib")}.

The variance is

$$V[X] = s^2$$

\texttt{std_normal (m,s)}  \hfill \text{[Function]}

Returns the standard deviation of a $\text{Normal}(m, s)$ random variable, with $s > 0$, namely $s$. To make use of this function, write first \texttt{load("distrib")}.

The standard deviation is

$$D[X] = s$$

\texttt{skewness_normal (m,s)}  \hfill \text{[Function]}

Returns the skewness coefficient of a $\text{Normal}(m, s)$ random variable, with $s > 0$. To make use of this function, write first \texttt{load("distrib")}.

The skewness coefficient is

$$SK[X] = 0$$

\texttt{kurtosis_normal (m,s)}  \hfill \text{[Function]}

Returns the kurtosis coefficient of a $\text{Normal}(m, s)$ random variable, with $s > 0$, which is always equal to 0. To make use of this function, write first \texttt{load("distrib")}.

The kurtosis coefficient is

$$KU[X] = 0$$

\texttt{random_normal (m,s)}  \hfill \text{[Function]}

\texttt{random_normal (m,s,n)}

Returns a $\text{Normal}(m, s)$ random variate, with $s > 0$. Calling \texttt{random\_normal} with a third argument $n$, a random sample of size $n$ will be simulated.
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This is an implementation of the Box-Mueller algorithm, as described in Knuth, D.E. (1981) *Seminumerical Algorithms. The Art of Computer Programming*. Addison-Wesley.

To make use of this function, write first load("distrib").

### 52.2.2 Student’s t Random Variable

Student’s t random variable is denoted by \( t(n) \) where \( n \) is the degrees of freedom with \( n > 0 \). If \( Z \) is a Normal\((0,1)\) variable and \( V \) is an independent \( \chi^2 \) random variable with \( n \) degrees of freedom, then

\[
\frac{Z}{\sqrt{V/n}}
\]

has a Student’s \( t \)-distribution with \( n \) degrees of freedom.

**pdf_student_t** \((x,n)\)

Returns the value at \( x \) of the density function of a Student random variable \( t(n) \), with \( n > 0 \) degrees of freedom. To make use of this function, write first load("distrib").

The pdf is

\[
f(x; n) = \left[ \sqrt{n} B \left( \frac{1}{2}, \frac{n}{2} \right) \right]^{-1} \left( 1 + \frac{x^2}{n} \right)^{-\frac{n+1}{2}}
\]

**cdf_student_t** \((x,n)\)

Returns the value at \( x \) of the distribution function of a Student random variable \( t(n) \), with \( n > 0 \) degrees of freedom.

The cdf is

\[
F(x; n) = \begin{cases} 
1 - \frac{1}{2} I_t \left( \frac{n}{2}, \frac{1}{2} \right) & x \geq 0 \\
\frac{1}{2} I_t \left( \frac{n}{2}, \frac{1}{2} \right) & x < 0 
\end{cases}
\]

where \( t = n/(n + x^2) \) and \( I_t(a,b) \) is the \[\text{beta_incomplete_regularized}\], page 309, function.

\begin{verbatim}
(%i1) load ("distrib")$
(%i2) cdf_student_t(1/2, 7/3);
7 1 28
beta_incomplete_regularized(-, -, --)  
6 2 31
---
2
(%i3) float(%);
(%o3) 0.6698450596140415
\end{verbatim}

**quantile_student_t** \((q,n)\)

Returns the \( q \)-quantile of a Student random variable \( t(n) \), with \( n > 0 \); in other words, this is the inverse of **cdf_student_t**. Argument \( q \) must be an element of \([0,1]\). To make use of this function, write first load("distrib").
mean_student_t (n) [Function]
Returns the mean of a Student random variable \( t(n) \), with \( n > 0 \). To make use of this function, write first \texttt{load("distrib")}.
The mean is
\[
E[X] = 0
\]

var_student_t (n) [Function]
Returns the variance of a Student random variable \( t(n) \), with \( n > 2 \).
The variance is
\[
V[X] = \frac{n}{n - 2}
\]

\[
(%i1) \text{load ("distrib")}$

\[
(%i2) \text{var_student_t(n);} \quad \frac{n}{n - 2}
\]

std_student_t (n) [Function]
Returns the standard deviation of a Student random variable \( t(n) \), with \( n > 2 \). To make use of this function, write first \texttt{load("distrib")}.
The standard deviation is
\[
D[X] = \sqrt{\frac{n}{n - 2}}
\]

skewness_student_t (n) [Function]
Returns the skewness coefficient of a Student random variable \( t(n) \), with \( n > 3 \), which is always equal to 0. To make use of this function, write first \texttt{load("distrib")}.
The skewness coefficient is
\[
SK[X] = 0
\]

kurtosis_student_t (n) [Function]
Returns the kurtosis coefficient of a Student random variable \( t(n) \), with \( n > 4 \). To make use of this function, write first \texttt{load("distrib")}.
The kurtosis coefficient is
\[
KU[X] = \frac{6}{n - 4}
\]

random_student_t (n)
random_student_t (n, m) [Function]
Returns a Student random variate \( t(n) \), with \( n > 0 \). Calling \texttt{random_student_t} with a second argument \( m \), a random sample of size \( m \) will be simulated.
The implemented algorithm is based on the fact that if \( Z \) is a normal random variable \texttt{Normal}(0,1) and \( S^2 \) is a \( \chi^2 \) random variable with \( n \) degrees of freedom, \( \chi^2(n) \) , then
\[
X = \frac{Z}{\sqrt{\frac{S^2}{n}}}
\]
is a Student random variable with \( n \) degrees of freedom, \( t(n) \).
To make use of this function, write first \texttt{load("distrib")}. 

52.2.3 Noncentral Student’s t Random Variable

Let \( ncp \) be the non-centrality parameter, \( n \) be the degrees of freedom for the non-central Student’s \( t \) random variable.

Then let \( X \) be a \( \text{Normal}(n, ncp) \) and \( S^2 \) be an independent \( \chi^2 \) random variable with \( n \) degrees of freedom, the random variable

\[
U = \frac{X}{\sqrt{S^2/n}}
\]

has a non-central Student’s \( t \) distribution with non-centrality parameter \( ncp \).

\[
\text{pdf\_noncentral\_student\_t}(x, n, ncp)
\]

Returns the value at \( x \) of the density function of a noncentral Student random variable \( nc\_t(n, ncp) \), with \( n > 0 \) degrees of freedom and noncentrality parameter \( ncp \). To make use of this function, write first \texttt{load("distrib")}.

The pdf is

\[
f(x; n, \mu) = \left[ \sqrt{n} B\left(\frac{1}{2}, \frac{n}{2}\right) \right]^{-1} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}} e^{-\mu^2/2} \left[ A_n(x; \mu) + B_n(x; \mu) \right]
\]

where

\[
A_n(x; \mu) = \text{}_1F_1\left(\frac{n+1}{2}; \frac{1}{2}; \frac{\mu^2 x^2}{2(x^2+n)}\right)
\]

\[
B_n(x; \mu) = \frac{\sqrt{2\mu x}}{\sqrt{x^2+n}} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)} \text{}_1F_1\left(\frac{n}{2}+1; \frac{3}{2}; \frac{\mu^2 x^2}{2(x^2+n)}\right)
\]

and \( \mu \) is the non-centrality parameter \( ncp \).

Sometimes an extra work is necessary to get the final result.

\texttt{(%%i1) load ("distrib")}$
(%i2) expand(pdf_noncentral_student_t(3,5,0.1));
rat: replaced 0.01889822365046136 by 15934951/843198350 = 0.01889822365046136
rat: replaced -8.734356480209641 by -294697965/33740089 = -8.734356480209641
rat: replaced 4.136255165816327 by 51033443/12338079 = 4.136255165816332
rat: replaced 1.08061432164203 by 56754827/52520891 = 1.08061432164203
rat: replaced 0.0565127306411839 by 5608717/99246965 = 0.05651273064118384
rat: replaced -300.8069396896258 by -79782423/265228 = -300.8069396896256
rat: replaced 160.6269176184973 by 178374907/1110492 = 160.626917618497
rat: replaced 7/2 7/2 0.04296414417400905 5 1.323650307289301e-6 5
rat: replaced 3/2 5/2 1.94793720435093e-4 5
rat: replaced 2 14 sqrt(%pi)
rat: replaced 7/2 + sqrt(%pi)
rat: replaced 7/2 + 1.94793720435093e-4 5
rat: replaced 1.94793720435093e-4 5
rat: replaced 0.942477796076938 5
rat: replaced 0.02080593159405671
(%o2) 0.02080593159405671

cdf_noncentral_student_t (x,n,ncp) [Function]
Returns the value at x of the distribution function of a noncentral Student random variable \( nc.t(n,ncp) \), with \( n > 0 \) degrees of freedom and noncentrality parameter \( ncp \). This function has no closed form and it is numerically computed.

(%i1) load ("distrib")$
(%i2) cdf_noncentral_student_t(-2,5,-5);
(%o2) 0.995203009331975

quantile_noncentral_student_t (q,n,ncp) [Function]
Returns the \( q \)-quantile of a noncentral Student random variable \( nc.t(n,ncp) \), with \( n > 0 \) degrees of freedom and noncentrality parameter \( ncp \); in other words, this is the inverse of cdf_noncentral_student_t. Argument \( q \) must be an element of \([0,1]\). To make use of this function, write first load("distrib").

mean_noncentral_student_t (n,ncp) [Function]
Returns the mean of a noncentral Student random variable \( nc.t(n,ncp) \), with \( n > 1 \) degrees of freedom and noncentrality parameter \( ncp \). To make use of this function, write first load("distrib").
The mean is

\[ E[X] = \frac{\mu \sqrt{n} \Gamma \left( \frac{n-1}{2} \right)}{\sqrt{2} \Gamma \left( \frac{n}{2} \right)} \]

where \( \mu \) is the noncentrality parameter \( ncp \).

\begin{verbatim}
(%i1) load ("distrib")$
(%i2) mean_noncentral_student_t(df,k);
    df - 1
   gamma(-----) sqrt(df) k
  2
-------------------------------
    df
   sqrt(2) gamma(--)
  2
(%o2) var_noncentral_student_t(n,ncp)

Returns the variance of a noncentral Student random variable \( nc_t(n,ncp) \), with
\( n > 2 \) degrees of freedom and noncentrality parameter \( ncp \). To make use of this
function, write first \texttt{load("distrib")}.

The variance is

\[ V[X] = \frac{n(\mu^2 + 1)}{n-2} - \frac{n\mu^2 \Gamma \left( \frac{n-1}{2} \right)^2}{2\Gamma \left( \frac{n}{2} \right)^2} \]

where \( \mu \) is the noncentrality parameter \( ncp \).

\begin{verbatim}

std_noncentral_student_t(n,ncp)

Returns the standard deviation of a noncentral Student random variable \( nc_t(n,ncp) \), with
\( n > 2 \) degrees of freedom and noncentrality parameter \( ncp \). To make use of
this function, write first \texttt{load("distrib")}.

The standard deviation is

\begin{verbatim}

skewness_noncentral_student_t(n,ncp)

Returns the skewness coefficient of a noncentral Student random variable \( nc_t(n,ncp) \), with
\( n > 3 \) degrees of freedom and noncentrality parameter \( ncp \). To make use of
this function, write first \texttt{load("distrib")}.

If \( U \) is a non-central Student’s \( t \) random variable with \( n \) degrees of freedom and a
noncentrality parameter \( \mu \), the skewness is

\[ SK[U] = \frac{\mu \sqrt{n} \Gamma \left( \frac{n-1}{2} \right)}{\sqrt{2\Gamma \left( \frac{n}{2} \right)} \sigma^3} \left( \frac{n(2n + \mu^2 - 3)}{(n-3)(n-2)} - 2\sigma^2 \right) \]

\[ \sigma^2 = \frac{n(\mu^2 + 1)}{n-2} - \frac{n\mu^2 \Gamma \left( \frac{n-1}{2} \right)^2}{2\Gamma \left( \frac{n}{2} \right)^2} \]
kurtosis_noncentral_student_t (n,ncp) [Function]
Returns the kurtosis coefficient of a noncentral Student random variable nc\_t(n, ncp) , with n > 4 degrees of freedom and noncentrality parameter ncp. To make use of this function, write first load("distrib").

If U is a non-central Student’s t random variable with n degrees of freedom and a noncentrality parameter \( \mu \), the kurtosis is

\[
K_U[U] = \frac{\mu^4}{\sigma^4} - 3
\]

\[
\mu_4 = \frac{(\mu^4 + 6\mu^2 + 3)n^2}{(n-4)(n-2)} - \left( \frac{n(3(3n-5)+\mu^2(n+1))}{(n-3)(n-2)} - 3\sigma^2 \right) F
\]

\[
\sigma^2 = \frac{n(\mu^2 + 1)}{n-2} - \frac{n\mu^2\Gamma\left(\frac{n-1}{2}\right)^2}{2\Gamma\left(\frac{n}{2}\right)^2}
\]

\[
F = \frac{n\mu^2\Gamma\left(\frac{n-1}{2}\right)^2}{2\sigma^4\Gamma\left(\frac{n}{2}\right)^2}
\]

random_noncentral_student_t (n,ncp) [Function]
random_noncentral_student_t (n,ncp,m)
Returns a noncentral Student random variate nc\_t(n, ncp) , with n > 0. Calling random_noncentral_student_t with a third argument m, a random sample of size m will be simulated.

The implemented algorithm is based on the fact that if X is a normal random variable Normal(ncp,1) and \( S^2 \) is a \( \chi^2 \) random variable with n degrees of freedom, \( \chi^2(n) \) , then

\[
U = \frac{X}{\sqrt{\frac{S^2}{n}}}
\]

is a noncentral Student random variable with n degrees of freedom and noncentrality parameter ncp, nc\_t(n, ncp).

To make use of this function, write first load("distrib").

52.2.4 Chi-squared Random Variable
Let \( X_1, X_2, \ldots, X_n \) be independent and identically distributed Normal(0,1) variables. Then

\[
X^2 = \sum_{i=1}^{n} X_i^2
\]

is said to follow a chi-square distribution with n degrees of freedom.

pdf\_chi2 (x,n) [Function]
Returns the value at x of the density function of a Chi-square random variable \( \chi^2(n) \) , with n > 0. The \( \chi^2(n) \) random variable is equivalent to the \( \Gamma\left(\frac{n}{2}, 2\right) \).

The pdf is
f(x; n) = \begin{cases} \frac{x^{n/2-1}e^{-x/2}}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}

(%i1) load ("distrib")$
(%i2) pdf_chi2(x,n); 
\quad \frac{n/2 - 1 - x/2}{x \ %e \ \unit_step(x)} \\
\quad \frac{\gamma(-)}{2}
\quad \frac{n \ n/2}{2}

\text{cdf\_chi2 (x,n)} \quad \quad \text{[Function]}
Returs the value at } x \text{ of the distribution function of a Chi-square random variable } \chi^2(n) \text{, with } n > 0.

The cdf is
\quad F(x; n) = \begin{cases} 1 - Q\left(\frac{n}{2}, \frac{x}{2}\right) & x > 0 \\ 0 & \text{otherwise} \end{cases}

where Q(a, z) is the [gamma\_incomplete\_regularized], page 303, function.

(%i1) load ("distrib")$
(%i2) cdf\_chi2(3, 4); 
\quad 1 - \gamma\_incomplete\_regularized(2, -)
\quad \frac{3}{2}

(%i3) float(%); 
\quad 0.4421745996289252

\text{quantile\_chi2 (q,n)} \quad \quad \text{[Function]}
Returs the } q \text{-quantile of a Chi-square random variable } \chi^2(n) \text{, with } n > 0; \text{ in other words, this is the inverse of cdf\_chi2. Argument } q \text{ must be an element of } [0, 1].

This function has no closed form and it is numerically computed.

(%i1) load ("distrib")$
(%i2) quantile\_chi2(0.99, 9); 
\quad 21.66599433346194

\text{mean\_chi2 (n)} \quad \quad \text{[Function]}
Returs the mean of a Chi-square random variable } \chi^2(n) \text{, with } n > 0.

The } \chi^2(n) \text{ random variable is equivalent to the } \Gamma\left(n/2, 2\right).

The mean is
\quad E[X] = n

(%i1) load ("distrib")$
(%i2) mean\_chi2(n); 
\quad n
var\_chi2 (n) \quad \text{[Function]}

Returns the variance of a Chi-square random variable $\chi^2(n)$, with $n > 0$.

The $\chi^2(n)$ random variable is equivalent to the $\Gamma(n/2, 2)$.

The variance is

$$V[X] = 2n$$

(\%i1) load ("distrib")$
(\%i2) var\_chi2(n);
(\%o2) \quad 2 \ n$

std\_chi2 (n) \quad \text{[Function]}

Returns the standard deviation of a Chi-square random variable $\chi^2(n)$, with $n > 0$.

The $\chi^2(n)$ random variable is equivalent to the $\Gamma(n/2, 2)$.

The standard deviation is

$$D[X] = \sqrt{2n}$$

(\%i1) load ("distrib")$
(\%i2) std\_chi2(n);
(\%o2) \quad \text{sqrt}(2) \ \text{sqrt}(n)$

skewness\_chi2 (n) \quad \text{[Function]}

Returns the skewness coefficient of a Chi-square random variable $\chi^2(n)$, with $n > 0$.

The $\chi^2(n)$ random variable is equivalent to the $\Gamma(n/2, 2)$.

The skewness coefficient is

$$SK[X] = \frac{\sqrt{8}}{n}$$

(\%i1) load ("distrib")$
(\%i2) skewness\_chi2(n);
(\%o2) \quad 3/2 \ 
2 \ 
-----
\text{sqrt}(n)$

kurtosis\_chi2 (n) \quad \text{[Function]}

Returns the kurtosis coefficient of a Chi-square random variable $\chi^2(n)$, with $n > 0$.

The $\chi^2(n)$ random variable is equivalent to the $\Gamma(n/2, 2)$.

The kurtosis coefficient is

$$KU[X] = \frac{12}{n}$$

(\%i1) load ("distrib")$
(\%i2) kurtosis\_chi2(n);
(\%o2) \quad 12 \ 
--
\ n
random\_chi2 (n)

random\_chi2 (n,m)

Returns a Chi-square random variate $\chi^2(n)$, with $n > 0$. Calling random\_chi2 with a second argument $m$, a random sample of size $m$ will be simulated.

The simulation is based on the Ahrens-Cheng algorithm. See random\_gamma for details.

To make use of this function, write first load("distrib").

52.2.5 Noncentral Chi-squared Random Variable

Let $X_1, X_2, ..., X_n$ be $n$ independent normally distributed random variables with means $\mu_k$ and unit variances. Then the random variable

$$\sum_{k=1}^{n} X_k^2$$

has a noncentral $\chi^2$ distribution. The number of degrees of freedom is $n$, and the noncentrality parameter is defined by

$$\sum_{k=1}^{n} \mu_k^2$$

pdf\_noncentral\_chi2 (x,n,ncp)

Returns the value at $x$ of the density function of a noncentral $\chi^2$ random variable m4\_noncentral\_chi2(n,ncp), with $n > 0$ and noncentrality parameter $ncp \geq 0$. To make use of this function, write first load("distrib").

For $x < 0$, the pdf is 0, and for $x \geq 0$ the pdf is

$$f(x; n, \lambda) = \frac{1}{2} e^{-(x+\lambda)/2} \left(\frac{x}{\lambda}\right)^{n/2-1} I_{\frac{n}{2}-1} \left(\sqrt{\lambda x}\right)$$

cdf\_noncentral\_chi2 (x,n,ncp)

Returns the value at $x$ of the distribution function of a noncentral Chi-square random variable m4\_noncentral\_chi2(n,ncp), with $n > 0$ and noncentrality parameter $ncp \geq 0$. To make use of this function, write first load("distrib").

quantile\_noncentral\_chi2 (q,n,ncp)

Returns the $q$-quantile of a noncentral Chi-square random variable m4\_noncentral\_chi2(n,ncp), with $n > 0$ and noncentrality parameter $ncp \geq 0$; in other words, this is the inverse of cdf\_noncentral\_chi2. Argument $q$ must be an element of $[0, 1]$.

This function has no closed form and it is numerically computed.

mean\_noncentral\_chi2 (n,ncp)

Returns the mean of a noncentral Chi-square random variable m4\_noncentral\_chi2(n,ncp), with $n > 0$ and noncentrality parameter $ncp \geq 0$.

The mean is

$$E[X] = n + \mu$$

where $\mu$ is the noncentrality parameter $ncp$. 
var_noncentral_chi2 (n,ncp)  
Returns the variance of a noncentral Chi-square random variable m4_noncentral_chi2(n,ncp), with \( n > 0 \) and noncentrality parameter \( ncp >= 0 \).

The variance is

\[ V[X] = 2(n + 2\mu) \]

where \( \mu \) is the noncentrality parameter \( ncp \).

std_noncentral_chi2 (n,ncp)  
Returns the standard deviation of a noncentral Chi-square random variable m4_noncentral_chi2(n,ncp), with \( n > 0 \) and noncentrality parameter \( ncp >= 0 \).

The standard deviation is

\[ D[X] = \sqrt{2(n + 2\mu)} \]

where \( \mu \) is the noncentrality parameter \( ncp \).

skewness_noncentral_chi2 (n,ncp)  
Returns the skewness coefficient of a noncentral Chi-square random variable m4_noncentral_chi2(n,ncp), with \( n > 0 \) and noncentrality parameter \( ncp >= 0 \).

The skewness coefficient is

\[ SK[X] = \frac{2^{3/2}(n + 3\mu)}{(n + 2\mu)^{3/2}} \]

where \( \mu \) is the noncentrality parameter \( ncp \).

kurtosis_noncentral_chi2 (n,ncp)  
Returns the kurtosis coefficient of a noncentral Chi-square random variable m4_noncentral_chi2(n,ncp), with \( n > 0 \) and noncentrality parameter \( ncp >= 0 \).

The kurtosis coefficient is

\[ KU[X] = \frac{12(n + 4\mu)}{(2 + 2\mu)^2} \]

where \( \mu \) is the noncentrality parameter \( ncp \).

random_noncentral_chi2 (n,ncp)  
random_noncentral_chi2 (n,ncp,m)
Returns a noncentral Chi-square random variate m4_noncentral_chi2(n,ncp), with \( n > 0 \) and noncentrality parameter \( ncp >= 0 \). Calling \texttt{random_noncentral_chi2} with a third argument \( m \), a random sample of size \( m \) will be simulated.

To make use of this function, write first \texttt{load("distrib")}. 
52.2.6 F Random Variable

Let $S_1$ and $S_2$ be independent random variables with a $\chi^2$ distribution with degrees of freedom $n$ and $m$, respectively. Then

$$F = \frac{S_1/n}{S_2/m}$$

has an $F$ distribution with $n$ and $m$ degrees of freedom.

\textbf{pdf\_f \ (x,m,n)} \textbf{[Function]}

Returns the value at $x$ of the density function of a $F$ random variable $F(m, n)$, with $m, n > 0$. To make use of this function, write first \texttt{load("distrib")}.

The pdf is

$$f(x; m, n) = \begin{cases} 
B\left(\frac{m}{2}, \frac{n}{2}\right)^{-1} \left(\frac{m}{n}\right)^{m/2} x^{m/2-1} \left(1 + \frac{m}{n}x\right)^{-(m+n)/2} & x > 0 \\
0 & \text{otherwise}
\end{cases}$$

\textbf{cdf\_f \ (x,m,n)} \textbf{[Function]}

Returns the value at $x$ of the distribution function of a $F$ random variable $F(m, n)$, with $m, n > 0$.

The cdf is

$$F(x; m, n) = \begin{cases} 
1 - I_z\left(\frac{m}{2}, \frac{n}{2}\right) & x > 0 \\
0 & \text{otherwise}
\end{cases}$$

where

$$z = \frac{n}{mx + n}$$

and $I_z(a, b)$ is the \texttt{beta\_incomplete\_regularized}, page 309, function.

\texttt{(\%i1) load ("distrib")}
\texttt{(\%i2) cdf\_f(2,3,9/4); 9 3 3
\texttt{1 - beta\_incomplete\_regularized(-, -, --)}
\texttt{8 2 11}
\texttt{(\%o2) 0.6675672817900802}
\texttt{(\%o3) float(\%o2);}
\texttt{\%o3) 0.6675672817900802}

\textbf{quantile\_f \ (q,m,n)} \textbf{[Function]}

Returns the $q$-quantile of a $F$ random variable $F(m, n)$, with $m, n > 0$; in other words, this is the inverse of \texttt{cdf\_f}. Argument $q$ must be an element of $[0, 1]$.

\texttt{(\%i1) load ("distrib")}$\$
\texttt{(\%i2) quantile\_f(2/5,sqrt(3),5); 9 3 3
\texttt{0.5189478385736904}
\texttt{(\%o2) 0.5189478385736904}

\textbf{mean\_f \ (m,n)} \textbf{[Function]}

Returns the mean of a $F$ random variable $F(m, n)$, with $m > 0, n > 2$. To make use of this function, write first \texttt{load("distrib")}. 


The mean is
\[ E[X] = \frac{n}{n - 2} \]

\textbf{var\_f (m,n)}

Returns the variance of a F random variable \( F(m,n) \), with \( m > 0, n > 4 \). To make use of this function, write first \texttt{load("distrib")}.

The variance is
\[ V[X] = \frac{2n^2(n + m - 2)}{m(n - 4)(n - 2)^2} \]

\textbf{std\_f (m,n)}

Returns the standard deviation of a F random variable \( F(m,n) \), with \( m > 0, n > 4 \). To make use of this function, write first \texttt{load("distrib")}.

The standard deviation is
\[ D[X] = \sqrt{\frac{2n}{n - 2} \sqrt{\frac{n + m - 2}{m(n - 4)}}} \]

\textbf{skewness\_f (m,n)}

Returns the skewness coefficient of a F random variable \( F(m,n) \), with \( m > 0, n > 6 \). To make use of this function, write first \texttt{load("distrib")}.

The skewness coefficient is
\[ SK[X] = \frac{(n + 2m - 2)\sqrt{8(n - 4)}}{(n - 6)\sqrt{m(n + m - 2)}} \]

\textbf{kurtosis\_f (m,n)}

Returns the kurtosis coefficient of a F random variable \( F(m,n) \), with \( m > 0, n > 8 \). To make use of this function, write first \texttt{load("distrib")}.

The kurtosis coefficient is
\[ KU[X] = \frac{12}{m(n + m - 2)(5n - 22) + (n - 4)(n - 2)^2}{m(n - 8)(n - 6)(n + m - 2)} \]

\textbf{random\_f (m,n)}

\textbf{random\_f (m,n,k)}

Returns a F random variate \( F(m,n) \), with \( m, n > 0 \). Calling \texttt{random\_f} with a third argument \( k \), a random sample of size \( k \) will be simulated.

The simulation algorithm is based on the fact that if \( X \) is a \( \chi^2(m) \) random variable and \( Y \) is a \( \chi^2(n) \) random variable, then
\[ F = \frac{nX}{mY} \]

is a F random variable with \( m \) and \( n \) degrees of freedom, \( F(m,n) \).

To make use of this function, write first \texttt{load("distrib")}. 

\[ \]
52.2.7 Exponential Random Variable

The *exponential distribution* is the probability distribution of the time between events in a process where the events occur continuously and independently at a constant average rate.

**pdf_exp (x,m)** [Function]

Returns the value at x of the density function of an Exponential(m) random variable, with m > 0.

The Exponential(m) random variable is equivalent to the Weibull(1,1/m).

The pdf is

\[ f(x; m) = \begin{cases} \me^{-mx} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

(%i1) load ("distrib")$
(%i2) pdf_exp(x,m);
   - m x
   m %e unit_step(x)

**cdf_exp (x,m)** [Function]

Returns the value at x of the distribution function of an Exponential(m) random variable, with m > 0.

The Exponential(m) random variable is equivalent to the Weibull(1,1/m).

The cdf is

\[ F(x; m) = \begin{cases} 1 - e^{-mx} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

(%i1) load ("distrib")$
(%i2) cdf_exp(x,m);
   - m x
   (1 - %e ) unit_step(x)

**quantile_exp (q,m)** [Function]

Returns the q-quantile of an Exponential(m) random variable, with m > 0; in other words, this is the inverse of cdf_exp. Argument q must be an element of [0,1].

The Exponential(m) random variable is equivalent to the Weibull(1,1/m).

(%i1) load ("distrib")$
(%i2) quantile_exp(0.56,5);
(%i2) 0.1641961104139661
(%i3) quantile_exp(0.56,5);
(%i3) 0.8209805520698303
   ------------------
   m

**mean_exp (m)** [Function]

Returns the mean of an Exponential(m) random variable, with m > 0.

The Exponential(m) random variable is equivalent to the Weibull(1,1/m).

The mean is

\[ E[X] = \frac{1}{m} \]
(%i1) load ("distrib")$
(%i2) mean_exp(m);
   1
(%o2) -
   m

var_exp (m) [Function]
Returns the variance of an Exponential(m) random variable, with m > 0.
The Exponential(m) random variable is equivalent to the Weibull(1, 1/m).
The variance is
\[ V[X] = \frac{1}{m^2} \]

(%i1) load ("distrib")$
(%i2) var_exp(m);
   1
(%o2) --
   2
   m

std_exp (m) [Function]
Returns the standard deviation of an Exponential(m) random variable, with m > 0.
The Exponential(m) random variable is equivalent to the Weibull(1, 1/m).
The standard deviation is
\[ D[X] = \frac{1}{m} \]

(%i1) load ("distrib")$
(%i2) std_exp(m);
   1
(%o2) -
   m

skewness_exp (m) [Function]
Returns the skewness coefficient of an Exponential(m) random variable, with m > 0.
The Exponential(m) random variable is equivalent to the Weibull(1, 1/m).
The skewness coefficient is
\[ SK[X] = 2 \]

(%i1) load ("distrib")$
(%i2) skewness_exp(m);
(%o2) 2

kurtosis_exp (m) [Function]
Returns the kurtosis coefficient of an Exponential(m) random variable, with m > 0.
The Exponential(m) random variable is equivalent to the Weibull(1, 1/m).
The kurtosis coefficient is
\[ KU[X] = 6 \]

(%i1) load ("distrib")$
(%i2) kurtosis_exp(m);
(%o2) 6
random_exp \(m\)  
random_exp \(m, k\)  
Returns an Exponential \((m)\) random variate, with \(m > 0\). Calling `random_exp` with a second argument \(k\), a random sample of size \(k\) will be simulated.

The simulation algorithm is based on the general inverse method.

To make use of this function, write first `load("distrib")`.

### 52.2.8 Lognormal Random Variable

The lognormal distribution is distribution for a random variable whose logarithm is normally distributed.

pdf_lognormal \((x, m, s)\)  
Returns the value at \(x\) of the density function of a Lognormal \((m, s)\) random variable, with \(s > 0\). To make use of this function, write first `load("distrib")`.

The pdf is

\[
f(x; m, s) = \begin{cases} 
\frac{1}{xs\sqrt{2\pi}} \exp \left(-\frac{(\log x - m)^2}{2s^2}\right) & \text{for } x \geq 0 \\
0 & \text{for } x < 0 
\end{cases}
\]

cdf_lognormal \((x, m, s)\)  
Returns the value at \(x\) of the distribution function of a Lognormal \((m, s)\) random variable, with \(s > 0\). This function is defined in terms of Maxima’s built-in error function `erf`.

The cdf is

\[
F(x; m, s) = \begin{cases} 
\frac{1}{2} \left[1 + \text{erf}\left(\frac{\log x - m}{s\sqrt{2}}\right)\right] & \text{for } x > 0 \\
0 & \text{for } x \leq 0 
\end{cases}
\]

```
(%i1) load("distrib")$
(%i2) cdf_lognormal(x,m,s);
log(x) - m
erf(----------)
\sqrt{2} s 1
unit_step(x) (--------------- + -)
2 2
(%o2) %e
```

See also `erf`.

quantile_lognormal \((q, m, s)\)  
Returns the \(q\)-quantile of a Lognormal \((m, s)\) random variable, with \(s > 0\); in other words, this is the inverse of `cdf_lognormal`.

Argument \(q\) must be an element of \([0, 1]\).

To make use of this function, write first `load("distrib")`.

```
(%i1) load("distrib")$
(%i2) quantile_lognormal(95/100,0,1);
sqrt(2) inverse_erf(9/10)
(%o2) %e
(%i3) float(%);
(%o3) 5.180251602233015
```
mean_lognormal (m,s)  [Function]
Returns the mean of a Lognormal(m, s) random variable, with s > 0. To make use of this function, write first load("distrib").
The mean is
\[ E[X] = \exp \left( m + \frac{s^2}{2} \right) \]

var_lognormal (m,s)  [Function]
Returns the variance of a Lognormal(m, s) random variable, with s > 0. To make use of this function, write first load("distrib").
The variance is
\[ V[X] = (\exp (s^2) - 1) \exp (2m + s^2) \]

std_lognormal (m,s)  [Function]
Returns the standard deviation of a Lognormal(m, s) random variable, with s > 0. To make use of this function, write first load("distrib").
The standard deviation is
\[ D[X] = \sqrt{\exp (s^2) - 1} \exp \left( m + \frac{s^2}{2} \right) \]

skewness_lognormal (m,s)  [Function]
Returns the skewness coefficient of a Lognormal(m, s) random variable, with s > 0. To make use of this function, write first load("distrib").
The skewness coefficient is
\[ SK[X] = (\exp (s^2) + 2) \sqrt{\exp (s^2) - 1} \]

kurtosis_lognormal (m,s)  [Function]
Returns the kurtosis coefficient of a Lognormal(m, s) random variable, with s > 0. To make use of this function, write first load("distrib").
The kurtosis coefficient is
\[ KU[X] = \exp (4s^2) + 2 \exp (3s^2) + 3 \exp (2s^2) - 3 \]

random_lognormal (m,s)  [Function]
random_lognormal (m,s,n)
Returns a Lognormal(m, s) random variate, with s > 0. Calling random_lognormal with a third argument n, a random sample of size n will be simulated.
Log-normal variates are simulated by means of random normal variates. See random_normal for details.
To make use of this function, write first load("distrib").
52.2.9 Gamma Random Variable

The *gamma distribution* is a two-parameter family of probability distributions. Maxima uses the parameterization using the shape and scale for the first and second parameters of the distribution.

**pdf Gamma (x,a,b)**

Returns the value at x of the density function of a $\Gamma(a,b)$ random variable, with $a,b > 0$. To make use of this function, write first `load("distrib")`.

The shape parameter is $a$, and the scale parameter is $b$.

The pdf is

$$f(x; a, b) = \frac{x^{a-1}e^{-x/b}}{b^a\Gamma(a)}$$

**cdf Gamma (x,a,b)**

Returns the value at x of the distribution function of a $\Gamma(a,b)$ random variable, with $a,b > 0$.

The cdf is

$$F(x; a, b) = \begin{cases} 
1 - Q(a, \frac{x}{b}) & \text{for } x \geq 0 \\
0 & \text{for } x < 0 
\end{cases}$$

where $Q(a,z)$ is the [gamma_incomplete_regularized], page 303, function.

```
(%i1) load ("distrib")$
(%i2) cdf_gamma(3,5,21);
(%o2) 1 - gamma_incomplete_regularized(5, -)
(%i3) float(%);
(%o3) 4.402663157376807e-7
```

**quantile Gamma (q,a,b)**

Returns the $q$-quantile of a $\Gamma(a,b)$ random variable, with $a,b > 0$; in other words, this is the inverse of `cdfGamma`. Argument $q$ must be an element of [0,1]. To make use of this function, write first `load("distrib")`.

**mean Gamma (a,b)**

Returns the mean of a $\Gamma(a,b)$ random variable, with $a,b > 0$. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = ab$$

**var Gamma (a,b)**

Returns the variance of a $\Gamma(a,b)$ random variable, with $a,b > 0$. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = ab^2$$
std_gamma (a,b)  
Returns the standard deviation of a $\Gamma(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first load("distrib").

The standard deviation is

$$D[X] = b\sqrt{a}$$

skewness_gamma (a,b)  
Returns the skewness coefficient of a $\Gamma(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first load("distrib").

The skewness coefficient is

$$SK[X] = \frac{2}{\sqrt{a}}$$

kurtosis_gamma (a,b)  
Returns the kurtosis coefficient of a $\Gamma(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first load("distrib").

The kurtosis coefficient is

$$KU[X] = \frac{6}{a}$$

cdf_beta (x,a,b)  
Returns the value at $x$ of the density function of a $Beta(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first load("distrib").

The pdf is

$$f(x; a, b) = \begin{cases} 
  \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} & \text{for } 0 \leq x \leq 1 \\
  0 & \text{otherwise}
\end{cases}$$

52.2.10 Beta Random Variable

The $beta$ distribution is a family of distributions defined over $[0,1]$ parameterized by two positive shape parameters $a$, and $b$.

pdf_beta (x,a,b)  
Returns the value at $x$ of the density function of a $Beta(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first load("distrib").

The pdf is

$$f(x; a, b) = \begin{cases} 
  \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} & \text{for } 0 \leq x \leq 1 \\
  0 & \text{otherwise}
\end{cases}$$

cdf_beta (x,a,b)  
Returns the value at $x$ of the distribution function of a $Beta(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first load("distrib").
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The cdf is
\[ F(x; a, b) = \begin{cases} 
0 & x < 0 \\
I_x(a, b) & 0 \leq x \leq 1 \\
1 & x > 1 
\end{cases} \]

(%i1) load ("distrib")$
(%i2) cdf_beta(1/3,15,2); 11
(%o2) \frac{11}{14348907}
(%i3) float(%);
(%o3) 7.666089131388195e-7

quantile_beta (q,a,b)  [Function]
Returns the q-quantile of a Beta(a,b) random variable, with a,b > 0; in other words, this is the inverse of cdf_beta. Argument q must be an element of [0,1]. To make use of this function, write first load("distrib").

mean_beta (a,b)  [Function]
Returns the mean of a Beta(a,b) random variable, with a,b > 0. To make use of this function, write first load("distrib").

The mean is
\[ E[X] = \frac{a}{a+b} \]

var_beta (a,b)  [Function]
Returns the variance of a Beta(a,b) random variable, with a,b > 0. To make use of this function, write first load("distrib").

The variance is
\[ V[X] = \frac{ab}{(a+b)^2(a+b+1)} \]

std_beta (a,b)  [Function]
Returns the standard deviation of a Beta(a,b) random variable, with a,b > 0. To make use of this function, write first load("distrib").

The standard deviation is
\[ D[X] = \frac{1}{a+b} \sqrt{ab} \frac{a+b+1}{a+b+1} \]

skewness_beta (a,b)  [Function]
Returns the skewness coefficient of a Beta(a,b) random variable, with a,b > 0. To make use of this function, write first load("distrib").

The skewness coefficient is
\[ SK[X] = \frac{2(b-a)\sqrt{a+b+1}}{(a+b+2)\sqrt{ab}} \]

kurtosis_beta (a,b)  [Function]
Returns the kurtosis coefficient of a Beta(a,b) random variable, with a,b > 0. To make use of this function, write first load("distrib").
The kurtosis coefficient is
\[
KU[X] = \frac{3(a + b + 1)(2(a + b)^2 + ab(a + b - 6))}{ab(a + b + 2)(a + b + 3)} - 3
\]

random_beta (a,b)

random_beta (a,b,n)

Returns a Beta(a, b) random variate, with \(a, b > 0\). Calling \texttt{random_beta} with a third argument \(n\), a random sample of size \(n\) will be simulated.


To make use of this function, write first \texttt{load("distrib")}.

52.2.11 Continuous Uniform Random Variable

The continuous uniform distribution is constant over the interval \([a, b]\) and is zero elsewhere.

pdf_continuous_uniform (x,a,b)

Returns the value at \(x\) of the density function of a ContinuousUniform\((a, b)\) random variable, with \(a < b\). To make use of this function, write first \texttt{load("distrib")}.

The pdf
\[
f(x; a, b) = \begin{cases}
\frac{1}{b - a} & \text{for } 0 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]
and is 0 otherwise.

cdf_continuous_uniform (x,a,b)

Returns the value at \(x\) of the distribution function of a ContinuousUniform\((a, b)\) random variable, with \(a < b\). To make use of this function, write first \texttt{load("distrib")}.

The cdf is
\[
F(x; a, b) = \begin{cases}
0 & \text{for } x < a \\
\frac{x - a}{b - a} & \text{for } a \leq x \leq b \\
1 & \text{for } x > b
\end{cases}
\]

quantile_continuous_uniform (q,a,b)

Returns the \(q\)-quantile of a ContinuousUniform\((a, b)\) random variable, with \(a < b\); in other words, this is the inverse of \texttt{cdf_continuous_uniform}. Argument \(q\) must be an element of \([0, 1]\). To make use of this function, write first \texttt{load("distrib")}.

mean_continuous_uniform (a,b)

Returns the mean of a ContinuousUniform\((a, b)\) random variable, with \(a < b\). To make use of this function, write first \texttt{load("distrib")}.

The mean is
\[
E[X] = \frac{a + b}{2}
\]
var_continuous_uniform \( (a,b) \) [Function]
Returns the variance of a \( \text{ContinuousUniform}(a,b) \) random variable, with \( a < b \). To make use of this function, write first \texttt{load("distrib")}.

The variance is
\[
V[X] = \frac{(b - a)^2}{12}
\]

std_continuous_uniform \( (a,b) \) [Function]
Returns the standard deviation of a \( \text{ContinuousUniform}(a,b) \) random variable, with \( a < b \). To make use of this function, write first \texttt{load("distrib")}.

The standard deviation is
\[
D[X] = \frac{b - a}{2\sqrt{3}}
\]

skewness_continuous_uniform \( (a,b) \) [Function]
Returns the skewness coefficient of a \( \text{ContinuousUniform}(a,b) \) random variable, with \( a < b \). To make use of this function, write first \texttt{load("distrib")}.

The skewness coefficient is
\[
SK[X] = 0
\]

kurtosis_continuous_uniform \( (a,b) \) [Function]
Returns the kurtosis coefficient of a \( \text{ContinuousUniform}(a,b) \) random variable, with \( a < b \). To make use of this function, write first \texttt{load("distrib")}.

The kurtosis coefficient is
\[
KU[X] = -\frac{6}{5}
\]

random_continuous_uniform \( (a,b) \) [Function]
\[
\text{random_continuous_uniform} \ (a,b,n)
\]
Returns a \( \text{ContinuousUniform}(a,b) \) random variate, with \( a < b \). Calling \texttt{random_continuous_uniform} with a third argument \( n \), a random sample of size \( n \) will be simulated.

This is a direct application of the \texttt{random} built-in Maxima function.
See also \texttt{random}. To make use of this function, write first \texttt{load("distrib")}.

52.2.12 Logistic Random Variable
The \textit{logistic} distribution is a continuous distribution where it’s cumulative distribution function is the logistic function.

pdf_logistic \( (x,a,b) \) [Function]
Returns the value at \( x \) of the density function of a \( \text{Logistic}(a,b) \) random variable , with \( b > 0 \). To make use of this function, write first \texttt{load("distrib")}.

\( a \) is the location parameter and \( b \) is the scale parameter.

The pdf is
\[
f(x; a, b) = \frac{e^{-(x-a)/b}}{b(1 + e^{-(x-a)/b})^2}
\]
cdf_logistic (x,a,b)
Returns the value at x of the distribution function of a Logistic(a, b) random variable , with b > 0. To make use of this function, write first load("distrib").
The cdf is
\[ F(x; a, b) = \frac{1}{1 + e^{-(x-a)/b}} \]

quantile_logistic (q,a,b)
Returns the q-quantile of a Logistic(a, b) random variable , with b > 0; in other words, this is the inverse of cdf_logistic. Argument q must be an element of [0, 1]. To make use of this function, write first load("distrib").

mean_logistic (a,b)
Returns the mean of a Logistic(a, b) random variable , with b > 0. To make use of this function, write first load("distrib").
The mean is
\[ E[X] = a \]

var_logistic (a,b)
Returns the variance of a Logistic(a, b) random variable , with b > 0. To make use of this function, write first load("distrib").
The variance is
\[ V[X] = \frac{\pi^2 b^2}{3} \]

std_logistic (a,b)
Returns the standard deviation of a Logistic(a, b) random variable , with b > 0. To make use of this function, write first load("distrib").
The standard deviation is
\[ D[X] = \frac{\pi b}{\sqrt{3}} \]

skewness_logistic (a,b)
Returns the skewness coefficient of a Logistic(a, b) random variable , with b > 0. To make use of this function, write first load("distrib").
The skewness coefficient is
\[ SK[X] = 0 \]

kurtosis_logistic (a,b)
Returns the kurtosis coefficient of a Logistic(a, b) random variable , with b > 0. To make use of this function, write first load("distrib").
The kurtosis coefficient is
\[ KU[X] = \frac{6}{5} \]

random_logistic (a,b)
Returns a Logistic(a, b) random variate, with b > 0. Calling random_logistic with a third argument n, a random sample of size n will be simulated.
The implemented algorithm is based on the general inverse method.
To make use of this function, write first load("distrib").
52.2.13 Pareto Random Variable

pdf_pareto (x,a,b)  [Function]
Returns the value at x of the density function of a Pareto(a,b) random variable, with a, b > 0. To make use of this function, write first load("distrib").

The pdf is

\[ f(x; a, b) = \begin{cases} \frac{ab^a}{x^{a+1}} & \text{for } x \geq b \\ 0 & \text{for } x < b \end{cases} \]

cdf_pareto (x,a,b)  [Function]
Returns the value at x of the distribution function of a Pareto(a,b) random variable, with a, b > 0. To make use of this function, write first load("distrib").

The cdf is

\[ F(x; a, b) = \begin{cases} 1 - \left( \frac{b}{x} \right)^a & \text{for } x \geq b \\ 0 & \text{for } x < b \end{cases} \]

quantile_pareto (q,a,b)  [Function]
Returns the q-quantile of a Pareto(a,b) random variable, with a, b > 0; in other words, this is the inverse of cdf_pareto. Argument q must be an element of [0, 1]. To make use of this function, write first load("distrib").

mean_pareto (a,b)  [Function]
Returns the mean of a Pareto(a,b) random variable, with a > 1, b > 0. To make use of this function, write first load("distrib").

The mean is

\[ E[X] = \frac{ab}{a-1} \]

var_pareto (a,b)  [Function]
Returns the variance of a Pareto(a,b) random variable, with a > 2, b > 0. To make use of this function, write first load("distrib").

The variance is

\[ V[X] = \frac{ab^2}{(a-2)(a-1)^2} \]

std_pareto (a,b)  [Function]
Returns the standard deviation of a Pareto(a,b) random variable, with a > 2, b > 0. To make use of this function, write first load("distrib").

The standard deviation is

\[ D[X] = \frac{b}{a-1} \sqrt{\frac{a}{a-2}} \]

skewness_pareto (a,b)  [Function]
Returns the skewness coefficient of a Pareto(a,b) random variable, with a > 3, b > 0. To make use of this function, write first load("distrib").
The skewness coefficient is

$$SK[X] = \frac{2(a + 1)}{a - 3} \sqrt{\frac{a - 2}{a}}$$

kurtosis_pareto (a,b)  [Function]
Returns the kurtosis coefficient of a Pareto(a,b) random variable, with \(a > 4, b > 0\).
To make use of this function, write first load("distrib").
The kurtosis coefficient is

$$KU[X] = \frac{6(a^3 + a^2 - 6a - 2)}{a(a - 3)(a - 4)} - 3$$

random_pareto (a,b)  [Function]
random_pareto (a,b,n)
Returns a Pareto(a,b) random variate, with \(a > 0, b > 0\). Calling random_pareto
with a third argument \(n\), a random sample of size \(n\) will be simulated.
The implemented algorithm is based on the general inverse method.
To make use of this function, write first load("distrib").

52.2.14 Weibull Random Variable

pdf_weibull (x,a,b)  [Function]
Returns the value at \(x\) of the density function of a Weibull(a,b) random variable,
with \(a, b > 0\). To make use of this function, write first load("distrib").
The pdf is

$$f(x;a,b) = \begin{cases} \frac{1}{b} \left(\frac{x}{b}\right)^{a-1} e^{-(x/b)^a} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

cdf_weibull (x,a,b)  [Function]
Returns the value at \(x\) of the distribution function of a Weibull(a,b) random variable,
with \(a, b > 0\). To make use of this function, write first load("distrib").
The cdf is

$$F(x;a,b) = \begin{cases} 1 - e^{-(x/b)^a} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

quantile_weibull (q,a,b)  [Function]
Returns the \(q\)-quantile of a Weibull(a,b) random variable, with \(a, b > 0\); in other
words, this is the inverse of cdf_weibull. Argument \(q\) must be an element of \([0, 1]\).
To make use of this function, write first load("distrib").

mean_weibull (a,b)  [Function]
Returns the mean of a Weibull(a,b) random variable, with \(a, b > 0\). To make use of
this function, write first load("distrib").
The mean is

$$E[X] = b\Gamma\left(1 + \frac{1}{a}\right)$$
**var_weibull (a,b)**  
Returns the variance of a $\text{Weibull}(a, b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.  
The variance is
\[ V[X] = b^2 \left[ \Gamma \left(1 + \frac{2}{a}\right) - \Gamma \left(1 + \frac{1}{a}\right)^2 \right] \]

**std_weibull (a,b)**  
Returns the standard deviation of a $\text{Weibull}(a, b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.  
The variance is
\[ D[X] = b \sqrt{\Gamma \left(1 + \frac{2}{a}\right) - \Gamma \left(1 + \frac{1}{a}\right)^2} \]

**skewness_weibull (a,b)**  
Returns the skewness coefficient of a $\text{Weibull}(a, b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.  
The skewness coefficient is
\[ SK[X] = \frac{\Gamma \left(1 + \frac{3}{a}\right) - 3\Gamma \left(1 + \frac{1}{a}\right) \Gamma \left(1 + \frac{2}{a}\right) + 2\Gamma \left(1 + \frac{1}{a}\right)^3}{\left[\Gamma \left(1 + \frac{2}{a}\right) - \Gamma \left(1 + \frac{1}{a}\right)^2\right]^{3/2}} \]

**kurtosis_weibull (a,b)**  
Returns the kurtosis coefficient of a $\text{Weibull}(a, b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.  
The kurtosis coefficient is
\[ KU[X] = \frac{\Gamma_4 - 4\Gamma_1\Gamma_3 + 6\Gamma_1^2\Gamma_2 - 3\Gamma_1^4}{[\Gamma_2 - \Gamma_1^2]^2} - 3 \]
where $\Gamma_k = \Gamma \left(1 + k/a\right)$.

**random_weibull (a,b)**  
**random_weibull (a,b,n)**  
Returns a $\text{Weibull}(a, b)$ random variate, with $a, b > 0$. Calling `random_weibull` with a third argument $n$, a random sample of size $n$ will be simulated.  
The implemented algorithm is based on the general inverse method.  
To make use of this function, write first `load("distrib")`. 
52.2.15 Rayleigh Random Variable

The Rayleigh distribution coincides with the \( \chi^2 \) distribution with two degrees of freedom.

**pdf\_rayleigh (x,b)**  
Returns the value at \( x \) of the density function of a Rayleigh\((b)\) random variable, with \( b > 0 \).

The Rayleigh\((b)\) random variable is equivalent to the Weibull\((2,1/b)\).

The pdf is

\[
 f(x; b) = \begin{cases} 
 2b^2 xe^{-b^2 x^2} & \text{for } x \geq 0 \\
 0 & \text{for } x < 0 
\end{cases}
\]

```
(%i1) load ("distrib")$
(%i2) pdf\_rayleigh(x,b);
```

```
2 2
2 b x %e
unit_step(x)
```

**cdf\_rayleigh (x,b)**  
Returns the value at \( x \) of the distribution function of a Rayleigh\((b)\) random variable, with \( b > 0 \).

The Rayleigh\((b)\) random variable is equivalent to the Weibull\((2,1/b)\).

The cdf is

\[
 F(x; b) = \begin{cases} 
 1 - e^{-b^2 x^2} & \text{for } x \geq 0 \\
 0 & \text{for } x < 0 
\end{cases}
\]

```
(%i1) load ("distrib")$
(%i2) cdf\_rayleigh(x,b);
```

```
(1 - %e ) unit_step(x)
```

**quantile\_rayleigh (q,b)**  
Returns the \( q \)-quantile of a Rayleigh\((b)\) random variable, with \( b > 0 \); in other words, this is the inverse of cdf\_rayleigh. Argument \( q \) must be an element of \([0,1]\).

The Rayleigh\((b)\) random variable is equivalent to the Weibull\((2,1/b)\).

```
(%i1) load ("distrib")$
(%i2) quantile\_rayleigh(0.99,b);
```

```
2.145966026289347
```

**mean\_rayleigh (b)**  
Returns the mean of a Rayleigh\((b)\) random variable, with \( b > 0 \).

The Rayleigh\((b)\) random variable is equivalent to the Weibull\((2,1/b)\).

The mean is

\[
 E[X] = \frac{\sqrt{\pi}}{2b} 
\]

```
(%i1) load ("distrib")$
```
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(mean_rayleigh(b);

(var_rayleigh(b);

Returns the variance of a Rayleigh(b) random variable, with b > 0.
The Rayleigh(b) random variable is equivalent to the Weibull(2, 1/b).
The variance is

\[ V[X] = \frac{1}{b^2} \left(1 - \frac{\pi}{4}\right) \]

(std_rayleigh(b);

Returns the standard deviation of a Rayleigh(b) random variable, with b > 0.
The Rayleigh(b) random variable is equivalent to the Weibull(2, 1/b).
The standard deviation is

\[ D[X] = \frac{1}{b} \sqrt{1 - \frac{\pi}{4}} \]

(skewness_rayleigh(b);

Returns the skewness coefficient of a Rayleigh(b) random variable, with b > 0.
The Rayleigh(b) random variable is equivalent to the Weibull(2, 1/b).
The skewness coefficient is

\[ SK[X] = \frac{2\sqrt{\pi}\left(\pi - 3\right)}{(4 - \pi)^{3/2}} \]
(%i2) skewness_rayleigh(b);
 3/2
%pi 3 sqrt(%pi)
-------- - -------------
 4 4

(%o2) ---------------------
 %pi 3/2
 (1 - ---)
 4

\textbf{kurtosis\_rayleigh (b)} \quad \text{[Function]}

Returns the kurtosis coefficient of a Rayleigh(b) random variable, with \( b > 0 \).

The Rayleigh(b) random variable is equivalent to the Weibull(2, 1/b).

The kurtosis coefficient is

\[
KU[X] = \frac{32 - 3\pi}{(4 - \pi)^2} - 3
\]

(%i1) load ("distrib")$
(%i2) kurtosis\_rayleigh(b);

2
%pi
---
16

(%o2) ---------- - 3
 %pi 2
 (1 - ---)
 4

\textbf{random\_rayleigh (b)} \quad \text{[Function]}

random\_rayleigh (b,n)

Returns a Rayleigh(b) random variate, with \( b > 0 \). Calling random\_rayleigh with a second argument \( n \), a random sample of size \( n \) will be simulated.

The implemented algorithm is based on the general inverse method.

To make use of this function, write first \texttt{load("distrib")}.

\subsection*{52.2.16 Laplace Random Variable}

The Laplace distribution is a continuous probability distribution that is sometimes called the double exponential distribution because it can be thought of as two exponential distributions spliced back to back.

\textbf{pdf\_laplace (x,a,b)} \quad \text{[Function]}

Returns the value at \( x \) of the density function of a Laplace(a, b) random variable, with \( b > 0 \). To make use of this function, write first \texttt{load("distrib")}.

Here, \( a \) is the location parameter (or mean), and \( b \) is the scale parameter, related to the variance.
The pdf is
\[
f(x; a, b) = \frac{1}{2b} \exp \left( -\frac{|x - a|}{b} \right)
\]

\text{cdf}_\text{laplace} (x,a,b) \quad \text{[Function]}
Returns the value at \( x \) of the distribution function of a \( \text{Laplace}(a, b) \) random variable, with \( b > 0 \). To make use of this function, write first \texttt{load("distrib")}.

The cdf is
\[
F(x; a, b) = \begin{cases} 
\frac{1}{2} \exp \left( \frac{x - a}{b} \right) & \text{for } x < a \\
1 - \frac{1}{2} \exp \left( \frac{x - a}{b} \right) & \text{for } x \geq a 
\end{cases}
\]

\text{quantile}_\text{laplace} (q,a,b) \quad \text{[Function]}
Returns the \( q \)-quantile of a \( \text{Laplace}(a, b) \) random variable, with \( b > 0 \); in other words, this is the inverse of \( \text{cdf}_\text{laplace} \). Argument \( q \) must be an element of \([0, 1]\). To make use of this function, write first \texttt{load("distrib")}.

\text{mean}_\text{laplace} (a,b) \quad \text{[Function]}
Returns the mean of a \( \text{Laplace}(a, b) \) random variable, with \( b > 0 \). To make use of this function, write first \texttt{load("distrib")}.

The mean is
\[
E[X] = a
\]

\text{var}_\text{laplace} (a,b) \quad \text{[Function]}
Returns the variance of a \( \text{Laplace}(a, b) \) random variable, with \( b > 0 \). To make use of this function, write first \texttt{load("distrib")}.

The variance is
\[
V[X] = 2b^2
\]

\text{std}_\text{laplace} (a,b) \quad \text{[Function]}
Returns the standard deviation of a \( \text{Laplace}(a, b) \) random variable, with \( b > 0 \). To make use of this function, write first \texttt{load("distrib")}.

The standard deviation is
\[
D[X] = \sqrt{2}b
\]

\text{skewness}_\text{laplace} (a,b) \quad \text{[Function]}
Returns the skewness coefficient of a \( \text{Laplace}(a, b) \) random variable, with \( b > 0 \). To make use of this function, write first \texttt{load("distrib")}.

The skewness coefficient is
\[
SK[X] = 0
\]

\text{kurtosis}_\text{laplace} (a,b) \quad \text{[Function]}
Returns the kurtosis coefficient of a \( \text{Laplace}(a, b) \) random variable, with \( b > 0 \). To make use of this function, write first \texttt{load("distrib")}.

The kurtosis coefficient is
\[
KU[X] = 3
\]
random_laplace (a,b)  
random_laplace (a,b,n)  
Returns a Laplace(a, b) random variate, with b > 0. Calling random_laplace with a third argument n, a random sample of size n will be simulated.

The implemented algorithm is based on the general inverse method.
To make use of this function, write first load("distrib").

52.2.17 Cauchy Random Variable
The Cauchy distribution (also known as the Lorentz distribution) is the distribution of of the ratio of two independent normally distributed random variables with mean zero.

Note that the mean, variance, standard deviation, skewness coefficient, and kurtosis coefficient are all undefined for the Cauchy distribution. The integrals do not converge in this case.

df_cauchy (x,a,b)  
Returns the value at x of the density function of a Cauchy(a, b) random variable, with b > 0. To make use of this function, write first load("distrib").

The pdf is
\[ f(x; a, b) = \frac{b}{\pi ((x - a)^2 + b^2)} \]

cdf_cauchy (x,a,b)  
Returns the value at x of the distribution function of a Cauchy(a, b) random variable, with b > 0. To make use of this function, write first load("distrib").

The cdf is
\[ F(x; a, b) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \frac{x - a}{b} \]

quantile_cauchy (q,a,b)  
Returns the q-quantile of a Cauchy(a, b) random variable, with b > 0; in other words, this is the inverse of cdf_cauchy. Argument q must be an element of [0, 1]. To make use of this function, write first load("distrib").

random_cauchy (a,b)  
random_cauchy (a,b,n)  
Returns a Cauchy(a, b) random variate, with b > 0. Calling random_cauchy with a third argument n, a random sample of size n will be simulated.

The implemented algorithm is based on the general inverse method.
To make use of this function, write first load("distrib").

52.2.18 Gumbel Random Variable
pdf_gumbel (x,a,b)  
Returns the value at x of the density function of a Gumbel(a, b) random variable, with b > 0. To make use of this function, write first load("distrib").

The pdf is
\[ f(x; a, b) = \frac{1}{b} \exp \left[ \frac{a - x}{b} - \exp \left( \frac{a - x}{b} \right) \right] \]
cdf_gumbel(x,a,b)  
Returns the value at x of the distribution function of a Gumbel(a,b) random variable, with b > 0. To make use of this function, write first load("distrib").  
The cdf is  
\[ F(x; a, b) = \exp \left[ -\exp \left( \frac{a - x}{b} \right) \right] \]

quantile_gumbel(q,a,b)  
Returns the q-quantile of a Gumbel(a,b) random variable, with b > 0; in other words, this is the inverse of cdf_gumbel. Argument q must be an element of [0,1]. To make use of this function, write first load("distrib").

mean_gumbel(a,b)  
Returns the mean of a Gumbel(a,b) random variable, with b > 0.  
The mean is  
\[ E[X] = a + b\gamma \]

\%i1) load("distrib")$  
\%i2) mean_gumbel(a,b);  
\%o2) %gamma b + a  
where symbol %gamma stands for the Euler-Mascheroni constant. See also %gamma.

var_gumbel(a,b)  
Returns the variance of a Gumbel(a,b) random variable, with b > 0. To make use of this function, write first load("distrib").  
The variance is  
\[ V[X] = \frac{\pi^2}{6} b^2 \]

std_gumbel(a,b)  
Returns the standard deviation of a Gumbel(a,b) random variable, with b > 0. To make use of this function, write first load("distrib").  
The standard deviation is  
\[ D[X] = \frac{\pi}{\sqrt{6}} b \]

skewness_gumbel(a,b)  
Returns the skewness coefficient of a Gumbel(a,b) random variable, with b > 0.  
The skewness coefficient is  
\[ SK[X] = \frac{12\sqrt{6}}{\pi^3} \zeta(3) \]

\%i1) load("distrib")$  
\%i2) skewness_gumbel(a,b);  
\%o2) (3/2) 2 6 zeta(3)  
-------------------  
%pi 3  
where zeta stands for the Riemann’s zeta function.
kurtosis_gumbel (a,b)  \hspace{1cm} [\text{Function}]  
Returns the kurtosis coefficient of a Gumbel$(a,b)$ random variable, with $b > 0$. To make use of this function, write first \texttt{load("distrib").}

The kurtosis coefficient is

$$KU[X] = \frac{12}{5}$$

random_gumbel (a,b) \hspace{1cm} [\text{Function}]  
random_gumbel (a,b,n)  
Returns a Gumbel$(a,b)$ random variate, with $b > 0$. Calling \texttt{random_gumbel} with a third argument $n$, a random sample of size $n$ will be simulated.

The implemented algorithm is based on the general inverse method.

To make use of this function, write first \texttt{load("distrib").}

52.3 Functions and Variables for discrete distributions
Maxima knows the following kinds of discrete distributions

52.3.1 General Finite Discrete Random Variable

pdf_general_finite_discrete (x,v)  \hspace{1cm} [\text{Function}]  
Returns the value at $x$ of the probability function of a general finite discrete random variable, with vector probabilities $v$, such that $\Pr(X=i) = v_i$. Vector $v$ can be a list of nonnegative expressions, whose components will be normalized to get a vector of probabilities. To make use of this function, write first \texttt{load("distrib").}

(\%i1) load ("distrib")$
(\%i2) pdf_general_finite_discrete(2, [1/7, 4/7, 2/7]);
4
(\%o2) -
7
(\%i3) pdf_general_finite_discrete(2, [1, 4, 2]);
4
(\%o3) -
7

cdf_general_finite_discrete (x,v)  \hspace{1cm} [\text{Function}]  
Returns the value at $x$ of the distribution function of a general finite discrete random variable, with vector probabilities $v$.

See \texttt{pdf_general_finite_discrete} for more details.

(\%i1) load ("distrib")$
(\%i2) cdf_general_finite_discrete(2, [1/7, 4/7, 2/7]);
5
(\%o2) -
7
(\%i3) cdf_general_finite_discrete(2, [1, 4, 2]);
5
(\%o3) -
7
quantile_general_finite_discrete \( (q,v) \) \hspace{1cm} \text{[Function]} \\
Returns the \( q \)-quantile of a general finite discrete random variable, with vector probabilities \( v \).
See \texttt{pdf_general_finite_discrete} for more details.

mean_general_finite_discrete \( (v) \) \hspace{1cm} \text{[Function]} \\
Returns the mean of a general finite discrete random variable, with vector probabilities \( v \).
See \texttt{pdf_general_finite_discrete} for more details.

var_general_finite_discrete \( (v) \) \hspace{1cm} \text{[Function]} \\
Returns the variance of a general finite discrete random variable, with vector probabilities \( v \).
See \texttt{pdf_general_finite_discrete} for more details.

std_general_finite_discrete \( (v) \) \hspace{1cm} \text{[Function]} \\
Returns the standard deviation of a general finite discrete random variable, with vector probabilities \( v \).
See \texttt{pdf_general_finite_discrete} for more details.

skewness_general_finite_discrete \( (v) \) \hspace{1cm} \text{[Function]} \\
Returns the skewness coefficient of a general finite discrete random variable, with vector probabilities \( v \).
See \texttt{pdf_general_finite_discrete} for more details.

kurtosis_general_finite_discrete \( (v) \) \hspace{1cm} \text{[Function]} \\
Returns the kurtosis coefficient of a general finite discrete random variable, with vector probabilities \( v \).
See \texttt{pdf_general_finite_discrete} for more details.

random_general_finite_discrete \( (v) \) \hspace{1cm} \text{[Function]} \\
random_general_finite_discrete \( (v,m) \) \\
Returns a general finite discrete random variate, with vector probabilities \( v \). Calling \texttt{random_general_finite_discrete} with a second argument \( m \), a random sample of size \( m \) will be simulated.
See \texttt{pdf_general_finite_discrete} for more details.

\begin{verbatim}
(\%i1) load ("distrib")$
(\%i2) random_general_finite_discrete([1,3,1,5]);
(\%o2) 4
(\%i3) random_general_finite_discrete([1,3,1,5], 10);
(\%o3) [3, 4, 3, 4, 4, 4, 4, 2, 4, 4]
\end{verbatim}
The *binomial distribution* with parameters $n$ and $p$ is a discrete probability distribution. It consists of $n$ independent experiments where each experiment consists of a Boolean-valued outcome where a success occurs with probability $p$.

For example, a biased coin that comes up heads with probability $p$ is tossed $n$ times. Then the probability of exactly $k$ heads in $n$ tosses is given by the binomial distribution.

**pdf_binomial** $(x,n,p)$

Returns the value at $x$ of the probability function of a $Binomial(n, p)$ random variable, with $0 \leq p \leq 1$ and $n$ a positive integer. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x; n, p) = \binom{n}{x} (1 - p)^{n-x} p^x$$

**cdf_binomial** $(x,n,p)$

Returns the value at $x$ of the distribution function of a $Binomial(n, p)$ random variable, with $0 \leq p \leq 1$ and $n$ a positive integer.

The cdf is

$$F(x; n, p) = I_{1-p}(n - \lfloor x \rfloor, \lfloor x \rfloor + 1)$$

where $I(z, a, b)$ is the [beta_incomplete_regularized], page 309, function.

```
(%i1) load ("distrib")
(%i2) cdf_binomial(5,7,1/6);
    7775
(%o2) ----
    7776
(%i3) float(%);
    0.9998713991769548
```  

**quantile_binomial** $(q,n,p)$

Returns the $q$-quantile of a $Binomial(n, p)$ random variable, with $0 \leq p \leq 1$ and $n$ a positive integer; in other words, this is the inverse of `cdf_binomial`. Argument $q$ must be an element of $[0, 1]$. To make use of this function, write first `load("distrib")`.

**mean_binomial** $(n,p)$

Returns the mean of a $Binomial(n, p)$ random variable, with $0 \leq p \leq 1$ and $n$ a positive integer. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = np$$

**var_binomial** $(n,p)$

Returns the variance of a $Binomial(n, p)$ random variable, with $0 \leq p \leq 1$ and $n$ a positive integer. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = np(1 - p)$$
std_binomial (n,p) [Function]
Returns the standard deviation of a Binomial(n,p) random variable, with 0 ≤ p ≤ 1 and n a positive integer. To make use of this function, write first load("distrib").
The standard deviation is
\[ D[X] = \sqrt{np(1-p)} \]

skewness_binomial (n,p) [Function]
Returns the skewness coefficient of a Binomial(n,p) random variable, with 0 ≤ p ≤ 1 and n a positive integer. To make use of this function, write first load("distrib").
The skewness coefficient is
\[ SK[X] = \frac{1-2p}{\sqrt{np(1-p)}} \]

kurtosis_binomial (n,p) [Function]
Returns the kurtosis coefficient of a Binomial(n,p) random variable, with 0 ≤ p ≤ 1 and n a positive integer. To make use of this function, write first load("distrib").
The kurtosis coefficient is
\[ KU[X] = \frac{1-6p(1-p)}{np(1-p)} \]

random_binomial (n,p) random_binomial (n,p,m) [Function]
Returns a Binomial(n,p) random variate, with 0 ≤ p ≤ 1 and n a positive integer. Calling random_binomial with a third argument m, a random sample of size m will be simulated.
To make use of this function, write first load("distrib").

52.3.3 Poisson Random Variable
The Poisson distribution is a discrete probability distribution. It is the probability that a given number of events occur in a fixed interval when the events occur independently of the time of the last event, and the events occur with a known constant rate.

pdf_poisson (x,m) [Function]
Returns the value at x of the probability function of a Poisson(m) random variable, with m > 0. To make use of this function, write first load("distrib").
The pdf is
\[ f(x;m) = \frac{m^re^{-m}}{x!} \]

cdf_poisson (x,m) [Function]
Returns the value at x of the distribution function of a Poisson(m) random variable, with m > 0.
The cdf is

\[ F(x; m) = Q(\lfloor x \rfloor + 1, m) \]

where \( Q(x, m) \) is the \([\text{gamma_incomplete_regularized}], \text{page 303}, \) function.

\[
\begin{align*}
(\%i1) & \text{ load ("distrib")$} \\
(\%i2) & \text{ cdf_poisson(3,5);} \\
(\%o2) & \text{ gamma_incomplete_regularized(4, 5)} \\
(\%i3) & \text{ float(\%);} \\
(\%o3) & \text{ 0.2650259152973619}
\end{align*}
\]

quantile_poisson \((q,m)\) \([\text{Function}]\)

Returns the \( q \)-quantile of a Poisson \((m) \) random variable, with \( m > 0 \); in other words, this is the inverse of \( \text{cdf_poisson} \). Argument \( q \) must be an element of \([0,1] \). To make use of this function, write first \text{load("distrib")}.

mean_poisson \((m)\) \([\text{Function}]\)

Returns the mean of a Poisson \((m) \) random variable, with \( m > 0 \). To make use of this function, write first \text{load("distrib")}.

The mean is

\[ E[X] = m \]

var_poisson \((m)\) \([\text{Function}]\)

Returns the variance of a Poisson \((m) \) random variable, with \( m > 0 \). To make use of this function, write first \text{load("distrib")}.

The variance is

\[ V[X] = m \]

std_poisson \((m)\) \([\text{Function}]\)

Returns the standard deviation of a Poisson \((m) \) random variable, with \( m > 0 \). To make use of this function, write first \text{load("distrib")}.

The standard deviation is

\[ V[X] = \sqrt{m} \]

skewness_poisson \((m)\) \([\text{Function}]\)

Returns the skewness coefficient of a Poisson \((m) \) random variable, with \( m > 0 \). To make use of this function, write first \text{load("distrib")}.

The skewness is

\[ SK[X] = \frac{1}{\sqrt{m}} \]

kurtosis_poisson \((m)\) \([\text{Function}]\)

Returns the kurtosis coefficient of a Poisson random variable \( \text{Poi}(m) \), with \( m > 0 \). To make use of this function, write first \text{load("distrib")}.

The kurtosis coefficient is

\[ KU[X] = \frac{1}{m} \]
random_poisson (m)
random_poisson (m,n)

Returns a Poisson(m) random variate, with \( m > 0 \). Calling \texttt{random_poisson} with a second argument \( n \), a random sample of size \( n \) will be simulated.


To make use of this function, write first \texttt{load("distrib").}

52.3.4 Bernoulli Random Variable

The Bernoulli distribution is a discrete probability distribution which takes on two values, 0 and 1. The value 1 occurs with probability \( p \), and 0 occurs with probability \( 1 - p \).

It is equivalent to the Binomial\((1, p)\) distribution (see Section 52.3.2 [Binomial Random Variable], page 802)

pdf_bernoulli (x,p)

Returns the value at \( x \) of the probability function of a Bernoulli\((p)\) random variable, with \( 0 \leq p \leq 1 \).

The Bernoulli\((p)\) random variable is equivalent to the Binomial\((1, p)\).

The mean is

\[
f(x; p) = p^x (1 - p)^{1-x}
\]

(\%i1) \texttt{load ("distrib")$}
(\%i2) pdf_bernoulli(1,p);
(\%o2) p

cdf_bernoulli (x,p)

Returns the value at \( x \) of the distribution function of a Bernoulli\((p)\) random variable, with \( 0 \leq p \leq 1 \). To make use of this function, write first \texttt{load("distrib")}.

The cdf is

\[
F(x; p) = I_{1-p}(1 - \lfloor x \rfloor, \lfloor x \rfloor + 1)
\]

quantile_bernoulli (q,p)

Returns the \( q \) -quantile of a Bernoulli\((p)\) random variable, with \( 0 \leq p \leq 1 \); in other words, this is the inverse of \texttt{cdf_bernoulli}. Argument \( q \) must be an element of \([0, 1]\).

To make use of this function, write first \texttt{load("distrib")}.

mean_bernoulli (p)

Returns the mean of a Bernoulli\((p)\) random variable, with \( 0 \leq p \leq 1 \).

The Bernoulli\((p)\) random variable is equivalent to the Binomial\((1, p)\).

The mean is

\[
E[X] = p
\]

(\%i1) \texttt{load ("distrib")$}
(\%i2) mean_bernoulli(p);
(\%o2) p
var_bernoulli (p)

Returns the variance of a Bernoulli(p) random variable, with 0 ≤ p ≤ 1.

The Bernoulli(p) random variable is equivalent to the Binomial(1,p).

The variance is

\[ V[X] = p(1 - p) \]

(%i1) load ("distrib")$
(%i2) var_bernoulli(p);
(%o2) (1 - p) p

std_bernoulli (p)

Returns the standard deviation of a Bernoulli(p) random variable, with 0 ≤ p ≤ 1.

The Bernoulli(p) random variable is equivalent to the Binomial(1,p).

The standard deviation is

\[ D[X] = \sqrt{p(1 - p)} \]

(%i1) load ("distrib")$
(%i2) std_bernoulli(p);
(%o2) sqrt((1 - p) p)

skewness_bernoulli (p)

Returns the skewness coefficient of a Bernoulli(p) random variable, with 0 ≤ p ≤ 1.

The Bernoulli(p) random variable is equivalent to the Binomial(1,p).

The skewness coefficient is

\[ SK[X] = \frac{1 - 2p}{\sqrt{p(1 - p)}} \]

(%i1) load ("distrib")$
(%i2) skewness_bernoulli(p);
(%o2) \frac{1 - 2p}{\sqrt((1 - p) p)}

kurtosis_bernoulli (p)

Returns the kurtosis coefficient of a Bernoulli(p) random variable, with 0 ≤ p ≤ 1.

The Bernoulli(p) random variable is equivalent to the Binomial(1,p).

The kurtosis coefficient is

\[ KU[X] = \frac{1 - 6p(1 - p)}{p(1 - p)} \]

(%i1) load ("distrib")$
(%i2) kurtosis_bernoulli(p);
(%o2) \frac{1 - 6p(1 - p)}{p(1 - p)}
random_bernoulli (p)
random_bernoulli (p,n)

Returns a Bernoulli(p) random variate, with $0 \leq p \leq 1$. Calling random_bernoulli with a second argument $n$, a random sample of size $n$ will be simulated.

This is a direct application of the random built-in Maxima function.

See also random. To make use of this function, write first load("distrib").

52.3.5 Geometric Random Variable

The Geometric distribution is a discrete probability distribution. It is the distribution of the number Bernoulli trials that fail before the first success.

Consider flipping a biased coin where heads occurs with probability $p$. Then the probability of $k - 1$ tails in a row followed by heads is given by the Geometric($p$) distribution.

pdf_geometric (x,p)

Returns the value at $x$ of the probability function of a Geometric($p$) random variable, with $0 < p \leq 1$.

The pdf is

$$f(x;p) = p(1 - p)^x$$

This is interpreted as the probability of $x$ failures before the first success.

load("distrib") loads this function.

cdf_geometric (x,p)

Returns the value at $x$ of the distribution function of a Geometric($p$) random variable, with $0 < p \leq 1$.

The cdf is

$$1 - (1 - p)^{1+\lfloor x \rfloor}$$

load("distrib") loads this function.

quantile_geometric (q,p)

Returns the $q$-quantile of a Geometric($p$) random variable, with $0 < p \leq 1$; in other words, this is the inverse of cdf_geometric. Argument $q$ must be an element of $[0,1]$.

The probability from which the quantile is derived is defined as $p(1 - p)^x$. This is interpreted as the probability of $x$ failures before the first success.

load("distrib") loads this function.

mean_geometric (p)

Returns the mean of a Geometric($p$) random variable, with $0 < p \leq 1$.

The mean is

$$E[X] = \frac{1}{p} - 1$$

The probability from which the mean is derived is defined as $p(1 - p)^x$. This is interpreted as the probability of $x$ failures before the first success.

load("distrib") loads this function.
var_geometric (p)  
Returns the variance of a Geometric(p) random variable, with $0 < p \leq 1$.

The variance is

$$V[X] = \frac{1 - p}{p^2}$$

load("distrib") loads this function.

std_geometric (p)  
Returns the standard deviation of a Geometric(p) random variable, with $0 < p \leq 1$.

$$D[X] = \frac{\sqrt{1 - p}}{p}$$

load("distrib") loads this function.

skewness_geometric (p)  
Returns the skewness coefficient of a Geometric(p) random variable, with $0 < p \leq 1$.

The skewness coefficient is

$$SK[X] = \frac{2 - p}{\sqrt{1 - p}}$$

load("distrib") loads this function.

kurtosis_geometric (p)  
Returns the kurtosis coefficient of a geometric random variable Geometric(p) , with $0 < p \leq 1$.

The kurtosis coefficient is

$$KU[X] = \frac{p^2 - 6p + 6}{1 - p}$$

load("distrib") loads this function.

random_geometric (p)  
random_geometric (p,n)  
random_geometric(p) returns one random sample from a Geometric(p) distribution, with $0 < p \leq 1$.

random_geometric(p, n) returns a list of n random samples.

The algorithm is based on simulation of Bernoulli trials.

The probability from which the random sample is derived is defined as $p(1-p)^x$. This is interpreted as the probability of $x$ failures before the first success.

load("distrib") loads this function.
52.3.6 Discrete Uniform Random Variable

The Discrete uniform distribution is a discrete probability distribution where a finite number of values are equally likely to occur. The values are 1, 2, 3, ..., $n$.

For example, throwing a fair die of 6 sides numbered 1 through 6 follows a $DiscreteUniform(1/6)$ distribution.

**pdf_discrete_uniform** ($x,n$)  
Returns the value at $x$ of the probability function of a $DiscreteUniform(n)$ random variable, with $n$ a strictly positive integer. To make use of this function, write first `load("distrib")`.  
The pdf is  
$$f(x,n) = \frac{1}{n}$$

**cdf_discrete_uniform** ($x,n$)  
Returns the value at $x$ of the distribution function of a $DiscreteUniform(n)$ random variable, with $n$ a strictly positive integer. To make use of this function, write first `load("distrib")`.  
The cdf is  
$$F(x; n) = \frac{\lfloor x \rfloor}{n}$$

**quantile_discrete_uniform** ($q,n$)  
Returns the $q$-quantile of a $DiscreteUniform(n)$ random variable, with $n$ a strictly positive integer; in other words, this is the inverse of `cdf_discrete_uniform`. Argument $q$ must be an element of $[0,1]$. To make use of this function, write first `load("distrib")`.  

**mean_discrete_uniform** ($n$)  
Returns the mean of a $DiscreteUniform(n)$ random variable, with $n$ a strictly positive integer. To make use of this function, write first `load("distrib")`.  
The mean is  
$$E[X] = \frac{n + 1}{2}$$

**var_discrete_uniform** ($n$)  
Returns the variance of a $DiscreteUniform(n)$ random variable, with $n$ a strictly positive integer. To make use of this function, write first `load("distrib")`.  
The variance is  
$$V[X] = \frac{n^2 - 1}{12}$$

**std_discrete_uniform** ($n$)  
Returns the standard deviation of a $DiscreteUniform(n)$ random variable, with $n$ a strictly positive integer. To make use of this function, write first `load("distrib")`.  
The standard deviation is  
$$D[X] = \frac{\sqrt{n^2 - 1}}{2\sqrt{3}}$$
skewness_discrete_uniform (n)  [Function]
Returns the skewness coefficient of a DiscreteUniform(n) random variable, with n a
strictly positive integer. To make use of this function, write first load("distrib").
The skewness coefficient is
\[ SK[X] = 0 \]

kurtosis_discrete_uniform (n)  [Function]
Returns the kurtosis coefficient of a DiscreteUniform(n) random variable, with n a
strictly positive integer. To make use of this function, write first load("distrib").
The kurtosis coefficient is
\[ KU[X] = -\frac{6(n^2 + 1)}{5(n^2 - 1)} \]

random_discrete_uniform (n)  random_discrete_uniform (n,m)
Returns a DiscreteUniform(n) random variate, with n a strictly positive integer. Calling random_discrete_uniform with a second argument m, a random sample of size m will be simulated.
This is a direct application of the random built-in Maxima function.
See also random. To make use of this function, write first load("distrib").

52.3.7 Hypergeometric Random Variable
The hypergeometric distribution is a discrete probability distribution.

Let \( n_1 \) be the number of objects of a class A and \( n_2 \) be the number of objects of class B. We take out \( n \) objects, without replacement. Then the hypergeometric distribution is
the probability that exactly \( k \) objects are from class A. Of course \( n \leq n_1 + n_2 \).

pdf_hypergeometric (x,n_1,n_2,n)
Returns the value at \( x \) of the probability function of a Hypergeometric(n1,n2,n)
random variable, with \( n_1, n_2 \) and \( n \) non negative integers and \( n \leq n_1 + n_2 \). Being
\( n_1 \) the number of objects of class A, \( n_2 \) the number of objects of class B, and \( n \) the
size of the sample without replacement, this function returns the probability of event
"exactly \( x \) objects are of class A".
To make use of this function, write first load("distrib").
The pdf is
\[
f(x; n_1, n_2, n) = \frac{\binom{n_1}{x} \binom{n_2}{n-x}}{\binom{n_2 + n_1}{n}}
\]

cdf_hypergeometric (x,n_1,n_2,n)
Returns the value at \( x \) of the distribution function of a Hypergeometric(n1,n2,n)
random variable, with \( n_1, n_2 \) and \( n \) non negative integers and \( n \leq n_1 + n_2 \). See pdf_hypergeometric for a more complete description.
To make use of this function, write first `load("distrib")`.

The cdf is

\[
F(x; n_1, n_2, n) = \left( \frac{n_2 + n_1}{n} \right)^{-1} \sum_{k=0}^{\lfloor x \rfloor} \binom{n_1}{k} \binom{n_2}{n - k}
\]

\texttt{quantile_hypergeometric(\texttt{q,n1,n2,n})}

Returns the \texttt{q}-quantile of a \texttt{Hypergeometric}(n1, n2, n) random variable, with \texttt{n1}, \texttt{n2} and \texttt{n} non negative integers and \texttt{n} \leq \texttt{n1} + \texttt{n2}; in other words, this is the inverse of \texttt{cdf_hypergeometric}. Argument \texttt{q} must be an element of \([0, 1]\). To make use of this function, write first \texttt{load("distrib")}.

\texttt{mean_hypergeometric(n_1,n_2,n) [Function]}

Returns the mean of a discrete uniform random variable \texttt{Hypergeometric}(n1, n2, n), with \texttt{n1}, \texttt{n2} and \texttt{n} non negative integers and \texttt{n} \leq \texttt{n1} + \texttt{n2}. To make use of this function, write first \texttt{load("distrib")}.

The mean is

\[
E[X] = \frac{nn_1}{n_2 + n_1}
\]

\texttt{var_hypergeometric(n1,n2,n) [Function]}

Returns the variance of a hypergeometric random variable \texttt{Hypergeometric}(n1, n2, n), with \texttt{n1}, \texttt{n2} and \texttt{n} non negative integers and \texttt{n} \leq \texttt{n1} + \texttt{n2}. To make use of this function, write first \texttt{load("distrib")}.

The variance is

\[
V[X] = \frac{nn_1n_2(n_1 + n_2 - n)}{(n_1 + n_2 - 1)(n_1 + n_2)^2}
\]

\texttt{std_hypergeometric(n_1,n_2,n) [Function]}

Returns the standard deviation of a \texttt{Hypergeometric}(n1, n2, n) random variable, with \texttt{n1}, \texttt{n2} and \texttt{n} non negative integers and \texttt{n} \leq \texttt{n1} + \texttt{n2}. To make use of this function, write first \texttt{load("distrib")}.

The standard deviation is

\[
D[X] = \frac{1}{n_1 + n_2} \sqrt{\frac{nn_1n_2(n_1 + n_2 - n)}{n_1 + n_2 - 1}}
\]

\texttt{skewness_hypergeometric(n_1,n_2,n) [Function]}

Returns the skewness coefficient of a \texttt{Hypergeometric}(n1, n2, n) random variable, with \texttt{n1}, \texttt{n2} and \texttt{n} non negative integers and \texttt{n} \leq \texttt{n1} + \texttt{n2}. To make use of this function, write first \texttt{load("distrib")}.

The skewness coefficient is

\[
SK[X] = \frac{(n_2 - n_2)(n_1 + n_2 - 2n)}{n_1 + n_2 - 2} \sqrt{\frac{n_1 + n_2 - 1}{nn_1n_2(n_1 + n_2 - n)}}
\]

\texttt{kurtosis_hypergeometric(n_1,n_2,n) [Function]}

Returns the kurtosis coefficient of a \texttt{Hypergeometric}(n1, n2, n) random variable, with \texttt{n1}, \texttt{n2} and \texttt{n} non negative integers and \texttt{n} \leq \texttt{n1} + \texttt{n2}. To make use of this function, write first \texttt{load("distrib")}.
The kurtosis coefficient is

\[ KU[X] = \left[ \frac{C(1)C(0)^2}{nn_1n_2C(3)C(2)C(n)} \times \left( \frac{3n_1n_2((n-2)C(0)^2 + 6nC(n) - n^2C(0))}{C(0)^2} - 6nC(n) + C(0)C(-1) \right) \right] - 3 \]

where \( C(k) = n_1 + n_2 - k \).

random_hypergeometric (n1,n2,n)  \hspace{1cm} \text{[Function]} \hspace{1cm} \text{random_hypergeometric (n1,n2,n,m)}

Returns a Hypergeometric \((n1,n2,n)\) random variate, with \(n1\), \(n2\) and \(n\) non negative integers and \(n <= n1+n2\). Calling \text{random_hypergeometric} with a fourth argument \(m\), a random sample of size \(m\) will be simulated.


To make use of this function, write first \text{load("distrib")}.

52.3.8 Negative Binomial Random Variable

The negative binomial distribution is a discrete probability distribution. Suppose we have a sequence of Bernoulli trials where each trial has two outcomes called “success” and “failure” where “success” occurs with probability \(p\) and “failure” with probability \(1-p\). We observe the sequence until a predefined number \(r\) of successes have occurred. Then the number of failures seen will have a \text{NegativeBinomial}(r,p)\) distribution.

pdf_negative_binomial (x,n,p)  \hspace{1cm} \text{[Function]} \hspace{1cm} \text{cdf_negative_binomial (x,n,p)}

Returns the value at \(x\) of the probability function of a \text{NegativeBinomial}(n,p) random variable, with \(0 < p \leq 1\) and \(n\) a positive number. To make use of this function, write first \text{load("distrib")}.

The pdf is

\[ f(x; n, p) = \binom{x + n - 1}{n - 1} (1-p)^x p^n \]

Returns the value at \(x\) of the distribution function of a \text{NegativeBinomial}(n,p) random variable, with \(0 < p \leq 1\) and \(n\) a positive number.

The cdf is

\[ F(x; n, p) = I_p(n, \lfloor x \rfloor + 1) \]

where \(I_p(a,b)\) is the \text{beta_incomplete_regularized}, page 309, function.

(%i1) load ("distrib")$
(%i2) cdf_negative_binomial(3,4,1/8);

(524288)
quantile_negative_binomial \( (q,n,p) \)  
Returns the \( q \)-quantile of a \( \text{NegativeBinomial}(n,p) \) random variable, with \( 0 < p \leq 1 \) and \( n \) a positive number; in other words, this is the inverse of \( \text{cdf_negative_binomial} \). Argument \( q \) must be an element of \([0,1]\). To make use of this function, write first \text{load("distrib")}.

mean_negative_binomial \( (n,p) \)  
Returns the mean of a \( \text{NegativeBinomial}(n,p) \) random variable, with \( 0 < p \leq 1 \) and \( n \) a positive number. To make use of this function, write first \text{load("distrib")}.

The mean is

\[
E[X] = \frac{n(1 - p)}{p}
\]

var_negative_binomial \( (n,p) \)  
Returns the variance of a \( \text{NegativeBinomial}(n,p) \) random variable, with \( 0 < p \leq 1 \) and \( n \) a positive number. To make use of this function, write first \text{load("distrib")}.

The variance is

\[
V[X] = \frac{n(1 - p)}{p^2}
\]

std_negative_binomial \( (n,p) \)  
Returns the standard deviation of a \( \text{NegativeBinomial}(n,p) \) random variable, with \( 0 < p \leq 1 \) and \( n \) a positive number. To make use of this function, write first \text{load("distrib")}.

The standard deviation is

\[
D[X] = \frac{\sqrt{n(1 - p)}}{p}
\]

skewness_negative_binomial \( (n,p) \)  
Returns the skewness coefficient of a \( \text{NegativeBinomial}(n,p) \) random variable, with \( 0 < p \leq 1 \) and \( n \) a positive number. To make use of this function, write first \text{load("distrib")}.

The skewness coefficient is

\[
SK[X] = \frac{2 - p}{\sqrt{n(1 - p)}}
\]

kurtosis_negative_binomial \( (n,p) \)  
Returns the kurtosis coefficient of a \( \text{NegativeBinomial}(n,p) \) random variable, with \( 0 < p \leq 1 \) and \( n \) a positive number. To make use of this function, write first \text{load("distrib")}.

The kurtosis coefficient is

\[
KU[X] = \frac{p^2 - 6p + 6}{n(1 - p)}
\]
random_negative_binomial (n,p)  
                    [Function]  
random_negative_binomial (n,p,m)  

Returns a \textit{NegativeBinomial}(n,p) random variate, with 0 < p \leq 1 and n a positive number. Calling \texttt{random_negative_binomial} with a third argument \textit{m}, a random sample of size \textit{m} will be simulated.


To make use of this function, write first \texttt{load("distrib").}
53 draw

53.1 Introduction to draw

draw is a Maxima-Gnuplot and a Maxima-VTK interface.

There are three main functions to be used at Maxima level:

- **draw2d**, draws a single 2D scene.
- **draw3d**, draws a single 3D scene.
- **draw**, can be filled with multiple **gr2d** and **gr3d** commands that each creates a draw scene all sharing the same window.

Each scene can contain any number of objects and **key=value** pairs with options for the scene or the following objects.

A selection of useful objects a scene can be made up from are:

- **explicit** plots a function.
- **implicit** plots all points an equation is true at.
- **points** plots points that are connected by lines if the option **pointsjoined** was set to **true** in a previous line of the current scene.
- **parametric** allows to specify separate expressions that calculate the x, y (and in 3d plots also for the z) variable.

A short description of all draw commands and options including example plots (in the html and pdf version of this manual) can be found in the section See Section 53.2 [Functions and Variables for draw], page 816. An online version of the html manual can be found at https://maxima.sourceforge.io/docs/manual/maxima_singlepage.html#draw. More elaborated examples of this package can be found at the following locations:

http://riotorto.users.sourceforge.net/Maxima/gnuplot/
http://riotorto.users.sourceforge.net/Maxima/vtk/

Example:

```maxima
(%i1) draw2d(
   title="Two simple plots",
   xlabel="x",ylabel="y",grid=true,
   color=red,key="A sinus",
   explicit(sin(x),x,1,10),
   color=blue,line_type=dots,key="A cosinus",
   explicit(cos(x),x,1,10)
)
```

)
Two simple plots

A sinus
A cosinus

You need Gnuplot 4.2 or newer to run draw; If you are using wxMaxima as a front end
wxdraw, wxdraw2d and wxdraw3d are drop-in replacements for draw that do the same as
\texttt{draw}, \texttt{draw2d} and \texttt{draw3d} but embed the resulting plot in the worksheet.

If you want to use VTK with draw, you need VTK with the Python interface installed
(the Chapter 55 \cite{dynamics-pkg}, page 937, uses VTK with the TCL interface!) and set the
variable:

\texttt{draw\_renderer: 'vtk $}

\section*{53.2 Functions and Variables for draw}

\subsection*{53.2.1 Scenes}

\texttt{gr2d (argument\_1, ...)} \hfill \textbf{[Scene constructor]}

Function \texttt{gr2d} builds an object describing a 2D scene. Arguments are \textit{graphic options},
\textit{graphic objects}, or lists containing both graphic options and objects. This scene is
interpreted sequentially: \textit{graphic options} affect those \textit{graphic objects} placed on its
right. Some \textit{graphic options} affect the global appearance of the scene.

This is the list of \textit{graphic objects} available for scenes in two dimensions: \texttt{bars},
\texttt{ellipse}, \texttt{explicit}, \texttt{image}, \texttt{implicit}, \texttt{label}, \texttt{parametric}, \texttt{points}, \texttt{polar}, \texttt{polygon},
\texttt{quadrilateral}, \texttt{rectangle}, \texttt{triangle}, \texttt{vector} and \texttt{geomap} (this one defined in pack-
age \texttt{worldmap}).

See also \texttt{draw} and \texttt{draw2d}.

\begin{verbatim}
(%i1) draw(
   gr2d(
       key="sin (x)",grid=[2,2],
       explicit(
           sin(x),
           x,0,2*%pi
       ),
   ),
   gr2d(
       key="cos (x)",grid=[2,2],
   )
)
\end{verbatim}
Chapter 53: draw

\[
\text{explicit(}
\begin{align*}
\cos(x), \\
x, 0, 2*\%pi
\end{align*}
\)}
\[
\text{);}
\]
\[
(%o1) \{\text{gr2d(explicit), gr2d(explicit)}\}
\]

\text{gr3d}\ (\text{argument}_1, \ldots) \quad \text{[Scene constructor]}

Function \text{gr3d} builds an object describing a 3d scene. Arguments are \textit{graphic options}, \textit{graphic objects}, or lists containing both graphic options and objects. This scene is interpreted sequentially: \textit{graphic options} affect those \textit{graphic objects} placed on its right. Some \textit{graphic options} affect the global appearance of the scene.

This is the list of \textit{graphic objects} available for scenes in three dimensions: \texttt{cylindrical}, \texttt{elevation_grid}, \texttt{explicit}, \texttt{implicit}, \texttt{label}, \texttt{mesh}, \texttt{parametric}, \texttt{parametric_surface}, \texttt{points}, \texttt{quadrilateral}, \texttt{spherical}, \texttt{triangle}, \texttt{tube}, \texttt{vector}, and \texttt{geomap} (this one defined in package \texttt{worldmap}).

See also \texttt{draw} and \texttt{draw3d}.

\textbf{53.2.2 Functions}

\texttt{draw}\ (\texttt{<arg}_1\), \ldots) \quad \text{[Function]}

Plots a series of scenes; its arguments are \texttt{gr2d} and/or \texttt{gr3d} objects, together with some options, or lists of scenes and options. By default, the scenes are put together in one column.

Besides scenes the function \texttt{draw} accepts the following global options: \texttt{terminal}, \texttt{columns}, \texttt{dimensions}, \texttt{file_name} and \texttt{delay}.

Functions \texttt{draw2d} and \texttt{draw3d} short cuts that can be used when only one scene is required, in two or three dimensions, respectively.

See also \texttt{gr2d} and \texttt{gr3d}. 
Examples:

(%i1) scene1: gr2d(title="Ellipse",
    nticks=300,
    parametric(2*cos(t),5*sin(t),t,0,2*%pi))$
(%i2) scene2: gr2d(title="Triangle",
    polygon([4,5,7],[6,4,2]))$
(%i3) draw(scene1, scene2, columns = 2)$

(%i1) scene1: gr2d(title="A sinus",
    grid=true,
    explicit(sin(t),t,0,2*%pi))$
(%i2) scene2: gr2d(title="A cosinus",
    grid=true,
    explicit(cos(t),t,0,2*%pi))$
(%i3) draw(scene1, scene2)$

The following two draw sentences are equivalent:
Chapter 53: draw

(%i1) draw(gr3d(explicit(x^2+y^2,x,-1,1,y,-1,1)));    
(%o1) [gr3d(explicit)]

(%i2) draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1));       
(%o2) [gr3d(explicit)]

Creating an animated gif file:

(%i1) draw(                                           
    delay = 100,                                          
    file_name = "zzz",                                   
    terminal = 'animated_gif,                            
    gr2d(explicit(x^2,x,-1,1)),                         
    gr2d(explicit(x^3,x,-1,1)),                         
    gr2d(explicit(x^4,x,-1,1)));                        
End of animation sequence                           
(%o1) [gr2d(explicit), gr2d(explicit), gr2d(explicit)]

See also gr2d, gr3d, draw2d and draw3d.

\textbf{draw2d (argument\_1, ...)} \textit{[Function]}
This function is a shortcut for \texttt{draw(gr2d(options, ..., graphic\_object, ...)).}
It can be used to plot a unique scene in 2d, as can be seen in most examples below.
See also \texttt{draw} and \texttt{gr2d}.

\textbf{draw3d (argument\_1, ...)} \textit{[Function]}
This function is a shortcut for \texttt{draw(gr3d(options, ..., graphic\_object, ...)).}
It can be used to plot a unique scene in 3d, as can be seen in many examples below.
See also \texttt{draw} and \texttt{gr3d}.

\textbf{draw\_file (graphic\_option, ..., graphic\_object, ...)} \textit{[Function]}
Saves the current plot into a file. Accepted graphics options are: \texttt{terminal}, \texttt{dimensions} and \texttt{file\_name}.
Example:

(%i1) /* screen plot */
draw(gr3d(explicit(x^2+y^2,x,-1,1,y,-1,1)))$

(%i2) /* same plot in eps format */
draw_file(terminal = eps,
        dimensions = [5,5])$

multiplot_mode (term)

This function enables Maxima to work in one-window multiplot mode with terminal term; accepted arguments for this function are screen, wxt, aquaterm, windows and none.

When multiplot mode is enabled, each call to draw sends a new plot to the same window, without erasing the previous ones. To disable the multiplot mode, write multiplot_mode(none).

When multiplot mode is enabled, global option terminal is blocked and you have to disable this working mode before changing to another terminal.

On Windows this feature requires Gnuplot 5.0 or newer. Note, that just plotting multiple expressions into the same plot doesn’t require multiplot: It can be done by just issuing multiple explicit or similar commands in a row.

Example:

(%i1) set_draw_defaults(
           xrange = [-1,1],
           yrange = [-1,1],
           grid   = true,
           title  = "Step by step plot" )$

(%i2) multiplot_mode(screen)$

(%i3) draw2d(color=blue, explicit(x^2,x,-1,1))$

(%i4) draw2d(color=red, explicit(x^3,x,-1,1))$

(%i5) draw2d(color=brown, explicit(x^4,x,-1,1))$

(%i6) multiplot_mode(none)$
set_draw_defaults (graphic option, ..., graphic object, ...)  [Function]
Sets user graphics options. This function is useful for plotting a sequence of graphics
with common graphics options. Calling this function without arguments removes user
defaults.

Example:

(\%i1) set_draw_defaults(
    xrange = [-10,10],
    yrange = [-2, 2],
    color = blue,
    grid = true)$
(\%i2) /* plot with user defaults */
    draw2d(explicit(((1+x)**2/(1+x*x))-1,x,-10,10))$
(\%i3) set_draw_defaults()$
(\%i4) /* plot with standard defaults */
    draw2d(explicit(((1+x)**2/(1+x*x))-1,x,-10,10))$

53.2.3 Plot options for draw programs

adapt_depth  [Graphic option]
Default value: 10

adapt_depth is the maximum number of splittings used by the adaptive plotting
routine.

This option is relevant only for 2d explicit functions.

See also nticks

allocation  [Graphic option]
Default value: false

With option allocation it is possible to place a scene in the output window at
will; this is of interest in multiplots. When false, the scene is placed automatically,
depending on the value assigned to option columns. In any other case, allocation
must be set to a list of two pairs of numbers; the first corresponds to the position of
the lower left corner of the scene, and the second pair gives the width and height of
the plot. All quantities must be given in relative coordinates, between 0 and 1.

Examples:
In site graphics.

(\%i1) draw( 
    gr2d( 
        explicit(x^2,x,-1,1)),
    gr2d( 
        allocation = [[1/4, 1/4],[1/2, 1/2]],
        explicit(x^3,x,-1,1),
        grid = true) )$
Multiplot with selected dimensions.

(%i1) draw(
   terminal = wxt,
   gr2d(
      grid=[5,5],
      allocation = [[0, 0],[1, 1/4]],
      explicit(x^2,x,-1,1)),
   gr3d(
      allocation = [[0, 1/4],[1, 3/4]],
      explicit(x^2+y^2,x,-1,1,y,-1,1) ))$

See also option `columns`.

**axis_3d**

[Graphic option]

Default value: `true`

If `axis_3d` is `true`, the x, y and z axis are shown in 3d scenes.

Since this is a global graphics option, its position in the scene description does not matter.
Example:

```
(%i1) draw3d(axis_3d = false,
             explicit(sin(x^2+y^2),x,-2,2,y,-2,2) )$
```

See also `axis_bottom`, `axis_left`, `axis_top`, and `axis_right` for axis in 2d.

**axis_bottom**

[Graphic option]

Default value: `true`

If `axis_bottom` is `true`, the bottom axis is shown in 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(axis_bottom = false,
             explicit(x^3,x,-1,1))$
```

See also `axis_left`, `axis_top`, `axis_right` and `axis_3d`. 
axis_left
Default value: true
If axis_left is true, the left axis is shown in 2d scenes.
Since this is a global graphics option, its position in the scene description does not matter.
Example:
   (%i1) draw2d(axis_left = false, explicit(x^3,x,-1,1))$

See also axis_bottom, axis_top, axis_right and axis_3d.

axis_right
Default value: true
If axis_right is true, the right axis is shown in 2d scenes.
Since this is a global graphics option, its position in the scene description does not matter.
Example:
   (%i1) draw2d(axis_right = false, explicit(x^3,x,-1,1))$

See also axis_bottom, axis_left, axis_top and axis_3d.

axis_top
Default value: true
If axis_top is true, the top axis is shown in 2d scenes.
Since this is a global graphics option, its position in the scene description does not matter.
Example:
   (%i1) draw2d(axis_top = false, explicit(x^3,x,-1,1))$

See also axis_bottom, axis_left, axis_right, and axis_3d.

background_color
Default value: white
Sets the background color for terminals. Default background color is white.
Since this is a global graphics option, its position in the scene description does not matter.
This option does not work with terminals epslatex and epslatex_standalone.
See also color

border
Default value: true
If border is true, borders of polygons are painted according to line_type and line_width.
This option affects the following graphic objects:
   • gr2d: polygon, rectangle and ellipse.
**Example:**

```lisp
(%i1) draw2d(color = brown,
              line_width = 8,
              polygon([[3,2],[7,2],[5,5]]),
              border = false,
              fill_color = blue,
              polygon([[5,2],[9,2],[7,5]]) )$
```

**capping**

[Graphic option]

Default value: [false, false]

A list with two possible elements, true and false, indicating whether the extremes of a graphic object tube remain closed or open. By default, both extremes are left open.

Setting capping = false is equivalent to capping = [false, false], and capping = true is equivalent to capping = [true, true].

**Example:**

```lisp
(%i1) draw3d(  capping = [false, true],
               tube(0, 0, a, 1,  
                    a, 0, 8) )$
```
cbrange  

Default value: auto

If cbrange is auto, the range for the values which are colored when enhanced3d is not false is computed automatically. Values outside of the color range use color of the nearest extreme.

When enhanced3d or colorbox is false, option cbrange has no effect.

If the user wants a specific interval for the colored values, it must be given as a Maxima list, as in cbrange=[-2, 3].

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw3d (  
    enhanced3d = true,  
    color = green,  
    cbrange = [-3,10],  
    explicit(x^2+y^2, x,-2,2,y,-2,2)) $
See also enhanced3d, colorbox and ctics.

**cbtics**

[Graphic option]

Default value: auto

This graphic option controls the way tic marks are drawn on the colorbox when option enhanced3d is not false.

When enhanced3d or colorbox is false, option cbtics has no effect.

See xtics for a complete description.

Example:

(\%i1) draw3d (  
    enhanced3d = true,  
    color = green,  
    cbtics = {"High",10},{"Medium",05},{"Low",0},  
    cbrange = [0, 10],  
    explicit(x^2+y^2, x,-2,2,y,-2,2) $

See also enhanced3d, colorbox and cbrange.

**color**

[Graphic option]

Default value: blue

color specifies the color for plotting lines, points, borders of polygons and labels.

Colors can be given as names or in hexadecimal rgb code. If a gnuplot version >= 5.0 is used and the terminal that is in use supports this rgba colors with transparency information are also supported.

Available color names are:

- white
- gray10
- gray30
- gray50
- gray70
- gray90
- black
- grey10
- grey30
- grey50
- grey70
- grey90
- gray0
- grey20
- grey40
- grey60
- grey80
- grey100
Cromatic components in hexadecimal code are introduced in the form "#rrggbb".

Example:

(%i1) draw2d(explicit(x^2,x,-1,1), /* default is black */
    color = red,
    explicit(0.5 + x^2,x,-1,1),
    color = blue,
    explicit(1 + x^2,x,-1,1),
    color = light_blue,
    explicit(1.5 + x^2,x,-1,1),
    color = "#23ab0f",
    label(["This is a label",0,1.2]) )$

(%i1) draw2d(
    line_width=50,
    color="#FF0000",
    explicit(sin(x),x,0,10),

\[ H(p, p_0) := \frac{i}{2\pi i (p - p_0)} \]

\(\text{draw2d(}
    \text{proportional_axes=xy,}
    \text{ip_grid=[150,150],}
    \text{grid=true,}
    \text{makelist(}
        \text{[}
            \text{color=printf(false,"#~2,0x~2,0x~2,0x",i*10,0,0),}
            \text{key_pos=top_left,}
            \text{key = if mod(i,5)=0 then sconcat("H=",i,"A/M") else "",}
            \text{implicit(}
                \text{cabs(H(x+%i*y,-1-%i)+H(x+%i*y,1+%i)-H(x+%i*y,1-%i)}
                \text{-H(x+%i*y,-1+%i))=i/10,}
                \text{x,-3,3,}
                \text{y,-3,3}
            \text{)}
        \text{],}
    \text{}],}
    \text{i,1,25}
\text{)}\)
(%i1) draw2d(
    "figures/draw_color4",
    makelist(
        [color=i,
         key=sconcat("color =",i),
         explicit(sin(i*x),x,0,1)
        ],
        i,0,17
    )
)
)

See also \texttt{fill\_color}.

\texttt{colorbox} \hspace{1cm} \text{[Graphic option]}

Default value: \texttt{true}
If \texttt{colorbox} is \texttt{true}, a color scale without label is drawn together with \texttt{image} 2D objects, or coloured 3d objects. If \texttt{colorbox} is \texttt{false}, no color scale is shown. If \texttt{colorbox} is a string, a color scale with label is drawn.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

Color scale and images.

\begin{verbatim}
(%i1) im: apply('matrix,
    makelist(makelist(random(200),i,1,30),i,1,30))$
(%i2) draw(
    gr2d(image(im,0,0,30,30)),
    gr2d(colorbox = false, image(im,0,0,30,30))
)
\end{verbatim}

Color scale and 3D coloured object.

\begin{verbatim}
(%i1) draw3d(
    colorbox = "Magnitude",
    enhanced3d = true,
    explicit(x^2+y^2,x,-1,1,y,-1,1))$
\end{verbatim}
columns [Graphic option]

Default value: 1

columns is the number of columns in multiple plots.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function draw.

Example:

(%i1) scene1: gr2d(title="Ellipse",
    nticks=30,
    parametric(2*cos(t),5*sin(t),t,0,2*%pi))$

(%i2) scene2: gr2d(title="Triangle",
    polygon([4,5,7],[6,4,2]))$

(%i3) draw(scene1, scene2, columns = 2)$

contour [Graphic option]

Default value: none
Option `contour` enables the user to select where to plot contour lines. Possible values are:

- **none**: no contour lines are plotted.
- **base**: contour lines are projected on the xy plane.
- **surface**: contour lines are plotted on the surface.
- **both**: two contour lines are plotted: on the xy plane and on the surface.
- **map**: contour lines are projected on the xy plane, and the view point is set just in the vertical.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d(explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),
    contour_levels = 15,
    contour       = both,
    surface_hide  = true)
```

```
(%i1) draw3d(explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),
    contour_levels = 15,
    contour       = map)
```
**contour_levels**

[Graphic option]

Default value: 5

This graphic option controls the way contours are drawn. `contour_levels` can be set to a positive integer number, a list of three numbers or an arbitrary set of numbers:

- When option `contour_levels` is bounded to positive integer \( n \), \( n \) contour lines will be drawn at equal intervals. By default, five equally spaced contours are plotted.
- When option `contour_levels` is bounded to a list of length three of the form \([\text{lowest}, s, \text{highest}]\), contour lines are plotted from `lowest` to `highest` in steps of `s`.
- When option `contour_levels` is bounded to a set of numbers of the form \(\{n_1, n_2, \ldots\}\), contour lines are plotted at values \(n_1, n_2, \ldots\)

Since this is a global graphics option, its position in the scene description does not matter.

Examples:
Ten equally spaced contour lines. The actual number of levels can be adjusted to give simple labels.

```plaintext
(%i1) draw3d(color = green,
    explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),
    contour_levels = 10,
    contour = both,
    surface_hide = true) $
```

From -8 to 8 in steps of 4.

```plaintext
(%i1) draw3d(color = green,
    explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),
    contour_levels = [-8,4,8],
    contour = both,
    surface_hide = true) $
```

Isolines at levels -7, -6, 0.8 and 5.

```plaintext
(%i1) draw3d(color = green,
```
explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),
contour_levels = {-7, -6, 0.8, 5},
contour = both,
surface_hide = true) $

See also contour.

data_file_name

[Graphic option]
Default value: "data.gnuplot"
This is the name of the file with the numeric data needed by Gnuplot to build the requested plot.
Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function draw.
See example in gnuplot_file_name.

delay

[Graphic option]
Default value: 5
This is the delay in 1/100 seconds of frames in animated gif files.
Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function draw.
Example:

(%i1) draw(  
    delay = 100,
    file_name = "zzz",
    terminal = 'animated_gif,
    gr2d(explicit(x^2,x,-1,1)),
    gr2d(explicit(x^3,x,-1,1)),
    gr2d(explicit(x^4,x,-1,1)));
End of animation sequence
(%o2) [gr2d(explicit), gr2d(explicit), gr2d(explicit)]

Option delay is only active in animated gif's; it is ignored in any other case.
See also terminal, and dimensions.

dimensions

[Graphic option]
Default value: [600,500]
Dimensions of the output terminal. Its value is a list formed by the width and the height. The meaning of the two numbers depends on the terminal you are working with.
With terminals gif, animated_gif, png, jpg, svg, screen, wxt, and aquaterm, the integers represent the number of points in each direction. If they are not integers, they are rounded.
With terminals eps, eps_color, pdf, and pdfcairo, both numbers represent hundredths of cm, which means that, by default, pictures in these formats are 6 cm in width and 5 cm in height.
Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function draw.
Examples:

Option `dimensions` applied to file output and to wxt canvas.

(\%i1) draw2d(
    dimensions = [300,300],
    terminal   = 'png,
    explicit(x^4,x,-1,1) )$

(\%i2) draw2d(
    dimensions = [300,300],
    terminal   = 'wxt,
    explicit(x^4,x,-1,1) )$

Option `dimensions` applied to eps output. We want an eps file with A4 portrait dimensions.

(\%i1) A4portrait: 100*[21, 29.7]$

(\%i2) draw3d(
    dimensions = A4portrait,
    terminal   = 'eps,
    explicit(x^2-y^2,x,-2,2,y,-2,2) )$

draw_realpart

[Graphic option]
Default value: true

When true, functions to be drawn are considered as complex functions whose real part value should be plotted; when false, nothing will be plotted when the function does not give a real value.

This option affects objects `explicit` and `parametric` in 2D and 3D, and `parametric_surface`.

Example:

(\%i1) draw2d(
    draw_realpart = false,
    explicit(sqrt(x^2 - 4*x) - x, x, -1, 5),
    color       = red,
    draw_realpart = true,
    parametric(x,sqrt(x^2 - 4*x) - x + 1, x, -1, 5) )

draw_realpart

[Graphic option]

enhanced3d

[Graphic option]
Default value: none

If `enhanced3d` is none, surfaces are not colored in 3D plots. In order to get a colored surface, a list must be assigned to option `enhanced3d`, where the first element is an expression and the rest are the names of the variables or parameters used in that expression. A list such [f(x,y,z), x, y, z] means that point [x,y,z] of the surface is assigned number f(x,y,z), which will be colored according to the actual palette. For those 3D graphic objects defined in terms of parameters, it is possible to define the color number in terms of the parameters, as in [f(u), u], as in objects `parametric` and `tube`, or [f(u,v), u, v], as in object `parametric_surface`. While all 3D objects admit the model based on absolute coordinates, [f(x,y,z), x, y, z], only two of them, namely `explicit` and `elevation_grid`, accept also models defined...
on the \([x,y]\) coordinates, \([f(x,y), x, y]\). 3D graphic object \textit{implicit} accepts only the \([f(x,y,z), x, y, z]\) model. Object \textit{points} accepts also the \([f(x,y,z), x, y, z]\) model, but when points have a chronological nature, model \([f(k), k]\) is also valid, being \(k\) an ordering parameter.

When \textit{enhanced3d} is assigned something different to \textit{none}, options \textit{color} and \textit{surface_hide} are ignored.

The names of the variables defined in the lists may be different to those used in the definitions of the graphic objects.

In order to maintain back compatibility, \textit{enhanced3d} = \textit{false} is equivalent to \textit{enhanced3d} = \textit{none}, and \textit{enhanced3d} = \textit{true} is equivalent to \textit{enhanced3d} = \([z, x, y, z]\). If an expression is given to \textit{enhanced3d}, its variables must be the same used in the surface definition. This is not necessary when using lists.

See option \textit{palette} to learn how palettes are specified.

Examples:

\textit{explicit} object with coloring defined by the \([f(x,y,z), x, y, z]\) model.

\begin{verbatim}
(%i1) draw3d(  
  enhanced3d = [x-z/10,x,y,z],  
  palette = gray,  
  explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3))$
\end{verbatim}

\textit{explicit} object with coloring defined by the \([f(x,y), x, y]\) model. The names of the variables defined in the lists may be different to those used in the definitions of the graphic objects; in this case, \(r\) corresponds to \(x\), and \(s\) to \(y\).

\begin{verbatim}
(%i1) draw3d(  
  enhanced3d = [sin(r*s),r,s],  
  explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3))$
\end{verbatim}
parametric object with coloring defined by the \([f(x,y,z), x, y, z]\) model.

(%i1) draw3d(
    nticks = 100,
    line_width = 2,
    enhanced3d = [if y>= 0 then 1 else 0, x, y, z],
    parametric(sin(u)^2,cos(u),u,u,0,4*%pi))$

parametric object with coloring defined by the \([f(u), u]\) model. In this case, \((u-1)^2\) is a shortcut for \([(u-1)^2,u]\).

(%i1) draw3d(
    nticks = 60,
    line_width = 3,
    enhanced3d = (u-1)^2,
    parametric(cos(5*u)^2,sin(7*u),u-2,u,0,2))$
**elevation_grid** object with coloring defined by the \([f(x,y), x, y]\) model.

```lisp
(%i1) m: apply(
    matrix,
    makelist(makelist(cos(i^2/80-k/30),k,1,30),i,1,20)) $
(%i2) draw3d(
    enhanced3d = [cos(x*y*10),x,y],
    elevation_grid(m,-1,-1,2,2),
    xlabel = "x",
    ylabel = "y");
```

**tube** object with coloring defined by the \([f(x,y,z), x, y, z]\) model.

```lisp
(%i1) draw3d(
    enhanced3d = [cos(x-y),x,y,z],
    palette = gray,
    xu_grid = 50,
    tube(cos(a), a, 0, 1, a, 0, 4*%pi) )$
```
tube object with coloring defined by the \([f(u), u]\) model. Here, \texttt{enhanced3d} = \texttt{-a} would be the shortcut for \texttt{enhanced3d} = \texttt{[-foo,foo]}.

\begin{verbatim}
(%i1) draw3d(
capping = [true, false],
palette = [26,15,-2],
enhanced3d = [-foo, foo],
tube(a, a, a^2, 1, a, -2, 2 )$
\end{verbatim}

\begin{verbatim}
(%i1) draw3d(
enhanced3d = [x-y,x,y,z],
implicit((x^2+y^2+z^2-1)*(x^2+(y-1.5)^2+z^2-0.5)=0.015,
x,-1,1,y,-1.2,2.3,z,-1,1)) $
\end{verbatim}

\begin{verbatim}
(%i2) m: makelist([random(1.0),random(1.0),random(1.0)],k,1,2000)$
\end{verbatim}
When points have a chronological nature, model \([f(k), k]\) is also valid, being \(k\) an ordering parameter.
error_type

Default value: y

Depending on its value, which can be x, y, or xy, graphic object errors will draw points with horizontal, vertical, or both, error bars. When error_type=boxes, boxes will be drawn instead of crosses.

See also errors.

file_name

Default value: "maxima_out"

This is the name of the file where terminals png, jpg, gif, eps, eps_color, pdf, pdfcairo and svg will save the graphic.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function draw.

Example:

(\%i1) draw2d(file_name = "myfile",
    explicit(x^2,x,-1,1),
    terminal = 'png)\$

See also terminal, dimensions_draw.

fill_color

Default value: "red"

fill_color specifies the color for filling polygons and 2d explicit functions.

See color to learn how colors are specified.

fill_density

Default value: 0

fill_density is a number between 0 and 1 that specifies the intensity of the fill_color in bars objects.

See bars for examples.
filled_func  
Default value: false

Option filled_func controls how regions limited by functions should be filled. When filled_func is true, the region bounded by the function defined with object explicit and the bottom of the graphic window is filled with fill_color. When filled_func contains a function expression, then the region bounded by this function and the function defined with object explicit will be filled. By default, explicit functions are not filled.

A useful special case is filled_func=0, which generates the region bond by the horizontal axis and the explicit function.

This option affects only the 2d graphic object explicit.

Example:

Region bounded by an explicit object and the bottom of the graphic window.

(%i1) draw2d(fill_color = red,  
     filled_func = true,  
     explicit(sin(x),x,0,10) )$

Region bounded by an explicit object and the function defined by option filled_func. Note that the variable in filled_func must be the same as that used in explicit.

(%i1) draw2d(fill_color = grey,  
     filled_func = sin(x),  
     explicit(-sin(x),x,0,%pi));
See also `fill_color` and `explicit`.  

`font`

[Graphic option]

Default value: "" (empty string)

This option can be used to set the font face to be used by the terminal. Only one font face and size can be used throughout the plot.

Since this is a global graphics option, its position in the scene description does not matter.

See also `font_size`.

Gnuplot doesn’t handle fonts by itself, it leaves this task to the support libraries of the different terminals, each one with its own philosophy about it. A brief summary follows:

- **x11**: Uses the normal x11 font server mechanism.
  
  Example:
  
  ```
  (%i1) draw2d(font = "Arial",
               font_size = 20,
               label(["Arial font, size 20",1,1]))$
  ```

- **windows**: The windows terminal doesn’t support changing of fonts from inside the plot. Once the plot has been generated, the font can be changed right-clicking on the menu of the graph window.

- **png, jpeg, gif**: The *libgd* library uses the font path stored in the environment variable `GDFONTPATH`; in this case, it is only necessary to set option `font` to the font’s name. It is also possible to give the complete path to the font file.

  Examples:
  
  Option `font` can be given the complete path to the font file:
  
  ```
  (%i1) path: "/usr/share/fonts/truetype/freefont/"$
  (%i2) file: "FreeSerifBoldItalic.ttf"$
  (%i3) draw2d(
          font = concat(path, file),
          font_size = 20,
          color = red,
          )
  ```
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```
label(["FreeSerifBoldItalic font, size 20",1,1]),
terminal = png)
```

If environment variable GDFONTPATH is set to the path where font files are allocated, it is possible to set graphic option font to the name of the font.

```
(%i1) draw2d(
    font = "FreeSerifBoldItalic",
    font_size = 20,
    color = red,
    label(["FreeSerifBoldItalic font, size 20",1,1]),
    terminal = png)
```


Example:

```
(%i1) draw2d(
    font = "Courier-Oblique",
    font_size = 15,
    label(["Courier-Oblique font, size 15",1,1]),
    terminal = eps)
```

- **pdf**: Uses same fonts as Postscript.
- **pdfcairo**: Uses same fonts as wxt.
- **wxt**: The pango library finds fonts via the fontconfig utility.
- **aqua**: Default is "Times-Roman".

The gnuplot documentation is an important source of information about terminals and fonts.

**font_size**

[Graphic option]

Default value: 10

This option can be used to set the font size to be used by the terminal. Only one font face and size can be used throughout the plot. **font_size** is active only when option **font** is not equal to the empty string.

Since this is a global graphics option, its position in the scene description does not matter.

See also **font**.

**gnuplot_file_name**

[Graphic option]

Default value: "maxout_xxx.gnuplot" with "xxx" being a number that is unique to each concurrently-running maxima process.

This is the name of the file with the necessary commands to be processed by Gnuplot.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function **draw**.
Example:

(%i1) draw2d(
    file_name = "my_file",
    gnuplot_file_name = "my_commands_for_gnuplot",
    data_file_name = "my_data_for_gnuplot",
    terminal = png,
    explicit(x^2,x,-1,1))$

See also data_file_name.

grid [Graphic option]

Default value: false

If grid is not false, a grid will be drawn on the xy plane. If grid is assigned true, one grid line per tick of each axis is drawn. If grid is assigned a list nx,ny with \([nx,ny] > [0,0]\) instead nx lines per tick of the x axis and ny lines per tick of the y axis are drawn.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw2d(grid = true,
    explicit(exp(u),u,-2,2))$

(%i1) draw2d(grid = [2,2],
    explicit(sin(x),x,0,2*%pi))$
head_angle

Default value: 45

head_angle indicates the angle, in degrees, between the arrow heads and the segment.

This option is relevant only for vector objects.

Example:

(%i1) draw2d(xrange = [0,10],
yrange = [0,9],
head_length = 0.7,
head_angle  = 10,
vector([1,1],[0,6]),
head_angle  = 20,
vector([2,1],[0,6]),
head_angle  = 30,
vector([3,1],[0,6]),
head_angle  = 40,
vector([4,1],[0,6]),
head_angle  = 60,
vector([5,1],[0,6]),
head_angle  = 90,
vector([6,1],[0,6]),
head_angle  = 120,
vector([7,1],[0,6]),
head_angle  = 160,
vector([8,1],[0,6]),
head_angle  = 180,
vector([9,1],[0,6]) )$
See also `head_both`, `head_length`, and `head_type`.

**head_both**

[Graphic option]

Default value: `false`

If `head_both` is `true`, vectors are plotted with two arrow heads. If `false`, only one arrow is plotted.

This option is relevant only for `vector` objects.

Example:

```
(%i1) draw2d(xrange = [0,8],
yrange = [0,8],
head_length = 0.7,
vector([1,1],[6,0]),
head_both = true,
vector([1,7],[6,0]))$
```

See also `head_length`, `head_angle`, and `head_type`. 
**head_length**

Default value: 2

*head_length* indicates, in x-axis units, the length of arrow heads.

This option is relevant only for *vector* objects.

Example:

```
(%i1) draw2d(xrange = [0,12],
            yrange = [0,8],
            vector([0,1],[5,5]),
            head_length = 1,
            vector([2,1],[5,5]),
            head_length = 0.5,
            vector([4,1],[5,5]),
            head_length = 0.25,
            vector([6,1],[5,5]))
```

See also *head_both*, *head_angle*, and *head_type*.

**head_type**

Default value: *filled*

*head_type* is used to specify how arrow heads are plotted. Possible values are: *filled* (closed and filled arrow heads), *empty* (closed but not filled arrow heads), and *nofilled* (open arrow heads).

This option is relevant only for *vector* objects.

Example:

```
(%i1) draw2d(xrange = [0,12],
            yrange = [0,10],
            head_length = 1,
            vector([0,1],[5,5]), /* default type */
            head_type = 'empty,
            vector([3,1],[5,5]),
```

```
See also head_both, head_angle, and head_length.

interpolate_color

[Graphic option]

Default value: false

This option is relevant only when enhanced3d is not false.

When interpolate_color is false, surfaces are colored with homogeneous quadrangles. When true, color transitions are smoothed by interpolation.

interpolate_color also accepts a list of two numbers, [m,n]. For positive m and n, each quadrangle or triangle is interpolated m times and n times in the respective direction. For negative m and n, the interpolation frequency is chosen so that there will be at least |m| and |n| points drawn; you can consider this as a special gridding function. Zeros, i.e. interpolate_color=[0,0], will automatically choose an optimal number of interpolated surface points.

Also, interpolate_color=true is equivalent to interpolate_color=[0,0].

Examples:

Color interpolation with explicit functions.

(\%i1) draw3d(
    enhanced3d = sin(x*y),
    explicit(20*exp(-x^2-y^2)-10, x, -3, 3, y, -3, 3))$
Chapter 53: draw

(%i2) draw3d(
   interpolate_color = true,
   enhanced3d = sin(x*y),
   explicit(20*exp(-x^2-y^2)-10, x , -3, 3, y, -3, 3)) $

(%i3) draw3d(
   interpolate_color = [-10,0],
   enhanced3d = sin(x*y),
   explicit(20*exp(-x^2-y^2)-10, x , -3, 3, y, -3, 3)) $
Color interpolation with the **mesh** graphic object.

Interpolating colors in parametric surfaces can give unexpected results.

(%i1) draw3d(
   enhanced3d = true,
   mesh([[1,1,3], [7,3,1],[12,-2,4], [15,0,5]],
      [[2,7,8], [4,3,1],[10,5,8], [12,7,1]],
      [[-2,11,10],[6,9,5],[6,15,1], [20,15,2]])) $

(%i2) draw3d(
   enhanced3d = true,
   interpolate_color = true,
   mesh([[1,1,3], [7,3,1],[12,-2,4], [15,0,5]],
      [[2,7,8], [4,3,1],[10,5,8], [12,7,1]],
      [[-2,11,10],[6,9,5],[6,15,1], [20,15,2]])) $
Chapter 53: draw

(%i3) draw3d(
    enhanced3d = true,
    interpolate_color = true,
    view = map,
    mesh([[1, 1, 3], [7, 3, 1], [12, -2, 4], [15, 0, 5]],
          [[2, 7, 8], [4, 3, 1], [10, 5, 8], [12, 7, 1]],
          [[-2, 11, 10], [6, 9, 5], [6, 15, 1], [20, 15, 2]])$

See also enhanced3d.

**ip_grid**

[Graphic option]

- Default value: [50, 50]

  **ip_grid** sets the grid for the first sampling in implicit plots.

  This option is relevant only for implicit objects.

**ip_grid_in**

[Graphic option]

- Default value: [5, 5]

  **ip_grid_in** sets the grid for the second sampling in implicit plots.
This option is relevant only for implicit objects.

**key**

[Graphic option]

Default value: "" (empty string)

key is the name of a function in the legend. If key is an empty string, no key is assigned to the function.

This option affects the following graphic objects:

- **gr2d**: points, polygon, rectangle, ellipse, vector, explicit, implicit, parametric and polar.
- **gr3d**: points, explicit, parametric and parametric_surface.

Example:

```maxima
(%i1) draw2d(key = "Sinus",
    explicit(sin(x),x,0,10),
    key = "Cosinus",
    color = red,
    explicit(cos(x),x,0,10) )$
```

**key_pos**

[Graphic option]

Default value: "" (empty string)

key_pos defines at which position the legend will be drawn. If key is an empty string, "top_right" is used. Available position specifiers are: top_left, top_center, top_right, center_left, center, center_right, bottom_left, bottom_center, and bottom_right.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```maxima
(%i1) draw2d(
    key_pos = top_left,
    key = "x",
    explicit(sin(x),x,0,10),
    key = "Cosinus",
    color = red,
    explicit(cos(x),x,0,10) )$
```
explicit(x, x,0,10),
color= red,
key  = "x squared",
explicit(x^2,x,0,10))$

(%i3) draw3d(
    key_pos = center,
    key = "x",
    explicit(x+y,x,0,10,y,0,10),
    color= red,
    key = "x squared",
    explicit(x^2+y^2,x,0,10,y,0,10))$

**label_alignment**

[Graphic option]

Default value: center

=label_alignment= is used to specify where to write labels with respect to the given coordinates. Possible values are: center, left, and right.

This option is relevant only for label objects.

Example:

(%i1) draw2d(xrange = [0,10],
yrange = [0,10],
pointsJoined = true,
points([[5,0],[5,10]]),
color = blue,
label(["Centered alignment (default)",5,2]),
label_alignment = 'left,
label(["Left alignment",5,5]),
label_alignment = 'right,
label(["Right alignment",5,8]))$
See also `label_orientation`, and `color`.

`label_orientation` [Graphic option]

Default value: `horizontal`

`label_orientation` is used to specify orientation of labels. Possible values are: `horizontal`, and `vertical`.

This option is relevant only for `label` objects.

Example:

In this example, a dummy point is added to get an image. Package `draw` needs always data to draw an scene.

```
(%i1) draw2d(xrange = [0,10],
    yrange = [0,10],
    point_size = 0,
    points([[5,5]]),
    color = navy,
    label("Horizontal orientation (default)",5,2),
    label_orientation = 'vertical,
    color = "#654321",
    label("Vertical orientation",1,5))$
```
See also `label_alignment` and `color`

**line_type**

[Graphic option]

Default value: `solid`

`line_type` indicates how lines are displayed; possible values are `solid` and `dots`, both available in all terminals, and `dashes`, `short_dashes`, `short_long_dashes`, `short_short_long_dashes`, and `dot_dash`, which are not available in `png`, `jpg`, and `gif` terminals.

This option affects the following graphic objects:

- **gr2d**: `points`, `polygon`, `rectangle`, `ellipse`, `vector`, `explicit`, `implicit`, `parametric` and `polar`.

- **gr3d**: `points`, `explicit`, `parametric` and `parametric_surface`.

Example:

```latex
(%i1) draw2d(line_type = dots,
   explicit(1 + x^2,x,-1,1),
   line_type = solid, /* default */
   explicit(2 + x^2,x,-1,1))$
```
See also `line_width`.

`line_width`  
[Graphic option]  

Default value: 1  

`line_width` is the width of plotted lines. Its value must be a positive number.  
This option affects the following graphic objects:  
- `gr2d`: `points`, `polygon`, `rectangle`, `ellipse`, `vector`, `explicit`, `implicit`, `parametric` and `polar`.  
- `gr3d`: `points` and `parametric`.  

Example:  
```plaintext  
(%i1) draw2d(explicit(x^2,x,-1,1), /* default width */  
    line_width = 5.5,  
    explicit(1 + x^2,x,-1,1),  
    line_width = 10,  
    explicit(2 + x^2,x,-1,1))$  
```  

See also `line_type`.
logcb

[Graphic option]

Default value: false

If logcb is true, the tics in the colorbox will be drawn in the logarithmic scale.

When enhanced3d or colorbox is false, option logcb has no effect.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```lisp
(%i1) draw3d(
    enhanced3d = true,
    color = green,
    logcb = true,
    logz = true,
    palette = [-15,24,-9],
    explicit(exp(x^2-y^2), x,-2,2,y,-2,2))$
```

See also enhanced3d, colorbox and cbrange.

logx

[Graphic option]

Default value: false

If logx is true, the x axis will be drawn in the logarithmic scale.

Since this is a global graphics option, its position in the scene description does not matter, with the exception that it should be written before any 2D explicit object, so that draw can produce a better plot.

Example:

```lisp
(%i1) draw2d(logx = true,
    explicit(log(x),x,0.01,5))$
```

See also logy, logx_secondary, logy_secondary, and logz.

logx_secondary

[Graphic option]

Default value: false
If \texttt{logx\_secondary} is \texttt{true}, the secondary x axis will be drawn in the logarithmic scale.

This option is relevant only for 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

\begin{verbatim}
(%i1) draw2d(
    grid = true,
    key="x^2, linear scale",
    color=red,
    explicit(x^2,x,1,100),
    xaxis_secondary = true,
    xtics_secondary = true,
    logx_secondary = true,
    key = "x^2, logarithmic x scale",
    color = blue,
    explicit(x^2,x,1,100) )$
\end{verbatim}

See also \texttt{logx\_draw}, \texttt{logy\_draw}, \texttt{logy\_secondary}, and \texttt{logz}.

\texttt{logy} \hspace{10cm} \texttt{[Graphic option]}

Default value: \texttt{false}

If \texttt{logy} is \texttt{true}, the y axis will be drawn in the logarithmic scale.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

\begin{verbatim}
(%i1) draw2d(logy = true,
    explicit(exp(x),x,0,5))$
\end{verbatim}

See also \texttt{logx\_draw}, \texttt{logx\_secondary}, \texttt{logy\_secondary}, and \texttt{logz}.
logy_secondary

Default value: false

If logy_secondary is true, the secondary y axis will be drawn in the logarithmic scale.

This option is relevant only for 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw2d(
   grid = true,
   key="x^2, linear scale",
   color=red,
   explicit(x^2,x,1,100),
   yaxis_secondary = true,
   ytics_secondary = true,
   logy_secondary = true,
   key = "x^2, logarithmic y scale",
   color = blue,
   explicit(x^2,x,1,100) )$

See also logx_draw, logy_draw, logx_secondary, and logz.

logz

Default value: false

If logz is true, the z axis will be drawn in the logarithmic scale.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw3d(logz = true,
   explicit(exp(u^2+v^2),u,-2,2,v,-2,2))$

See also logx_draw and logy_draw.

nticks

Default value: 29

In 2d, nticks gives the initial number of points used by the adaptive plotting routine for explicit objects. It is also the number of points that will be shown in parametric and polar curves.

This option affects the following graphic objects:

- gr2d: ellipse, explicit, parametric and polar.
- gr3d: parametric.

See also adapt_depth

Example:

(%i1) draw2d(transparent = true,
   ellipse(0,0,2,0,180),
\texttt{nticks = 5,}
\texttt{ellipse(0,0,4,2,180,180) }$

\textbf{palette}

Default value: \texttt{color}

\texttt{palette} indicates how to map gray levels onto color components. It works together with option \texttt{enhanced3d} in 3D graphics, who associates every point of a surfaces to a real number or gray level. It also works with gray images. With \texttt{palette}, levels are transformed into colors.

There are two ways for defining these transformations.

First, \texttt{palette} can be a vector of length three with components ranging from -36 to +36; each value is an index for a formula mapping the levels onto red, green and blue colors, respectively:

\begin{align*}
0: & \quad 0 \\
3: & \quad x \\
6: & \quad x^4 \\
9: & \quad \sin(90x) \\
12: & \quad (2x-1)^2 \\
15: & \quad \sin(360x) \\
18: & \quad |\cos(360x)| \\
21: & \quad 3x \\
24: & \quad |3x-1| \\
27: & \quad (3x-2)/2 \\
30: & \quad x/0.32-0.78125 \\
33: & \quad |2x - 0.5| \\
36: & \quad 2x - 1 \\
1: & \quad 0.5 \\
4: & \quad x^2 \\
7: & \quad \sqrt{x} \\
10: & \quad \cos(90x) \\
13: & \quad \sin(180x) \\
16: & \quad \cos(360x) \\
19: & \quad |\sin(720x)| \\
22: & \quad 3x-1 \\
25: & \quad |3x-2| \\
28: & \quad |(3x-1)/2| \\
31: & \quad 2x-1.84 \\
34: & \quad 2x \\
37: & \quad 2x - 0.5 \\
2: & \quad 1 \\
5: & \quad x^3 \\
8: & \quad \sqrt{\sqrt{x}} \\
11: & \quad |x-0.5| \\
14: & \quad |\cos(180x)| \\
17: & \quad |\sin(360x)| \\
20: & \quad |\cos(720x)| \\
23: & \quad 3x-2 \\
26: & \quad (3x-1)/2 \\
29: & \quad |(3x-2)/2| \\
32: & \quad 4x;1;-2x+1.84;x/0.08-11.5 \\
35: & \quad 2x - 0.5
\end{align*}

negative numbers mean negative colour component. \texttt{palette = gray} and \texttt{palette = color} are short cuts for \texttt{palette = [3,3,3]} and \texttt{palette = [7,5,15]}, respectively.

Second, \texttt{palette} can be a user defined lookup table. In this case, the format for building a lookup table of length \texttt{n} is \texttt{palette=[color\_1, color\_2, \ldots, color\_n]}.
n], where \texttt{color}_i\ is a well formed color (see option \texttt{color}) such that \texttt{color}_1\ is assigned to the lowest gray level and \texttt{color}_n\ to the highest. The rest of colors are interpolated.

Since this is a global graphics option, its position in the scene description does not matter.

Examples:

It works together with option \texttt{enhanced3d} in 3D graphics.

\begin{verbatim}
(%i1) draw3d(
    enhanced3d = [z-x+2*y,x,y,z],
    palette = [32, -8, 17],
    explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3))$
\end{verbatim}

It also works with gray images.

\begin{verbatim}
(%i1) im: apply(
    'matrix,
    makelist(makelist(random(200),i,1,30),i,1,30))$
(%i2) /* palette = color, default */
    draw2d(image(im,0,0,30,30))$
(%i3) draw2d(palette = gray, image(im,0,0,30,30))$
(%i4) draw2d(palette = [15,20,-4],
    colorbox=false,
    image(im,0,0,30,30))$
\end{verbatim}
palette can be a user defined lookup table. In this example, low values of \(x\) are colored in red, and higher values in yellow.

\[
\text{(\%i1) } \text{draw3d(}
\text{palette = [red, blue, yellow],}
\text{enhanced3d = x,}
\text{explicit(x^2+y^2,x,-1,1,y,-1,1)) }
\]

See also colorbox and enhanced3d.

point_size

[Graphic option]

Default value: 1

point_size sets the size for plotted points. It must be a non negative number.

This option has no effect when graphic option point_type is set to dot.

This option affects the following graphic objects:

- \text{gr2d: points.}
- \text{gr3d: points.}
Example:

```plaintext
(%i1) draw2d(points(makelist([random(20),random(50)],k,1,10)),
    point_size = 5,
    points(makelist(k,k,1,20),makelist(random(30),k,1,20)))$
```

**Point_type**

default value: 1

Point_type indicates how isolated points are displayed; the value of this option can be any integer index greater or equal than -1, or the name of a point style: $\text{none (-1), dot (0), plus (1), multiply (2), asterisk (3), square (4), filled_square (5), circle (6), filled_circle (7), up_triangle (8), filled_up_triangle (9), down_triangle (10), filled_down_triangle (11), diamant (12) and filled_diamant (13).}$

This option affects the following graphic objects:

- **gr2d**: points.
- **gr3d**: points.

Example:

```plaintext
(%i1) draw2d(xrange = [0,10],
    yrange = [0,10],
    point_size = 3,
    point_type = diamant,
    points([[1,1],[5,1],[9,1]]),
    point_type = filled_down_triangle,
    points([[1,2],[5,2],[9,2]]),
    point_type = asterisk,
    points([[1,3],[5,3],[9,3]]),
    point_type = filled_diamant,
    points([[1,4],[5,4],[9,4]]),
    point_type = 5,
    points([[1,5],[5,5],[9,5]]),
    point_type = 6,
```
points_joined

[Graphic option]

Default value: false

When `points_joined` is `true`, points are joined by lines; when `false`, isolated points are drawn. A third possible value for this graphic option is `impulses`; in such case, vertical segments are drawn from points to the x-axis (2D) or to the xy-plane (3D).

This option affects the following graphic objects:

- `gr2d`: `points`.
- `gr3d`: `points`.

Example:

```
(%i1) draw2d(xrange = [0,10],
    yrange = [0,4],
    point_size = 3,
    point_type = up_triangle,
    color = blue,
    points([[1,1],[5,1],[9,1]]),
    points_joined = true,
    point_type = square,
    line_type = dots,
    points([[1,2],[5,2],[9,2]]),
    point_type = circle,
    color = red,
    line_width = 7,
)`
proportional_axes

[Graphic option]

Default value: none

When proportional_axes is equal to xy or xyz, the aspect ratio of the axis units will be set to 1:1 resulting in a 2D or 3D scene that will be drawn with axes proportional to their relative lengths.

Since this is a global graphics option, its position in the scene description does not matter.

This option works with Gnuplot version 4.2.6 or greater.

Examples:

Single 2D plot.

(%i1) draw2d(
    ellipse(0,0,1,1,0,360),
    transparent=true,
    color=blue,
    line_width=4,
    ellipse(0,0,2,1/2,0,360),
    proportional_axes='xy)$
Multiplot.

(%i1) draw(
    terminal = wxt,
    gr2d(proportional_axes = 'xy,
        explicit(x^2, x, 0, 1)),
    gr2d(explicit(x^2, x, 0, 1),
        xrange = [0, 1],
        yrange = [0, 2],
        proportional_axes='xy),
    gr2d(explicit(x^2, x, 0, 1))))$

surface_hide

Default value: false

If surface_hide is true, hidden parts are not plotted in 3d surfaces.

Since this is a global graphics option, its position in the scene description does not matter.

Example:
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(%i1) draw(columns=2,
    gr3d(explicit(exp(sin(x)+cos(x^2)),x,-3,3,y,-3,3)),
    gr3d(surface_hide = true,
        explicit(exp(sin(x)+cos(x^2)),x,-3,3,y,-3,3)) )$

[Graphic option]

terminal

Default value: screen

Selects the terminal to be used by Gnuplot; possible values are: screen (default), png, pngcairo, jpg, gif, eps, eps_color, epslatex, epslatex_standalone, svg, canvas, dumb, dumb_file, pdf, pdfcairo, wxt, animated_gif, multipage_pdfcairo, multipage_pdf, multipage_eps, multipage_eps_color, and aqua.cerm.

Terminals screen, wxt, windows and aqua.cerm can be also defined as a list with two elements: the name of the terminal itself and a non negative integer number. In this form, multiple windows can be opened at the same time, each with its corresponding number. This feature does not work in Windows platforms.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function draw.

N.B. pdfcairo requires Gnuplot 4.3 or newer. pdf requires Gnuplot to be compiled with the option --enable-pdf and libpdf must be installed. The pdf library is available from: http://www.pdflib.com/en/download/pdflib-family/pdflib-lite/

Examples:

(%i1) /* screen terminal (default) */
   draw2d(explicit(x^2,x,-1,1))$
(%i2) /* png file */
   draw2d(terminal = 'png,
           explicit(x^2,x,-1,1))$
(%i3) /* jpg file */
   draw2d(terminal = 'jpg,
           dimensions = [300,300],
           explicit(x^2,x,-1,1))$
explicit(x^2,x,-1,1))$

(%i4) /* eps file */
   draw2d(file_name = "myfile",
          explicit(x^2,x,-1,1),
          terminal = 'eps)$

(%i5) /* pdf file */
   draw2d(file_name = "mypdf",
          dimensions = 100*[12.0,8.0],
          explicit(x^2,x,-1,1),
          terminal = 'pdf)$

(%i6) /* wxwidgets window */
   draw2d(explicit(x^2,x,-1,1),
          terminal = 'wxt)$

Multiple windows.

(%i1) draw2d(explicit(x^5,x,-2,2), terminal=[screen, 3])$
(%i2) draw2d(explicit(x^2,x,-2,2), terminal=[screen, 0])$

An animated gif file.

(%i1) draw(
   delay = 100,
   file_name = "zzz",
   terminal = 'animated_gif,
   gr2d(explicit(x^2,x,-1,1)),
   gr2d(explicit(x^3,x,-1,1)),
   gr2d(explicit(x^4,x,-1,1)));

End of animation sequence

(%o1) [gr2d(explicit), gr2d(explicit), gr2d(explicit)]

Option delay is only active in animated gif's; it is ignored in any other case.

Multipage output in eps format.

(%i1) draw(
   file_name = "parabol",
   terminal = multipage_eps,
   dimensions = 100*[10,10],
   gr2d(explicit(x^2,x,-1,1)),
   gr3d(explicit(x^2+y^2,x,-1,1,y,-1,1)) )$

See also file_name, dimensions_draw and delay.

title

[Graphic option]

Default value: "" (empty string)

Option title, a string, is the main title for the scene. By default, no title is written.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw2d(explicit(exp(u),u,-2,2),
   title = "Exponential function")$
transform

[Graphic option]

Default value: none

If transform is none, the space is not transformed and graphic objects are drawn as defined. When a space transformation is desired, a list must be assigned to option transform. In case of a 2D scene, the list takes the form \([f1(x,y), f2(x,y), x, y]\). In case of a 3D scene, the list is of the form \([f1(x,y,z), f2(x,y,z), f3(x,y,z), x, y, z]\).

The names of the variables defined in the lists may be different to those used in the definitions of the graphic objects.

Examples:

Rotation in 2D.

(%i1) th : %pi / 4$
(%i2) draw2d(
   color = "#e245f0",
   proportional_axes = 'xy,
   line_width = 8,
   triangle([3,2],[7,2],[5,5]),
   border = false,
   fill_color = yellow,
   transform = [cos(th)*x - sin(th)*y,
                sin(th)*x + cos(th)*y, x, y],
   triangle([3,2],[7,2],[5,5]) )$

Exponential function
Translation in 3D.

(%i1) draw3d(
    color = "#a2c00",
    explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3),
    transform = [x+10,y+10,z+10,x,y,z],
    color = blue,
    explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3) )$

**transparent**

[Graphic option]

Default value: *false*

If **transparent** is *false*, interior regions of polygons are filled according to **fill_color**.

This option affects the following graphic objects:

- **gr2d**: *polygon*, *rectangle* and *ellipse*.

Example:

(%i1) draw2d(polygon([[3,2],[7,2],[5,5]]),
    transparent = true,
    color = blue,
    polygon([[5,2],[9,2],[7,5]]))$
unit_vectors  
Default value: **false**

If `unit_vectors` is **true**, vectors are plotted with module 1. This is useful for plotting vector fields. If `unit_vectors` is **false**, vectors are plotted with its original length.

This option is relevant only for **vector** objects.

Example:

```latex
(%i1) draw2d(xrange = [-1,6],
yrange = [-1,6],
head_length = 0.1,
vector([0,0],[5,2]),
unit_vectors = true,
color = red,
vector([0,3],[5,2]))$
```

user_preamble  
Default value: **""** (empty string)
Expert Gnuplot users can make use of this option to fine tune Gnuplot’s behaviour by writing settings to be sent before the `plot` or `splot` command.

The value of this option must be a string or a list of strings (one per line).

Since this is a global graphics option, its position in the scene description does not matter.

Example:

Tell Gnuplot to draw axes and grid on top of graphics objects,

```
(%i1) draw2d(
    xaxis =true, xaxis_type=solid,
    yaxis =true, yaxis_type=solid,
    user_preamble="set grid front",
    region(x^2+y^2<1 ,x,-1.5,1.5,y,-1.5,1.5))$
```

Tell gnuplot to draw all contour lines in black

```
(%i1) draw3d(
    contour=both,
    surface_hide=true,enhanced3d=true,wired_surface=true,
    contour_levels=10,
    user_preamble="set for [i=1:8] linetype i dashtype i linecolor 0",
    explicit(sin(x)*cos(y),x,1,10,y,1,10)
    )
);
view

[Graphic option]

Default value: [60, 30]

A pair of angles, measured in degrees, indicating the view direction in a 3D scene. The first angle is the vertical rotation around the x axis, in the range [0, 360]. The second one is the horizontal rotation around the z axis, in the range [0, 360].

If option view is given the value map, the view direction is set to be perpendicular to the xy-plane.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw3d(view = [170, 50],
    enhanced3d = true,
    explicit(sin(x^2+y^2),x,-2,2,y,-2,2) )$

(%i2) draw3d(view = map,
    enhanced3d = true,
%i1) draw3d(
  enhanced3d  = [sin(x),x,y],
  wired_surface = true,
  explicit(x^2+y^2,x,-1,1,y,-1,1)) $

wired_surface
   [Graphic option]
   Default value: false
   Indicates whether 3D surfaces in enhanced3d mode show the grid joining the points or not.
   Since this is a global graphics option, its position in the scene description does not matter.

x_voxel
   [Graphic option]
   Default value: 10
x_voxel is the number of voxels in the x direction to be used by the *marching cubes algorithm* implemented by the 3d *implicit* object. It is also used by graphic object *region*.

**xaxis**

[Graphic option]

Default value: *false*

If *xaxis* is *true*, the x axis is drawn.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```lisp
(%i1) draw2d(explicit(x^3,x,-1,1),
    xaxis  = true,
    xaxis_color = blue)$
```

See also *xaxis_width*, *xaxis_type* and *xaxis_color*.

**xaxis_color**

[Graphic option]

Default value: "black"

*xaxis_color* specifies the color for the x axis. See *color* to know how colors are defined.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```lisp
(%i1) draw2d(explicit(x^3,x,-1,1),
    xaxis  = true,
    xaxis_color = red)$
```

See also *xaxis*, *xaxis_width* and *xaxis_type*.

**xaxis_secondary**

[Graphic option]

Default value: *false*
If \texttt{xaxis\_secondary} is \texttt{true}, function values can be plotted with respect to the second x axis, which will be drawn on top of the scene.

Note that this is a local graphics option which only affects to 2d plots.

Example:

\begin{verbatim}
(%i1) draw2d(
   key = "Bottom x-axis",
   explicit(x+1,x,1,2),
   color = red,
   key = "Above x-axis",
   xtics\_secondary = true,
   xaxis\_secondary = true,
   explicit(x^2,x,-1,1)) $
\end{verbatim}

See also \texttt{xrange\_secondary}, \texttt{xtics\_secondary}, \texttt{xtics\_rotate\_secondary}, \texttt{xtics\_axis\_secondary} and \texttt{xaxis\_secondary}.

\texttt{xaxis\_type} \hspace{1cm} [Graphic option]

Default value: \texttt{dots}

\texttt{xaxis\_type} indicates how the x axis is displayed; possible values are \texttt{solid} and \texttt{dots}

Since this is a global graphics option, its position in the scene description does not matter.

Example:

\begin{verbatim}
(%i1) draw2d(explicit(x^3,x,-1,1),
   xaxis = true,
   xaxis\_type = solid)$
\end{verbatim}

See also \texttt{xaxis}, \texttt{xaxis\_width} and \texttt{xaxis\_color}.

\texttt{xaxis\_width} \hspace{1cm} [Graphic option]

Default value: 1

\texttt{xaxis\_width} is the width of the x axis. Its value must be a positive number.
Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(explicit(x^3,x,-1,1),
    xaxis  = true,
    xaxis_width = 3)$
```

See also `xaxis`, `xaxis_type` and `xaxis_color`.

**xlabel**

Default value: ""

Option `xlabel`, a string, is the label for the x axis. By default, the axis is labeled with string "x".

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(xlabel = "Time",
    explicit(exp(u),u,-2,2),
    ylabel = "Population")$
```

See also `xlabel_secondary`, `ylabel`, `ylabel_secondary` and `zlabel_draw`.

**xlabel_secondary**

Default value: "" (empty string)

Option `xlabel_secondary`, a string, is the label for the secondary x axis. By default, no label is written.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(
    xaxis_secondary=true,yaxis_secondary=true,
    xtics_secondary=true,ytics_secondary=true,
    xlabel_secondary="t[s]",
    ylabel_secondary="U[V]",
    explicit(sin(t),t,0,10))$
```
See also xlabel_draw, ylabel_draw, ylabel_secondary and zlabel_draw.

**xtics**  
[Graphic option]  
Default value: true  
This graphic option controls the way tic marks are drawn on the x axis.  
- When option xtics is bounded to symbol true, tic marks are drawn automatically.  
- When option xtics is bounded to symbol false, tic marks are not drawn.
• When option `xtics` is bounded to a positive number, this is the distance between
two consecutive tic marks.
• When option `xtics` is bounded to a list of length three of the form
`[start,incr,end]`, tic marks are plotted from `start` to `end` at intervals of
length `incr`.
• When option `xtics` is bounded to a set of numbers of the form `{n1, n2, ...}`,
tic marks are plotted at values `n1, n2, ...`
• When option `xtics` is bounded to a set of pairs of the form `{{"label1", n1},
{"label2", n2}, ...}`, tic marks corresponding to values `n1, n2, ...` are labeled
with "label1", "label2", ..., respectively.

Since this is a global graphics option, its position in the scene description does not
matter.

Examples:
Disable tics.

```
(%i1) draw2d(xtics = 'false,
            explicit(x^3,x,-1,1) )$
```

Tics every 1/4 units.

```
(%i1) draw2d(xtics = 1/4,
            explicit(x^3,x,-1,1) )$
```

Tics from -3/4 to 3/4 in steps of 1/8.

```
(%i1) draw2d(xtics = [-3/4,1/8,3/4],
            explicit(x^3,x,-1,1) )$
```

Tics at points -1/2, -1/4 and 3/4.

```
(%i1) draw2d(xtics = {-1/2,-1/4,3/4},
            explicit(x^3,x,-1,1) )$
```

Labeled tics.

```
(%i1) draw2d(xtics = {["High",0.75],["Medium",0],["Low",-0.75]},
            explicit(x^3,x,-1,1) )$
```

See also `ytics_draw`, and `ztics_draw`.

`xtics_axis`  
[Graphic option]

Default value: `false`

If `xtics_axis` is `true`, tic marks and their labels are plotted just along the x axis, if
it is `false` tics are plotted on the border.

Since this is a global graphics option, its position in the scene description does not
matter.

`xtics_rotate`  
[Graphic option]

Default value: `false`

If `xtics_rotate` is `true`, tic marks on the x axis are rotated 90 degrees.

Since this is a global graphics option, its position in the scene description does not
matter.
xtics_rotate_secondary

Default value: false

If xtics_rotate_secondary is true, tic marks on the secondary x axis are rotated 90 degrees.

Since this is a global graphics option, its position in the scene description does not matter.

xtics_secondary

Default value: auto

This graphic option controls the way tic marks are drawn on the second x axis.

See xtics_draw for a complete description.

xtics_secondary_axis

Default value: false

If xtics_secondary_axis is true, tic marks and their labels are plotted just along the secondary x axis, if it is false tics are plotted on the border.

Since this is a global graphics option, its position in the scene description does not matter.

xu_grid

Default value: 30

xtics_secondary_axis is the number of coordinates of the first variable (x in explicit and u in parametric 3d surfaces) to build the grid of sample points.

This option affects the following graphic objects:

• gr3d: explicit and parametric_surface.

Example:

(%i1) draw3d(xu_grid = 10,
   yv_grid = 50,
   explicit(x^2+y^2,x,-3,3,y,-3,3) )$

See also yv_grid.

xy_file

Default value: "" (empty string)

xy_file is the name of the file where the coordinates will be saved after clicking with the mouse button and hitting the 'x' key. By default, no coordinates are saved.

Since this is a global graphics option, its position in the scene description does not matter.

xyplane

Default value: false

Allocates the xy-plane in 3D scenes. When xyplane is false, the xy-plane is placed automatically; when it is a real number, the xy-plane intersects the z-axis at this level. This option has no effect in 2D scenes.

Since this is a global graphics option, its position in the scene description does not matter.
Example:

```lisp
(%i1) draw3d(xyplane = %e-2,
    explicit(x^2+y^2,x,-1,1,y,-1,1))$
```

**y_voxel**

Default value: 10

`y_voxel` is the number of voxels in the y direction to be used by the *marching cubes algorithm* implemented by the 3d `implicit` object. It is also used by graphic object `region`.

**yaxis**

Default value: `false`

If `yaxis` is `true`, the y axis is drawn.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```lisp
(%i1) draw2d(explicit(x^3,x,-1,1),
    yaxis = true,
    yaxis_color = blue)$
```

See also `yaxis_width`, `yaxis_type` and `yaxis_color`.

**yaxis_color**

Default value: "black"

`yaxis_color` specifies the color for the y axis. See `color` to know how colors are defined.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```lisp
(%i1) draw2d(explicit(x^3,x,-1,1),
    yaxis = true,
    yaxis_color = red)$
```

See also `yaxis`, `yaxis_width` and `yaxis_type`.

**yaxis_secondary**

Default value: `false`

If `yaxis_secondary` is `true`, function values can be plotted with respect to the second y axis, which will be drawn on the right side of the scene.

Note that this is a local graphics option which only affects to 2d plots.

Example:

```lisp
(%i1) draw2d(
    explicit(sin(x),x,0,10),
    yaxis_secondary = true,
    ytics_secondary = true,
    color = blue,
    explicit(100*sin(x+0.1)+2,x,0,10));
```
See also `yrange_secondary`, `ytics_secondary`, `ytics_rotate_secondary` and `ytics_axis_secondary`

**yaxis_type**

Default value: `dots`

`yaxis_type` indicates how the $y$ axis is displayed; possible values are `solid` and `dots`. Since this is a global graphics option, its position in the scene description does not matter.

Example:

```plaintext
(%i1) draw2d(explicit(x^3,x,-1,1),
    yaxis = true,
    yaxis_type = solid)$
```

See also `yaxis`, `yaxis_width` and `yaxis_color`.

**yaxis_width**

Default value: 1

`yaxis_width` is the width of the $y$ axis. Its value must be a positive number. Since this is a global graphics option, its position in the scene description does not matter.

Example:

```plaintext
(%i1) draw2d(explicit(x^3,x,-1,1),
    yaxis = true,
    yaxis_width = 3)$
```

See also `yaxis`, `yaxis_type` and `yaxis_color`.

**ylabel**

Default value: ""

Option `ylabel`, a string, is the label for the $y$ axis. By default, the axis is labeled with string "y". Since this is a global graphics option, its position in the scene description does not matter.

Example:

```plaintext
(%i1) draw2d(xlabel = "Time",
    ylabel = "Population",
    explicit(exp(u),u,-2,2) )$
```

See also `xlabel_draw`, `xlabel_secondary`, `ylabel_secondary`, and `zlabel_draw`.

**ylabel_secondary**

Default value: "" (empty string)

Option `ylabel_secondary`, a string, is the label for the secondary $y$ axis. By default, no label is written. Since this is a global graphics option, its position in the scene description does not matter.
Example:

(%i1) draw2d(  
    key_pos=bottom_right,  
    key="current",  
    xlabel="t[s]",  
    ylabel="I[A]", ylabel_secondary="P[W]",  
    explicit(sin(t),t,0,10),  
    yaxis_secondary=true,  
    ytics_secondary=true,  
    color=red, key="Power",  
    explicit((sin(t))^2,t,0,10)
)

See also xlabel_draw, xlabel_secondary, ylabel_draw and zlabel_draw.

**yrange**

[Graphic option]

Default value: auto

If **yrange** is auto, the range for the y coordinate is computed automatically.

If the user wants a specific interval for y, it must be given as a Maxima list, as in yrange=[-2, 3].

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw2d(yrange = [-2,3],  
    explicit(x^2,x,-1,1),  
    xrange = [-3,3])$

See also xrange, yrange_secondary and zrange.

**yrange_secondary**

[Graphic option]

Default value: auto

If **yrange_secondary** is auto, the range for the second y axis is computed automatically.

If the user wants a specific interval for the second y axis, it must be given as a Maxima list, as in yrange_secondary=[-2, 3].

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw2d(  
    explicit(sin(x),x,0,10),  
    yaxis_secondary = true,  
    ytics_secondary = true,  
    yrange = [-3, 3],  
    yrange_secondary = [-20, 20],  
    color = blue,  
    explicit(100*sin(x+0.1)+2,x,0,10)) $

See also xrange, yrange and zrange.
ytics

[Graphic option]

Default value: true

This graphic option controls the way tic marks are drawn on the y axis.

See xtics for a complete description.

ytics_axis

[Graphic option]

Default value: false

If ytics_axis is true, tic marks and their labels are plotted just along the y axis, if it is false tics are plotted on the border.

Since this is a global graphics option, its position in the scene description does not matter.

ytics_rotate

[Graphic option]

Default value: false

If ytics_rotate is true, tic marks on the y axis are rotated 90 degrees.

Since this is a global graphics option, its position in the scene description does not matter.

ytics_rotate_secondary

[Graphic option]

Default value: false

If ytics_rotate_secondary is true, tic marks on the secondary y axis are rotated 90 degrees.

Since this is a global graphics option, its position in the scene description does not matter.

ytics_secondary

[Graphic option]

Default value: auto

This graphic option controls the way tic marks are drawn on the second y axis.

See xtics for a complete description.

ytics_secondary_axis

[Graphic option]

Default value: false

If ytics_secondary_axis is true, tic marks and their labels are plotted just along the secondary y axis, if it is false tics are plotted on the border.

Since this is a global graphics option, its position in the scene description does not matter.

yv_grid

[Graphic option]

Default value: 30

yv_grid is the number of coordinates of the second variable (y in explicit and v in parametric 3d surfaces) to build the grid of sample points.

This option affects the following graphic objects:

• gr3d: explicit and parametric_surface.
Example:

(%i1) draw3d(xu_grid = 10,
   yv_grid = 50,
   explicit(x^2+y^2,x,-3,3,y,-3,3) )$

See also xu_grid.

z_voxel

[Graphic option]
Default value: 10

z_voxel is the number of voxels in the z direction to be used by the marching cubes algorithm implemented by the 3d implicit object.

zaxis

[Graphic option]
Default value: false

If zaxis is true, the z axis is drawn in 3D plots. This option has no effect in 2D scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1),
   zaxis = true,
   zaxis_type = solid,
   zaxis_color = blue)$

See also zaxis_width, zaxis_type and zaxis_color.

zaxis_color

[Graphic option]
Default value: "black"

zaxis_color specifies the color for the z axis. See color to know how colors are defined. This option has no effect in 2D scenes.

Since this is a global graphics option, its position in the scene description does not matter.
Example:

(\%i1) \texttt{draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1),}
\texttt{zaxis = true,}
\texttt{zaxis\_type = solid,}
\texttt{zaxis\_color = red)}$

See also \texttt{zaxis, zaxis\_width} and \texttt{zaxis\_type}.

\textbf{zaxis\_type} \hspace{1cm} \textbf{[Graphic option]}
Default value: \texttt{dots}
\texttt{zaxis\_type} indicates how the z axis is displayed; possible values are \texttt{solid} and \texttt{dots}. This option has no effect in 2D scenes.
Since this is a global graphics option, its position in the scene description does not matter.
Example:

(\%i1) \texttt{draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1),}
\texttt{zaxis = true,}
\texttt{zaxis\_type = solid)}$

See also \texttt{zaxis, zaxis\_width} and \texttt{zaxis\_color}.

\textbf{zaxis\_width} \hspace{1cm} \textbf{[Graphic option]}
Default value: \texttt{1}
\texttt{zaxis\_width} is the width of the z axis. Its value must be a positive number. This option has no effect in 2D scenes.
Since this is a global graphics option, its position in the scene description does not matter.
Example:

(\%i1) \texttt{draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1),}
\texttt{zaxis = true,}
\texttt{zaxis\_type = solid,}
\texttt{zaxis\_width = 3)}$

See also \texttt{zaxis, zaxis\_type} and \texttt{zaxis\_color}.

\textbf{zlabel} \hspace{1cm} \textbf{[Graphic option]}
Default value: ""
Option \texttt{zlabel}, a string, is the label for the z axis. By default, the axis is labeled with string "z".
Since this is a global graphics option, its position in the scene description does not matter.
Example:

(\%i1) \texttt{draw3d(zlabel = "Z variable",}
\texttt{ylabel = "Y variable",}
\texttt{explicit(sin(x^2+y^2),x,-2,2,y,-2,2),}
\texttt{xlabel = "X variable")}$

See also \texttt{xlabel\_draw, ylabel\_draw, and zlabel\_rotate}.
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zlabel_rotate

Default value: "auto"

This graphics option allows to choose if the z axis label of 3d plots is drawn horizontal (false), vertical (true) or if maxima automatically chooses an orientation based on the length of the label (auto).

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw3d(
   explicit(sin(x)*sin(y),x,0,10,y,0,10),
   zlabel_rotate=false
)

See also zlabel_draw.

zrange

Default value: auto

If zrange is auto, the range for the z coordinate is computed automatically.

If the user wants a specific interval for z, it must be given as a Maxima list, as in zrange=[-2, 3].

Since this is a global graphics option, its position in the scene description does not matter.

Example:

(%i1) draw3d(yrange = [-3,3],
   zrange = [-2,5],
   explicit(x^2+y^2,x,-1,1,y,-1,1),
   xrange = [-3,3])$

See also xrange and yrange.

ztics

Default value: true

This graphic option controls the way tic marks are drawn on the z axis.

See xtics_draw for a complete description.

ztics_axis

Default value: false

If ztics_axis is true, tic marks and their labels are plotted just along the z axis, if it is false tics are plotted on the border.

Since this is a global graphics option, its position in the scene description does not matter.

ztics_rotate

Default value: false

If ztics_rotate is true, tic marks on the z axis are rotated 90 degrees.

Since this is a global graphics option, its position in the scene description does not matter.
53.2.4 Graphics objects

bars ([x1,h1,w1], [x2,h2,w2, ...]) [Graphic object]
Draws vertical bars in 2D.

2D
bars ([x1,h1,w1], [x2,h2,w2, ...]) draws bars centered at values x1, x2, ... with
heights h1, h2, ... and widths w1, w2, ...
This object is affected by the following graphic options: key, fill_color, fill_density and line_width.
Example:
(%i1) draw2d(
   key = "Group A",
   fill_color = blue,
   fill_density = 0.2,
   bars([0.8,5,0.4],[1.8,7,0.4],[2.8,-4,0.4]),
   key = "Group B",
   fill_color = red,
   fill_density = 0.6,
   line_width = 4,
   bars([1.2,4,0.4],[2.2,-2,0.4],[3.2,5,0.4]),
   xaxis = true);

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{bar_graph}
\caption{Bar graph with two groups, Group A and Group B.}
\end{figure}

cylindrical (radius, z, minz, maxz, azi, minazi, maxazi) [Graphic object]
Draws 3D functions defined in cylindrical coordinates.

3D
\begin{itemize}
\item cylindrical(radius, z, minz, maxz, azi, minazi, maxazi) plots the function
radius(z, azi) defined in cylindrical coordinates, with variable z taking values from minz to maxz and azimuth azi taking values from minazi to maxazi.
\item This object is affected by the following graphic options: xu_grid, yv_grid, line_type, key, wired_surface, enhanced3d and color
\end{itemize}
Example:

(%i1) draw3d(cylindrical(1,z,-2,2,az,0,2*%pi))$

```
[Graphics object]
```

elevation_grid (mat,x0,y0,width,height)  [Graphic object]

Draws matrix mat in 3D space. z values are taken from mat, the abscissas range from x0 to x0 + width and ordinates from y0 to y0 + height. Element a(1,1) is projected on point (x0,y0 + height), a(1,n) on (x0 + width,y0 + height), a(m,1) on (x0,y0), and a(m,n) on (x0 + width,y0).

This object is affected by the following graphic options: line_type, line_width key, wired_surface, enhanced3d and color

In older versions of Maxima, elevation_grid was called mesh. See also mesh.

Example:

(%i1) m: apply(
    matrix,
    makelist(makelist(random(10.0),k,1,30),i,1,20)) $

(%i2) draw3d(
    color = blue,
    elevation_grid(m,0,0,3,2),
    xlabel = "x",
    ylabel = "y",
    surface_hide = true);
ellipse \((xc, yc, a, b, ang1, ang2)\)

[Graphic object]

Draws ellipses and circles in 2D.

2D

\(\text{ellipse}(xc, yc, a, b, ang1, ang2)\) plots an ellipse centered at \([xc, yc]\) with horizontal and vertical semi axis \(a\) and \(b\), respectively, starting at angle \(ang1\) with an amplitude equal to angle \(ang2\).

This object is affected by the following graphic options: nticks, transparent, fill_color, border, line_width, line_type, key and color.

Example:

\[
\%i11 \text{draw2d(transparent = false, fill_color = red, color = gray30, transparent = false, line_width = 5, ellipse(0,6,3,2,270,-270), /* center (x,y), a, b, start & end in degrees */ transparent = true, color = blue, line_width = 3, ellipse(2.5,6,2,3,30,-90), xrange = [-3,6], yrange = [2,9] )}$
\]
errors ([x1, x2, ...], [y1, y2, ...])

Draws points with error bars, horizontally, vertically or both, depending on the value of option error_type.

2D

If error_type = x, arguments to errors must be of the form [x, y, xdelta] or [x, y, xlow, xhigh]. If error_type = y, arguments must be of the form [x, y, ydelta] or [x, y, ylow, yhigh]. If error_type = xy or error_type = boxes, arguments to errors must be of the form [x, y, xdelta, ydelta] or [x, y, xlow, xhigh, ylow, yhigh].

See also error_type.

This object is affected by the following graphic options: error_type, points Joined, line_width, key, line_type, color fill_density, xaxis_secondary and yaxis_secondary.

Option fill_density is only relevant when error_type=boxes.

Examples:

Horizontal error bars.

(\%i1) draw2d(
    error_type = 'y,
    errors([[1,2,1], [3,5,3], [10,3,1], [17,6,2]]))$

[Graphic object]
Vertical and horizontal error bars.

(%i1) draw2d(
    error_type = 'xy,
    points_joined = true,
    color = blue,
    errors([[1,2,1,2], [3,5,2,1], [10,3,1,1], [17,6,1/2,2]]);

Draws explicit functions in 2D and 3D.

2D

explicit(fcn, var, minval, maxval) plots explicit function fcn, with variable var taking values from minval to maxval.
This object is affected by the following graphic options: \texttt{nticks}, \texttt{adapt\_depth}, \texttt{draw\_realpart}, \texttt{line\_width}, \texttt{line\_type}, \texttt{key}, \texttt{filled\_func}, \texttt{fill\_color} and \texttt{color}

Example:

(\texttt{\textbackslash i1}) \texttt{draw2d(line\_width = 3, color = blue, explicit(x\^2,x,-3,3) )}$

3D

\texttt{explicit(fcn, var1, minval1, maxval1, var2, minval2, maxval2)} plots the explicit function \texttt{fcn}, with variable \texttt{var1} taking values from \texttt{minval1} to \texttt{maxval1} and variable \texttt{var2} taking values from \texttt{minval2} to \texttt{maxval2}.
This object is affected by the following graphic options: draw_realpart, xu_grid, yv_grid, line_type, line_width, key, wired_surface, enhanced3d and color.

Example:

```
(%i1) draw3d(key = "Gauss",
    color = "#a02c00",
    explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3),
    yv_grid = 10,
    color = blue,
    key = "Plane",
    explicit(x+y,x,-5,5,y,-5,5),
surface_hide = true)$
```

See also filled_func for filled functions.

**image (im,x0,y0,width,height)**

Renders images in 2D.

**2D**

image (im,x0,y0,width,height) plots image im in the rectangular region from vertex (x0,y0) to (x0+width,y0+height) on the real plane. Argument im must be a matrix of real numbers, a matrix of vectors of length three or a picture object.

If im is a matrix of real numbers or a levels picture object, pixel values are interpreted according to graphic option palette, which is a vector of length three with components ranging from -36 to +36; each value is an index for a formula mapping the levels onto red, green and blue colors, respectively:

```
0: 0  1: 0.5  2: 1
3: x  4: x^2  5: x^3
6: x^4 7: sqrt(x) 8: sqrt(sqrt(x))
9: sin(90x) 10: cos(90x) 11: |x-0.5|
12: (2x-1)^2 13: sin(180x) 14: |cos(180x)|
15: sin(360x) 16: cos(360x) 17: |sin(360x)|
18: |cos(360x)| 19: |sin(720x)| 20: |cos(720x)|
```
negative numbers mean negative colour component.

\texttt{palette = gray} and \texttt{palette = color} are short cuts for \texttt{palette = [3,3,3]} and \texttt{palette = [7,5,15]}, respectively.

If $\texttt{im}$ is a matrix of vectors of length three or an \texttt{rgb picture} object, they are interpreted as red, green and blue color components.

Examples:

If $\texttt{im}$ is a matrix of real numbers, pixel values are interpreted according to graphic option \texttt{palette}.

\begin{verbatim}
(%i1) im: apply(
   'matrix,
   makelist(makelist(random(200),i,1,30),i,1,30))$
(%i2) /* palette = color, default */
   draw2d(image(im,0,0,30,30))$
\end{verbatim}

\begin{verbatim}
(%i3) draw2d(palette = gray, image(im,0,0,30,30))$
\end{verbatim}
See also \texttt{colorbox}.

If \textit{im} is a matrix of vectors of length three, they are interpreted as red, green and blue color components.

\begin{verbatim}
(%i1) im: apply(
    'matrix,
    makelist(
        makelist([random(300),
            random(300),
            random(300)],i,1,30),i,1,30))$
\end{verbatim}

\begin{verbatim}
(%i2) draw2d(image(im,0,0,30,30))$
\end{verbatim}
Package **draw** automatically loads package **picture**. In this example, a level picture object is built by hand and then rendered.

```lisp
(%i1) im: make_level_picture([45,87,2,134,204,16],3,2);
(%o1) picture(level, 3, 2, {Array: #(45 87 2 134 204 16)})
(%i2) /* default color palette */
   draw2d(image(im,0,0,30,30))$
```

```lisp
(%i3) /* gray palette */
   draw2d(palette = gray,
          image(im,0,0,30,30))$
```
An xpm file is read and then rendered.

```
(%i1) im: read_xpm("myfile.xpm")$
(%i2) draw2d(image(im,0,0,10,7))$
```

See also `make_level_picture`, `make_rgb_picture` and `read_xpm`.

http://www.telefonica.net/web2/biomates/maxima/gpdraw/image contains more elaborated examples.

**implicit**

```
implicit (fcn,x,xmin,xmax,y,ymin,ymax)
implicit (fcn,x,xmin,xmax,y,ymin,ymax,z,zmin,zmax)
```

Draws implicit functions in 2D and 3D.

2D

`implicit(fcn,x,xmin,xmax,y,ymin,ymax)` plots the implicit function defined by `fcn`, with variable `x` taking values from `xmin` to `xmax`, and variable `y` taking values from `ymin` to `ymax`.

This object is affected by the following graphic options: `ip_grid`, `ip_grid_in`, `line_width`, `line_type`, `key` and `color`.

Example:

```
(%i1) draw2d(grid = true,
    line_type = solid,
    key = "y^2=x^3-2*x+1",
    implicit(y^2=x^3-2*x+1, x, -4,4, y, -4,4),
    line_type = dots,
    key = "x^3+y^3 = 3*x*y^2-x-1",
    implicit(x^3+y^3 = 3*x*y^2-x-1, x,-4,4, y,-4,4),
    title = "Two implicit functions" )$
```
Two implicit functions

\[ y^2 = x^3 - 2x + 1 \]
\[ x^3 + y^3 = 3x^2y - x - 1 \]

3D implicit \((f_{cn}, x, x_{min}, x_{max}, y, y_{min}, y_{max}, z, z_{min}, z_{max})\) plots the implicit surface defined by \(f_{cn}\), with variable \(x\) taking values from \(x_{min}\) to \(x_{max}\), variable \(y\) taking values from \(y_{min}\) to \(y_{max}\) and variable \(z\) taking values from \(z_{min}\) to \(z_{max}\). This object implements the marching cubes algorithm.

This object is affected by the following graphic options: \(x\_voxel\), \(y\_voxel\), \(z\_voxel\), \(line\_width\), \(line\_type\), \(key\), \(wired\_surface\), \(enhanced3d\) and \(color\).

Example:

```lisp
(%i1) draw3d(
    color=blue,
    implicit((x^2+y^2+z^2-1)*(x^2+(y-1.5)^2+z^2-0.5)=0.015,
        x,-1,1,y,-1.2,2.3,z,-1,1),
    surface_hide=true);
```

```console
(%i1)
```

```
```
label

  label ([string, x, y], ...)
  label ([string, x, y, z], ...)

Writes labels in 2D and 3D.

Colored labels work only with Gnuplot 4.3 and up.

This object is affected by the following graphic options: label_alignment, label_orientation and color.

2D

label([string, x, y]) writes the string at point [x, y].

Example:

(%i1) draw2d(yrange = [0.1,1.4],
  color = red,
  label(["Label in red",0,0.3]),
  color = "#0000ff",
  label(["Label in blue",0,0.6]),
  color = light_blue,
  label(["Label in light-blue",0,0.9],
    ["Another light-blue",0,1.2]) )$

3D

label([string, x, y, z]) writes the string at point [x, y, z].

Example:

(%i1) draw3d(explicit(exp(sin(x)+cos(x^2)),x,-3,3,y,-3,3),
  color = red,
  label(["UP 1",-2,0,3], ["UP 2",1.5,0,4]),
  color = blue,
  label(["DOWN 1",2,0,-3]) )$
mesh \((\text{row}_1, \text{row}_2, \ldots)\)

Draws a quadrangular mesh in 3D.

3D

Argument \(\text{row}_i\) is a list of \(n\) 3D points of the form \([x_{i1}, y_{i1}, z_{i1}], \ldots, [x_{i_n}, y_{i_n}, z_{i_n}]\], and all rows are of equal length. All these points define an arbitrary surface in 3D and in some sense it's a generalization of the \texttt{elevation\_grid} object.

This object is affected by the following \textit{graphic options}: \texttt{line\_type}, \texttt{line\_width}, \texttt{color}, \texttt{key}, \texttt{wired\_surface}, \texttt{enhanced3d} and \texttt{transform}.

Examples:

A simple example.

```lisp
(%i1) draw3d(
   mesh([[1,1,3], [7,3,1], [12,-2,4], [15,0,5]],
        [[2,7,8], [4,3,1], [10,5,8], [12,7,1]],
        [[-2,11,10], [6,9,5], [6,15,1], [20,15,2]]))$
```

Plotting a triangle in 3D.
(%i1) draw3d(
   line_width = 2,
   mesh([[1,0,0],[0,1,0]],
        [[0,0,1],[0,0,1]]))$

Two quadrilaterals.

(%i1) draw3d(
   surface_hide = true,
   line_width = 3,
   color = red,
   mesh([[0,0,0], [0,1,0]],
        [[2,0,2], [2,2,2]],
        color = blue,
        mesh([[0,0,2], [0,1,2]],
              [[2,0,4], [2,2,4]]))$
parametric

parametric ([x,y,par,parmin,parmax])
parametric ([x,y,z,par,parmin,parmax])

Draws parametric functions in 2D and 3D.

This object is affected by the following graphic options: ntics, line_width, line_type, key, color and enhanced3d.

2D

The command `parametric(xfun, yfun, par, parmin, parmax)` plots the parametric function `[xfun, yfun]`, with parameter `par` taking values from `parmin` to `parmax`.

Example:

```lisp
(%i1) draw2d(explicit(exp(x),x,-1,3),
   color = red,
   key = "This is the parametric one!!",
   parametric(2*cos(rrr),rrr^2,rrr,0,2*%pi))$
```

3D

`parametric([xfun, yfun, zfun, par, parmin, parmax])` plots the parametric curve `[xfun, yfun, zfun]`, with parameter `par` taking values from `parmin` to `parmax`.

Example:

```lisp
(%i1) draw3d(explicit(exp(sin(x)+cos(x^2)),x,-3,3,y,-3,3),
   color = royalblue,
   parametric(cos(5*u)^2,sin(7*u),u-2,u,0,2),
   color = turquoise,
   line_width = 2,
   parametric(t^2,sin(t),2+t,t,0,2),
   surface_hide = true,
   title = "Surface & curves")$
```
parametric_surface(xfun, yfun, zfun, par1, par1min, par1max, par2, par2min, par2max)

Draws parametric surfaces in 3D.

3D

The command `parametric_surface(xfun, yfun, zfun, par1, par1min, par1max, par2, par2min, par2max)` plots the parametric surface \([xfun, yfun, zfun]\), with parameter \(par1\) taking values from \(par1min\) to \(par1max\) and parameter \(par2\) taking values from \(par2min\) to \(par2max\).

This object is affected by the following graphic options: `draw_realpart`, `xu_grid`, `yv_grid`, `line_type`, `line_width`, `key`, `wired_surface`, `enhanced3d` and `color`.

Example:

```plaintext
(%i1) draw3d(title = "Sea shell",
    xu_grid = 100,
    yv_grid = 25,
    view = [100,20],
    surface_hide = true,
    parametric_surface(0.5*u*cos(u)*(cos(v)+1),
                      0.5*u*sin(u)*(cos(v)+1),
                      u*sin(v) - ((u+3)/8*%pi)^2 - 20,
                      u, 0, 13*%pi, v, -%pi, %pi))$
```

[Graphic object]
Chapter 53: draw

Sea shell

Draws points in 2D and 3D.

This object is affected by the following graphic options: point_size, point_type, points_joined, line_width, key, line_type and color. In 3D mode, it is also affected by enhanced3d

2D

points ([[x1,y1], [x2,y2],...]) or points ([x1,x2,...], [y1,y2,...]) plots points [x1,y1], [x2,y2], etc. If abscissas are not given, they are set to consecutive positive integers, so that points ([y1,y2,...]) draws points [1,y1], [2,y2], etc. If matrix is a two-column or two-row matrix, points (matrix) draws the associated points. If matrix is an one-column or one-row matrix, abscissas are assigned automatically.

If 1d_y_array is a 1D lisp array of numbers, points (1d_y_array) plots them setting abscissas to consecutive positive integers. points (1d_x_array, 1d_y_array) plots points with their coordinates taken from the two arrays passed as arguments. If 2d_xy_array is a 2D array with two columns, or with two rows, points (2d_xy_array) plots the corresponding points on the plane.

Examples:

Two types of arguments for points, a list of pairs and two lists of separate coordinates.

(%i1) draw2d(points ([(x1,y1), (x2,y2),...]),
   ...);
key = "Small points",
points(makelist([random(20),random(50)],k,1,10)),
point_type = circle,
point_size = 3,
pointsJoined = true,
key = "Great points",
points(makelist(k,k,1,20),makelist(random(30),k,1,20)),
point_type = filled_down_triangle,
key = "Automatic abscissas",
color = red,
points([2,12,8])

Drawing impulses.

(%i11) draw2d(
    pointsJoined = impulses,
    line_width = 2,
    color = red,
    points(makelist([random(20),random(50)],k,1,10)))$
Array with ordinates.

```
(%i1) a: make_array (flonum, 100) $
(%i2) for i:0 thru 99 do a[i]: random(1.0) $
(%i3) draw2d(points(a)) $
```

Two arrays with separate coordinates.

```
(%i1) x: make_array (flonum, 100) $
(%i2) y: make_array (fixnum, 100) $
(%i3) for i:0 thru 99 do (  
    x[i]: float(i/100),  
    y[i]: random(10) ) $
(%i4) draw2d(points(x, y)) $
```
A two-column 2D array.

\begin{verbatim}
(%i1) xy: make_array(flonum, 100, 2) $
(%i2) for i:0 thru 99 do (  
    xy[i, 0]: float(i/100),  
    xy[i, 1]: random(10) ) $
(%i3) draw2d(points(xy)) $
\end{verbatim}

Drawing an array filled with function \texttt{read_array}.

\begin{verbatim}
(%i1) a: make_array(flonum,100) $
(%i2) read_array (file_search ("pidigits.data"), a) $
(%i3) draw2d(points(a)) $
\end{verbatim}

3D

\texttt{points([[x1, y1, z1], [x2, y2, z2], ...])} or \texttt{points([x1, x2, ...], [y1, y2, ...], [z1, z2, ...])} plots points \([x1, y1, z1], [x2, y2, z2], \text{etc.}\) If \texttt{matrix} is a three-column or three-row matrix, \texttt{points (matrix)} draws the associated points.
When arguments are lisp arrays, `points (1d_x_array, 1d_y_array, 1d_z_array)` takes coordinates from the three 1D arrays. If `2d_xyz_array` is a 2D array with three columns, or with three rows, `points (2d_xyz_array)` plots the corresponding points.

Examples:

One tridimensional sample,

```lisp
(%i1) load ("numericalio")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) draw3d(title = "Daily average wind speeds",
    point_size = 2,
    points(args(submatrix (s2, 4, 5))) )$
```

Two tridimensional samples,

```lisp
(%i1) load ("numericalio")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) draw3d(
    title = "Daily average wind speeds. Two data sets",
    point_size = 2,
    key = "Sample from stations 1, 2 and 3",
    points(args(submatrix (s2, 4, 5))),
    point_type = 4,
    key = "Sample from stations 1, 4 and 5",
    points(args(submatrix (s2, 2, 3))) )$
```

Unidimensional arrays,

```lisp
(%i1) x: make_array (fixnum, 10) $
(%i2) y: make_array (fixnum, 10) $
(%i3) z: make_array (fixnum, 10) $
(%i4) for i:0 thru 9 do ( 
    x[i]: random(10),
    y[i]: random(10),
    z[i]: random(10) )$
(%i5) draw3d(points(x,y,z))$
```
Bidimensional colored array,

(%i1) xyz: make_array(fixnum, 10, 3) $
(%i2) for i:0 thru 9 do ( 
    xyz[i, 0]: random(10),
    xyz[i, 1]: random(10),
    xyz[i, 2]: random(10) )$
(%i3) draw3d( 
    enhanced3d = true,
    points_joined = true,
    points(xyz))$

Color numbers explicitly specified by the user.

(%i1) pts: makelist([t,t^2,cos(t)], t, 0, 15)$
(%i2) col_num: makelist(k, k, 1, length(pts))$
(%i3) draw3d( 
    enhanced3d = ['part(col_num,k),k],
    point_size = 3,
    point_type = filled_circle,
    points(pts))$
polar \( (radius, \text{ang}, \text{minang}, \text{maxang}) \) [Graphic object]

Draws 2D functions defined in polar coordinates.

2D

\text{polar} \( (radius, \text{ang}, \text{minang}, \text{maxang}) \) plots function \( radius(\text{ang}) \) defined in polar coordinates, with variable \( \text{ang} \) taking values from \( \text{minang} \) to \( \text{maxang} \).

This object is affected by the following \textit{graphic options}: \texttt{nticks}, \texttt{line_width}, \texttt{line_type}, \texttt{key} and \texttt{color}.

Example:

\begin{verbatim}
(\%i1) draw2d(user_preamble = "set grid polar",
    nticks = 200,
    xrange = [-5,5],
    yrange = [-5,5],
    color = blue,
    line_width = 3,
    title = "Hyperbolic Spiral",
    polar(10/theta,theta,1,10*\%pi) )$
\end{verbatim}
polygon

polygon([[x1, y1], [x2, y2], ...])
polygon([[x1, x2, ...], [y1, y2, ...])

Draws polygons in 2D.

2D

The commands polygon([[x1, y1], [x2, y2], ...]) or polygon([[x1, x2, ...], [y1, y2, ...]) plot on the plane a polygon with vertices [x1, y1], [x2, y2], etc.

This object is affected by the following graphic options: transparent, fill_color, border, line_width, key, line_type and color.

Example:

(%i1) draw2d(
color = "#e245f0",
line_width = 8,
polygon([[3,2],[7,2],[5,5]]),
border = false,
fill_color = yellow,
polygon([[5,2],[9,2],[7,5]]) )$

quadrilateral (point_1, point_2, point_3, point_4)

Draws a quadrilateral.

2D

quadrilateral([[x1, y1], [x2, y2], [x3, y3], [x4, y4]) draws a quadrilateral with vertices [x1, y1], [x2, y2], [x3, y3], and [x4, y4].

This object is affected by the following graphic options: transparent, fill_color, border, line_width, key, xaxis_secondary, yaxis_secondary, line_type, transform and color.

Example:

(%i1) draw2d(
quadrilateral([[1,1],[2,2],[3,-1],[2,-2]])$
3D

quadrilateral([x1, y1, z1], [x2, y2, z2], [x3, y3, z3], [x4, y4, z4]) draws a quadrilateral with vertices [x1, y1, z1], [x2, y2, z2], [x3, y3, z3], and [x4, y4, z4].

This object is affected by the following graphic options: line_type, line_width, color, key, enhanced3d and transform.

rectangle ([x1,y1], [x2,y2])
Draws rectangles in 2D.

2D

rectangle ([x1,y1], [x2,y2]) draws a rectangle with opposite vertices [x1,y1] and [x2,y2].

This object is affected by the following graphic options: transparent, fill_color, border, line_width, key, line_type and color.

Example:

(%i1) draw2d(fill_color = red,
line_width = 6,
line_type = dots,
transparent = false,
fill_color = blue,
rectangle([-2,-2],[8,-1]), /* opposite vertices */
transparent = true,
line_type = solid,
line_width = 1,
rectangle([9,4],[2,-1.5]),
xrange = [-3,10],
yrange = [-3,4.5]$
region(expr, var1, minval1, maxval1, var2, minval2, maxval2)  [Graphic object]
Plots a region on the plane defined by inequalities.

2D expr is an expression formed by inequalities and boolean operators **and**, **or**, and **not**. The region is bounded by the rectangle defined by \([minval1, maxval1]\) and \([minval2, maxval2]\).

This object is affected by the following **graphic options**: **fill_color**, **key**, **x_voxel** and **y_voxel**.

Example:

```plaintext
(%i1) draw2d(
    x_voxel = 30,
    y_voxel = 30,
    region(x^2+y^2<1 and x^2+y^2 > 1/2,
        x, -1.5, 1.5, y, -1.5, 1.5));
```

![Graphical representation of region](image_url)
spherical \((radius, azi, minazi, maxazi, zen, minzen, maxzen)\) [Graphic object]

Draws 3D functions defined in spherical coordinates.

3D

\[\text{spherical}(radius, azi, minazi, maxazi, zen, minzen, maxzen)\] plots the function \(radius(azi, zen)\) defined in spherical coordinates, with azimuth \(azi\) taking values from \(minazi\) to \(maxazi\) and zenith \(zen\) taking values from \(minzen\) to \(maxzen\).

This object is affected by the following graphic options: \(\text{xu_grid}, \text{yv_grid}, \text{line_type}, \text{key}, \text{wired_surface}, \text{enhanced3d} \) and \text{color}.

Example:

\[
(\%i1) \text{draw3d(spherical(1,a,0,2*%pi,z,0,%pi))}$

triangle \((point_1, point_2, point_3)\) [Graphic object]

Draws a triangle.

2D

triangle \(([x1,y1], [x2,y2], [x3,y3])\) draws a triangle with vertices \([x1,y1]\), \([x2,y2]\), and \([x3,y3]\).

This object is affected by the following graphic options: \(\text{transparent}, \text{fill_color}, \text{border}, \text{line_width}, \text{key}, \text{xaxis_secondary}, \text{yaxis_secondary}, \text{line_type}, \text{transform} \) and \text{color}.

Example:

\[
(\%i1) \text{draw2d(}
\quad \text{triangle}([1,1],[2,2],[3,-1]))$

\]
3D

triangle ([x1,y1,z1], [x2,y2,z2], [x3,y3,z3]) draws a triangle with vertices [x1,y1,z1], [x2,y2,z2], and [x3,y3,z3].

This object is affected by the following graphic options: line_type, line_width, color, key, enhanced3d and transform.

tube (xfun,yfun,zfun,rfun,p,pmin,pmax) [Graphic object]
Draws a tube in 3D with varying diameter.

3D

[xfun,yfun,zfun] is the parametric curve with parameter p taking values from pmin to pmax. Circles of radius rfun are placed with their centers on the parametric curve and perpendicular to it.

This object is affected by the following graphic options: xu_grid, yv_grid, line_type, line_width, key, wired_surface, enhanced3d, color and capping.

Example:

(%i1) draw3d(
   enhanced3d = true,
   xu_grid = 50,
   tube(cos(a), a, 0, cos(a/10)^2,
       a, 0, 4*%pi))$
vector

vector ([x,y], [dx,dy])
vector ([x,y,z], [dx,dy,dz])

Draws vectors in 2D and 3D.

This object is affected by the following graphic options: head_both, head_length, head_angle, head_type, line_width, line_type, key and color.

2D

vector([x,y], [dx,dy]) plots vector [dx,dy] with origin in [x,y].

Example:

(%i1) draw2d(xrange = [0,12],
yrange = [0,10],
head_length = 1,
vector([0,1],[5,5]), /* default type */
head_type = 'empty,
vector([3,1],[5,5]),
head_both = true,
head_type = 'nofilled,
line_type = dots,
vector([6,1],[5,5]))$
3D

`vector([x, y, z], [dx, dy, dz])` plots vector `[dx, dy, dz]` with origin in `[x, y, z]`.

Example:

```
(%i1) draw3d(color = cyan,
    vector([0,0,0],[1,1,1]/sqrt(3)),
    vector([0,0,0],[1,-1,0]/sqrt(2)),
    vector([0,0,0],[1,1,-2]/sqrt(6)) )$
```

`draw_renderer` [Variable]

Default value: `gnuplot_pipes`

When `draw_renderer` is set to `'vtk`, the VTK interface is used for draw.

53.3 Functions and Variables for pictures

`get_pixel (pic,x,y)` [Function]

Returns pixel from picture. Coordinates x and y range from 0 to `width-1` and `height-1`, respectively.
make_level_picture  
make_level_picture (data)  
make_level_picture (data, width, height)

Returns a levels picture object. make_level_picture (data) builds the picture object from matrix data. make_level_picture (data, width, height) builds the object from a list of numbers; in this case, both the width and the height must be given.

The returned picture object contains the following four parts:

1. symbol level
2. image width
3. image height
4. an integer array with pixel data ranging from 0 to 255. Argument data must contain only numbers ranged from 0 to 255; negative numbers are substituted by 0, and those which are greater than 255 are set to 255.

Example:

Level picture from matrix.

(%i1) make_level_picture(matrix([3,2,5],[7,-9,3000]));
  (%o1) picture(level, 3, 2, {Array: #(3 2 5 7 0 255)})

Level picture from numeric list.

(%i1) make_level_picture([-2,0,54,%pi],2,2);
  (%o1) picture(level, 2, 2, {Array: #(0 0 54 3)})

make_rgb_picture (redlevel, greenlevel, bluelevel)  

Returns an rgb-coloured picture object. All three arguments must be levels picture; with red, green and blue levels.

The returned picture object contains the following four parts:

1. symbol rgb
2. image width
3. image height
4. an integer array of length 3*width*height with pixel data ranging from 0 to 255. Each pixel is represented by three consecutive numbers (red, green, blue).

Example:

(%i1) red: make_level_picture(matrix([3,2],[7,260]));
  (%o1) picture(level, 2, 2, {Array: #(3 2 7 255)})
(%i2) green: make_level_picture(matrix([54,23],[73,-9]));
  (%o2) picture(level, 2, 2, {Array: #(54 23 73 0)})
(%i3) blue: make_level_picture(matrix([123,82],[45,32.5698]));
  (%o3) picture(level, 2, 2, {Array: #(123 82 45 33)})
(%i4) make_rgb_picture(red,green,blue);
  (%o4) picture(rgb, 2, 2,
            {Array: #(3 54 123 2 23 82 7 73 45 255 0 33)})

negative_picture (pic)  

Returns the negative of a (level or rgb) picture.
picture_equalp (x, y)
   Returns true in case of equal pictures, and false otherwise.

picturep (x)
   Returns true if the argument is a well formed image, and false otherwise.

read_xpm (xpm_file)
   Reads a file in xpm and returns a picture object.

rgb2level (pic)
   Transforms an rgb picture into a level one by averaging the red, green and blue channels.

take_channel (im, color)
   If argument color is red, green or blue, function take_channel returns the corresponding color channel of picture im. Example:
   (%i1) red: make_level_picture(matrix([3,2],[7,260]));
   (%o1) picture(level, 2, 2, {Array: #(3 2 7 255)})
   (%i2) green: make_level_picture(matrix([54,23],[73,-9]));
   (%o2) picture(level, 2, 2, {Array: #(54 23 73 0)})
   (%i3) blue: make_level_picture(matrix([123,82],[45,32.5698]));
   (%o3) picture(level, 2, 2, {Array: #(123 82 45 33)})
   (%i4) make_rgb_picture(red, green, blue);
   (%o4) picture(rgb, 2, 2,
            {Array: #(3 54 123 2 23 82 7 73 45 255 0 33)})
   (%i5) take_channel(%,'green); /* simple quote!!! */
   (%o5) picture(level, 2, 2, {Array: #(54 23 73 0)})

53.4 Functions and Variables for worldmap

53.4.1 Variables and Functions

boundaries_array
   Default value: false
   boundaries_array is where the graphic object geomap looks for boundaries coordinates.
   Each component of boundaries_array is an array of floating point quantities, the coordinates of a polygonal segment or map boundary.
   See also geomap.

numbered_boundaries (nlist)
   Draws a list of polygonal segments (boundaries), labeled by its numbers (boundaries_array coordinates). This is of great help when building new geographical entities.
   Example:
   Map of Europe labeling borders with their component number in boundaries_array.
   (%i1) load("worldmap")$
   (%i2) european_borders:
      region_boundaries(-31.81,74.92,49.84,32.06)$
   (%i3) numbered_boundaries(european_borders)$
make_poly_continent

make_poly_continent (continent_name)
make_poly_continent (country_list)

Makes the necessary polygons to draw a colored continent or a list of countries.

Example:

(%i1) load("worldmap")$
(%i2) /* A continent */
make_poly_continent(Africa)$
(%i3) apply(draw2d, %)$

(%i4) /* A list of countries */
make_poly_continent([Germany,Denmark,Poland])$
(%i5) apply(draw2d, %)$
**make_poly_country (country_name)**

Makes the necessary polygons to draw a colored country. If islands exist, one country can be defined with more than just one polygon.

Example:

```
(%i1) load("worldmap")$
(%i2) make_poly_country(India)$
(%i3) apply(draw2d, %)$
```

**make_polygon (nlist)**

Returns a polygon object from boundary indices. Argument nlist is a list of components of `boundaries_array`.

Example:

Bhutan is defined by boundary numbers 171, 173 and 1143, so that `make_polygon([171,173,1143])` appends arrays of coordinates `boundaries_array[171]`, `boundaries_array[173]` and `boundaries_array[1143]` and returns a polygon object suited to be plotted by `draw`. To avoid an error message, arrays must be compatible in the sense that any two consecutive arrays have two coordinates in the extremes in common. In this example, the two first components of `boundaries_array[171]` are equal to the last two coordinates of `boundaries_array[173]`, and the two first of `boundaries_array[173]` are equal to the two first of `boundaries_array[1143]`; in conclusion, boundary numbers 171, 173 and 1143 (in this order) are compatible and the colored polygon can be drawn.

```
(%i1) load("worldmap")$
(%i2) Bhutan;
(%o2) [[171, 173, 1143]]
(%i3) boundaries_array[171];
(%o3) {Array:
    #(88.750549 27.14727 88.806351 27.25305 88.901367 27.282221
     88.917877 27.321039)}
(%i4) boundaries_array[173];
Chapter 53: draw

```lisp
(%o4) {Array:
    #(91.659554 27.76511 91.6008 27.66666 91.598022 27.62499
     91.631348 27.536381 91.765533 27.45694 91.775253 27.4161
     92.007751 27.471939 92.11441 27.28583 92.015259 27.168051
     92.015533 27.08083 92.083313 27.02277 92.112183 26.920271
     92.069977 26.86194 91.997192 26.85194 91.915253 26.893881
     91.916924 26.85416 91.8358 26.863331 91.712479 26.799999
     91.542191 26.80444 91.492188 26.87472 91.418854 26.873329
     91.371353 26.800831 91.307457 26.778049 90.682457 26.77417
     90.392197 26.903601 90.344131 26.894159 90.143044 26.75333
     89.98996 26.73583 89.841919 26.70138 89.618301 26.72694
     89.636093 26.771111 89.360786 26.859989 89.22081 26.81472
     89.110237 26.829161 88.917877 26.873329 88.873016 26.95499
     88.867737 27.080549 88.843307 27.108601 88.750549
     27.14727)}

(%i5) boundaries_array[1143];
(%o5) {Array:
    #(91.666924 27.88888 91.65831 27.94805
     91.338028 28.05249 91.314972 28.096661 91.108856 27.971109
     91.015808 27.97777 90.896927 28.05055 90.382462 28.07972
     90.396088 28.23555 90.360786 28.257771 89.996353 28.32333
     89.83165 28.24888 89.58609 28.139999 89.35997 27.87166
     89.225517 27.795 89.125793 27.56749 88.971077 27.47361
     88.917877 27.321039)}

(%i6) Bhutan_polygon: make_polygon([171,173,1143])$
(%i7) draw2d(Bhutan_polygon)$
```

**region_boundaries (x1,y1,x2,y2)**

[Function]

Detects polygonal segments of global variable **boundaries_array** fully contained in the rectangle with vertices \((x1,y1)\) -upper left- and \((x2,y2)\) -bottom right-.

Example:
Returns segment numbers for plotting southern Italy.

(%i1) load("worldmap")$
(%i2) region_boundaries(10.4,41.5,20.7,35.4);
(%o2) [1846, 1863, 1864, 1881, 1888, 1894]
(%i3) draw2d(geomap(%))$

region_boundaries_plus (x1,y1,x2,y2) [Function]
Detects polygonal segments of global variable boundaries_array containing at least one vertex in the rectangle defined by vertices (x1,y1) -upper left- and (x2,y2) -bottom right-.

Example:

(%i1) load("worldmap")$
(%i2) region_boundaries_plus(10.4,41.5,20.7,35.4);
(%o2) [1060, 1062, 1076, 1835, 1839, 1844, 1846, 1858, 1861, 1863, 1864, 1871, 1881, 1888, 1894, 1897]
(%i3) draw2d(geomap(%))$
53.4.2 Graphic objects

**geomap**

geomap (numlist)

geomap (numlist,3Dprojection)

Draws cartographic maps in 2D and 3D.

**2D**

This function works together with global variable `boundaries_array`.

Argument `numlist` is a list containing numbers or lists of numbers. All these numbers must be integers greater or equal than zero, representing the components of global array `boundaries_array`.

Each component of `boundaries_array` is an array of floating point quantities, the coordinates of a polygonal segment or map boundary.

`geomap (numlist)` flattens its arguments and draws the associated boundaries in `boundaries_array`.

This object is affected by the following graphic options: `line_width`, `line_type` and `color`.

Examples:

A simple map defined by hand:

```latex
(\%i1) load("worldmap")$

(\%i2) /* Vertices of boundary #0: \{(1,1),(2,5),(4,3)\} */

(\%i3) /* Vertices of boundary #1: \{(4,3),(5,4),(6,4),(5,1)\} */

(\%i4) /* Vertices of boundary #2: \{(5,1),(3,0),(1,1)\} */
```
bnd2[0]:5.0, bnd2[1]:1.0, bnd2[2]:3.0, 
bnd2[3]:0.0, bnd2[4]:1.0, bnd2[5]:1.0 )$

(%i5) /* Vertices of boundary #3: {(1,1), (4,3)} */
  ( bnd3: make_array(flonum,4),
    bnd3[0]:1.0, bnd3[1]:1.0, bnd3[2]:4.0, bnd3[3]:3.0)$

(%i6) /* Vertices of boundary #4: {(4,3), (5,1)} */
  ( bnd4: make_array(flonum,4),
    bnd4[0]:4.0, bnd4[1]:3.0, bnd4[2]:5.0, bnd4[3]:1.0)$

(%i7) /* Pack all together in boundaries_array */
  ( boundaries_array: make_array(any,5),
    boundaries_array[0]: bnd0, boundaries_array[1]: bnd1, 
    boundaries_array[2]: bnd2, boundaries_array[3]: bnd3, 
    boundaries_array[4]: bnd4 )$

(%i8) draw2d(geomap([0,1,2,3,4]))$

The auxiliary package worldmap sets the global variable boundaries_array to real world boundaries in (longitude, latitude) coordinates. These data are in the public domain and come from https://web.archive.org/web/20100310124019/http://www-cger.nies.go.jp/grid-e/gridtxt/grid19.html. Package worldmap defines also boundaries for countries, continents and coastlines as lists with the necessary components of boundaries_array (see file share/draw/worldmap.mac for more information). Package worldmap automatically loads package worldmap.

(%i11) load("worldmap")$

(%i12) c1: gr2d(geomap([Canada,United_States, Mexico,Cuba]))$

(%i13) c2: gr2d(geomap(Africa))$

(%i14) c3: gr2d(geomap([Oceania,China,Japan]))$

(%i15) c4: gr2d(geomap([France,Portugal,Spain, Morocco,Western_Sahara]))$

(%i16) draw(columns = 2, 
   c1,c2,c3,c4)$
Package `worldmap` is also useful for plotting countries as polygons. In this case, graphic object `geomap` is no longer necessary and the `polygon` object is used instead. Since lists are now used and not arrays, maps rendering will be slower. See also `make_poly_country` and `make_poly_continent` to understand the following code.

```
(%i1) load("worldmap")
(%i2) mymap: append(
    [color = white], /* borders are white */
    [fill_color = red], make_poly_country(Bolivia),
    [fill_color = cyan], make_poly_country(Paraguay),
    [fill_color = green], make_poly_country(Colombia),
    [fill_color = blue], make_poly_country(Chile),
    [fill_color = "#23ab0f"], make_poly_country(Brazil),
    [fill_color = goldenrod], make_poly_country(Argentina),
    [fill_color = "midnight-blue"], make_poly_country(Uruguay))$
(%i3) apply(draw2d, mymap)$
```

3D
geomap (numlist) projects map boundaries on the sphere of radius 1 centered at (0,0,0). It is possible to change the sphere or the projection type by using geomap (numlist,3Dprojection).

Available 3D projections:

• [spherical_projection,x,y,z,r]: projects map boundaries on the sphere of radius r centered at (x,y,z).

\[
\begin{align*}
(\%i1) \text{load("worldmap")}$
(\%i2) \text{draw3d(geomap(Australia), /* default projection */}
\quad \text{geomap(Australia,}
\quad \quad [\text{spherical_projection,2,2,2,3}]))$
\end{align*}
\]

• [cylindrical_projection,x,y,z,r,rc]: re-projects spherical map boundaries on the cylinder of radius rc and axis passing through the poles of the globe of radius r centered at (x,y,z).

\[
\begin{align*}
(\%i1) \text{load("worldmap")}$
(\%i2) \text{draw3d(geomap([America_coastlines,Eurasia_coastlines],}
\quad [\text{cylindrical_projection,2,2,3,4}]))$
\end{align*}
\]
• \([\text{conic_projection}, x, y, z, r, \alpha]\): re-projects spherical map boundaries on the cones of angle \(\alpha\), with axis passing through the poles of the globe of radius \(r\) centered at \((x, y, z)\). Both the northern and southern cones are tangent to sphere.

(%i1) load("worldmap");
(%i2) draw3d(geomap(World_coastlines,
    [conic_projection,0,0,0,1,90]));

See also \url{http://riotorto.users.sf.net/gnuplot/geomap} for more elaborated examples.
54 drawdf

54.1 Introduction to drawdf

The function drawdf draws the direction field of a first-order Ordinary Differential Equation (ODE) or a system of two autonomous first-order ODE’s.

Since this is an additional package, in order to use it you must first load it with load("drawdf"). Drawdf is built upon the draw package, which requires Gnuplot 4.2.

To plot the direction field of a single ODE, the ODE must be written in the form:

\[ \frac{dy}{dx} = F(x, y) \]

and the function F should be given as the argument for drawdf. If the independent and dependent variables are not x, and y, as in the equation above, then those two variables should be named explicitly in a list given as an argument to the drawdf command (see the examples).

To plot the direction field of a set of two autonomous ODE’s, they must be written in the form

\[ \frac{dx}{dt} = G(x, y) \quad \frac{dy}{dt} = F(x, y) \]

and the argument for drawdf should be a list with the two functions G and F, in that order; namely, the first expression in the list will be taken to be the time derivative of the variable represented on the horizontal axis, and the second expression will be the time derivative of the variable represented on the vertical axis. Those two variables do not have to be x and y, but if they are not, then the second argument given to drawdf must be another list naming the two variables, first the one on the horizontal axis and then the one on the vertical axis.

If only one ODE is given, drawdf will implicitly admit \(x=t\), and \(G(x, y)=1\), transforming the non-autonomous equation into a system of two autonomous equations.

54.2 Functions and Variables for drawdf

54.2.1 Functions

drawdf

drawdf (dydx, ...options and objects...)  
**drawdf**(dydx, ...options and objects...)  
drawdf (dvdu, [u,v], ...options and objects...)  
**drawdf**(dvdu, [u,v], ...options and objects...)  
drawdf (dvdu, [u,umin,umax], [v,vmin,vmax], ...options and objects...)  
**drawdf**(dvdu, [u,umin,umax], [v,vmin,vmax], ...options and objects...)  
drawdf ([dxdt,dydt], ...options and objects...)  
**drawdf**([dxdt,dydt], ...options and objects...)  
drawdf ([dudt,dvdt], [u,v], ...options and objects...)  
**drawdf**([dudt,dvdt], [u,v], ...options and objects...)  
drawdf ([dudt,dvdt], [u,umin,umax], [v,vmin,vmax], ...options and objects...)  
**drawdf**([dudt,dvdt], [u,umin,umax], [v,vmin,vmax], ...options and objects...)  

Function drawdf draws a 2D direction field with optional solution curves and other graphics using the draw package.
The first argument specifies the derivative(s), and must be either an expression or a list of two expressions. \( dydx \), \( dxdt \) and \( dydt \) are expressions that depend on \( x \) and \( y \). \( dvdu \), \( dudt \) and \( dvdt \) are expressions that depend on \( u \) and \( v \).

If the independent and dependent variables are not \( x \) and \( y \), then their names must be specified immediately following the derivative(s), either as a list of two names \([u,v]\), or as two lists of the form \([u,umin,umax]\) and \([v,vmin,vmax]\).

The remaining arguments are graphic options, graphic objects, or lists containing graphic options and objects, nested to arbitrary depth. The set of graphic options and objects supported by \texttt{drawdf} is a superset of those supported by \texttt{draw2d} and \texttt{gr2d} from the \texttt{draw} package.

The arguments are interpreted sequentially: graphic options affect all following graphic objects. Furthermore, graphic objects are drawn on the canvas in order specified, and may obscure graphics drawn earlier. Some graphic options affect the global appearance of the scene.

The additional graphic objects supported by \texttt{drawdf} include: \texttt{solns_at}, \texttt{points_at}, \texttt{saddles_at}, \texttt{soln_at}, \texttt{point_at}, and \texttt{saddle_at}.

The additional graphic options supported by \texttt{drawdf} include: \texttt{field_degree}, \texttt{soln_arrows}, \texttt{field_arrows}, \texttt{field_grid}, \texttt{field_color}, \texttt{show_field}, \texttt{tstep}, \texttt{nsteps}, \texttt{duration}, \texttt{direction}, \texttt{field_tstep}, \texttt{field_nsteps}, and \texttt{field_duration}.

Commonly used graphic objects inherited from the \texttt{draw} package include: \texttt{explicit}, \texttt{implicit}, \texttt{parametric}, \texttt{polygon}, \texttt{points}, \texttt{vector}, \texttt{label}, and all others supported by \texttt{draw2d} and \texttt{gr2d}.

Commonly used graphic options inherited from the \texttt{draw} package include: \texttt{points_joined}, \texttt{color}, \texttt{point_type}, \texttt{point_size}, \texttt{line_width}, \texttt{line_type}, \texttt{key}, \texttt{title}, \texttt{xlabel}, \texttt{ylabel}, \texttt{user_preamble}, \texttt{terminal}, \texttt{dimensions}, \texttt{file_name}, and all others supported by \texttt{draw2d} and \texttt{gr2d}.

See also \texttt{draw2d}, \texttt{rk}, \texttt{desolve} and \texttt{ode2}.

Users of wxMaxima or Imaxima may optionally use \texttt{wxdrawdf}, which is identical to \texttt{drawdf} except that the graphics are drawn within the notebook using \texttt{wxdraw}.

To make use of this function, write first \texttt{load("drawdf")}.

Examples:

```maxima
(%i1) load("drawdf")$
(%i2) drawdf(exp(-x)+y)$  /* default vars: x,y */
(%i3) drawdf(exp(-t)+y, [t,y])$  /* default range: \([-10,10]\] */
(%i4) drawdf([y,-9*sin(x)-y/5], [x,1,5], [y,-2,2])$
(%i5) drawdf(2*cos(t)-1+y, [t,-5,10], [y,-4,9], soln_at([0,0.1],[0,-0.1]),
            solns_at([0,0.1],[0,-0.1]),
            trajectory_at,0,0])$
```

For backward compatibility, \texttt{drawdf} accepts most of the parameters supported by \texttt{plotdf}.

```maxima
(%i5) drawdf(2*cos(t)-1+y, [t,-5,10], [y,-4,9],
            [trajectory_at,0,0])$
```

\texttt{soln_at} and \texttt{solns_at} draw solution curves passing through the specified points, using a slightly enhanced 4th-order Runge Kutta numerical integrator.

```maxima
(%i6) drawdf(2*cos(t)-1+y, [t,-5,10], [y,-4,9],
            solns_at([0,0.1],[0,-0.1]),
```
field_degree=2 causes the field to be composed of quadratic splines, based on the first and second derivatives at each grid point. field_grid=[COLS,ROWS] specifies the number of columns and rows in the grid.

(%i7) drawdf(2*cos(t)-1+y, [t,-5,10], [y,-4,9],
           field_degree=2, field_grid=[20,15],
           solns_at([0,0.1],[0,-0.1]),
           color=blue, soln_at(0,0))$

soln_arrows=true adds arrows to the solution curves, and (by default) removes them from the direction field. It also changes the default colors to emphasize the solution curves.

(%i8) drawdf(2*cos(t)-1+y, [t,-5,10], [y,-4,9],
           soln_arrows=true,
           solns_at([0,0.1],[0,-0.1],[0,0]),
           duration=40)$

duration=40 specifies the time duration of numerical integration (default 10). Integration will also stop automatically if the solution moves too far away from the plotted region, or if the derivative becomes complex or infinite. Here we also specify field_degree=2 to plot quadratic splines. The equations below model a predator-prey system.

(%i9) drawdf([x*(1-x-y), y*(3/4-y-x/2)], [x,0,1.1], [y,0,1],
           field_degree=2, duration=40,
           soln_arrows=true, point_at(1/2,1/2),
           solns_at([0.1,0.2], [0.2,0.1], [1,0.8],
                     [0.8,1], [0.1,0.1], [0.6,0.05],
                     [0.05,0.4], [1,0.01], [0.01,0.75]),
           field_degree='solns)$

drawdf([x*(1-x-y), y*(3/4-y-x/2)], [x,0,1.1], [y,0,1],
           field_degree='solns, duration=40,
           soln_arrows=true, point_at(1/2,1/2),
           solns_at([0.1,0.2], [0.2,0.1], [1,0.8],
                     [0.8,1], [0.1,0.1], [0.6,0.05],
                     [0.05,0.4], [1,0.01], [0.01,0.75]))$

saddles_at attempts to automatically linearize the equation at each saddle, and to plot a numerical solution corresponding to each eigenvector, including the separatrices. tstep=0.05 specifies the maximum time step for the numerical integrator (the default is 0.1). Note that smaller time steps will sometimes be used in order to keep the x and y steps small. The equations below model a damped pendulum.

(%i10) drawdf([y,-9*sin(x)-y/5], tstep=0.05,
          soln_arrows=true, point_size=0.5,
          points_at([0,0], [2*%pi,0], [-2*%pi,0]),
          field_degree='solns,
          saddles_at([%pi,0], [-%pi,0]))$

show_field=false suppresses the field entirely.

(%i12) drawdf([y,-9*sin(x)-y/5], tstep=0.05,
\begin{verbatim}
show_field=false, soln_arrows=true,
point_size=0.5,
points_at([0,0], [2*%pi,0], [-2*%pi,0]),
saddles_at([3*%pi,0], [-3*%pi,0],
            [%pi,0], [-%pi,0]))
\end{verbatim}

\texttt{drawdf} passes all unrecognized parameters to \texttt{draw2d} or \texttt{gr2d}, allowing you to combine the full power of the \texttt{draw} package with \texttt{drawdf}.

\begin{verbatim}
(%i13) drawdf(x^2+y^2, [x,-2,2], [y,-2,2], field_color=gray,
         key="soln 1", color=black, soln_at(0,0),
         key="soln 2", color=red, soln_at(0,1),
         key="isocline", color=green, line_width=2,
         nticks=100, parametric(cos(t),sin(t),t,0,2*%pi))$
\end{verbatim}

\texttt{drawdf} accepts lists of graphic options and objects, allowing convenient use of \texttt{makelist} and other function calls to generate graphics.

\begin{verbatim}
(%i14) colors : ['red,'blue,'purple,'orange,'green]$
(%i15) drawdf([(x-x*y/2, (x*y - 3*y)/4],
            [x,2.5,3.5], [y,1.5,2.5],
            field_color = gray,
            makelist([ key = concat("soln",k),
                       color = colors[k],
                       soln_at(3, 2 + k/20 ) ],
                      k,1,5))$
\end{verbatim}
55 dynamics

55.1 The dynamics package

Package dynamics includes functions for 3D visualization, animations, graphical analysis of differential and difference equations and numerical solution of differential equations. The functions for differential equations are described in the section on Chapter 22 [Numerical], page 403 and the functions to plot the Mandelbrot and Julia sets are described in the section on Chapter 12 Plotting.

All the functions in this package will be loaded automatically the first time they are used.

55.2 Graphical analysis of discrete dynamical systems

chaosgame ([[x1, y1]|...|xm, ym]], [x0, y0], b, n, options, ...); [Function]
Implement the so-called chaos game: the initial point \((x0, y0)\) is plotted and then one of the \(m\) points \([x1, y1]|...|xm, ym]\) will be selected at random. The next point plotted will be on the segment from the previous point plotted to the point chosen randomly, at a distance from the random point which will be \(b\) times that segment’s length. The procedure is repeated \(n\) times. The options are the same as for plot2d.

Example. A plot of Sierpinsky’s triangle:

(%i1) chaosgame([[0, 0], [1, 0], [0.5, sqrt(3)/2]], [0.1, 0.1], 1/2, 30000, [style, dots]);

evolution \((F, y0, n, ..., options, ...); [Function]
Draws \(n+1\) points in a two-dimensional graph, where the horizontal coordinates of the points are the integers 0, 1, 2, ..., \(n\), and the vertical coordinates are the corresponding values \(y(n)\) of the sequence defined by the recurrence relation

\[ y_{n+1} = F(y_n) \]
With initial value $y(0)$ equal to $y_0$. $F$ must be an expression that depends only on one variable (in the example, it depend on $y$, but any other variable can be used), $y_0$ must be a real number and $n$ must be a positive integer. This function accepts the same options as \texttt{plot2d}.

**Example.**

\begin{verbatim}
(%i1) evolution(cos(y), 2, 11);
\end{verbatim}

\begin{verbatim}
(%i1) evolution(\cos(y), 2, 11);
\end{verbatim}

\begin{verbatim}
evolution2d ([F, G], [u, v], [u0, y0], n, options, . . .);
\end{verbatim}

\begin{verbatim}
evolution2d ([\texttt{f}, \texttt{g}], [\texttt{x}, \texttt{y}], [-0.5,0], 50000, [\texttt{style}, \texttt{dots}]);
\end{verbatim}
And an enlargement of a small region in that fractal:

\[(%)9\) evolution2d([f, g], [x, y], [-0.5, 0], 300000, [x, -0.8, -0.6], [y, -0.4, -0.2], [style, dots]);

\[\text{ifs } ([r1, \ldots, rm], [A1, \ldots, Am], [[x1, y1], \ldots, [xm, ym]], [x0, y0], n, \quad [\text{Function}]\]

\[\text{options, } \ldots);\]

Implements the Iterated Function System method. This method is similar to the method described in the function \texttt{chaosgame}. but instead of shrinking the segment from the current point to the randomly chosen point, the 2 components of that segment will be multiplied by the 2 by 2 matrix \(Ai\) that corresponds to the point chosen randomly.

The random choice of one of the \(m\) attractive points can be made with a non-uniform probability distribution defined by the weights \(r1, \ldots, rm\). Those weights are given in cumulative form; for instance if there are 3 points with probabilities 0.2, 0.5 and 0.3, the weights \(r1\), \(r2\) and \(r3\) could be 2, 7 and 10. The options are the same as for \texttt{plot2d}. 
Example. Barnsley's fern, obtained with 4 matrices and 4 points:

\begin{verbatim}
(%i1) a1: matrix([0.85,0.04],
[-0.04,0.85])$
(%i2) a2: matrix([0.2,-0.26],[0.23,0.22])$
(%i3) a3: matrix([-0.15,0.28],[0.26,0.24])$
(%i4) a4: matrix([0,0],[0,0.16])$
(%i5) p1: [0,1.6]$
(%i6) p2: [0,1.6]$
(%i7) p3: [0,0.44]$
(%i8) p4: [0,0]$
(%i9) w: [85,92,99,100]$
(%i10) ifs(w, [a1,a2,a3,a4], [p1,p2,p3,p4], [5,0], 50000, [style,dots]);
\end{verbatim}

\begin{verbatim}
Example. Barnsley's fern, obtained with 4 matrices and 4 points:

(%i11) orbits(x^2+a, 0, 50, 200, [a, -2, 0.25], [style, dots]);
\end{verbatim}

The function $F(y)$ defines a sequence with a starting value of $y_0$, as in the case of the function \texttt{evolution}, but in this case that function will also depend on a parameter $x$ that will take values in the interval from $x_0$ to $x_f$ with increments of $x_{step}$. Each value used for the parameter $x$ is shown on the horizontal axis. The vertical axis will show the $n_2$ values of the sequence $y(n_1+1),\ldots, y(n_1+n_2+1)$ obtained after letting the sequence evolve $n_1$ iterations. In addition to the options accepted by \texttt{plot2d}, it accepts an option \texttt{pixels} that sets up the maximum number of different points that will be represented in the vertical direction.

Example. Orbits diagram of the quadratic map, with a parameter $a$:
To enlarge the region around the lower bifurcation near $x = -1.25$ use:

\[
\texttt{(\%i2) orbits(x^2+a, 0, 100, 400, [a,-1,-1.53], [x,-1.6,-0.8],}
\]

\[
\texttt{[nticks, 400], [style,dots]);}
\]

The interpretation and allowed values of the input parameters is the same as for the function \texttt{evolution}. A staircase diagram consists of a plot of the function $F(y)$, together with the line $G(y) = y$. A vertical segment is drawn from the point $(y0, y0)$ on that line until the point where it intersects the function $F$. From that point a horizontal segment is drawn until it reaches the point $(y1, y1)$ on the line, and the procedure is repeated $n$ times until the point $(yn, yn)$ is reached. The options are the same as for \texttt{plot2d}. 

\texttt{staircase (F, y0, n, options, ...); [Function]}

Draws a staircase diagram for the sequence defined by the recurrence relation

\[
y_{n+1} = F(y_n)
\]
Example.

(%i1) staircase(cos(y), 1, 11, [y, 0, 1.2]);

55.3 Visualization with VTK

Function scene creates 3D images and animations using the Visualization ToolKit (VTK) software. In order to use that function, Xmaxima and VTK should be installed in your system (including the TCL bindings of VTK, which in some system might come in a separate package).

scene (objects, ..., options, ...):

Accepts an empty list or a list of several [scene_objects], page 945 and [scene_options], page 944. The program launches Xmaxima, which opens an external window representing the given objects in a 3-dimensional space and applying the options given. Each object must belong to one of the following 4 classes: sphere, cube, cylinder or cone (see [scene_objects], page 945). Objects are identified by giving their name or by a list in which the first element is the class name and the following elements are options for that object.

Example. A hexagonal pyramid with a blue background:

(%i1) scene(cone, [background,"#9980e5"])$
By holding down the left button of the mouse while it is moved on the graphics window, the camera can be rotated showing different views of the pyramid. The two plot options [scene_elevation], page 944 and [scene_azimuth], page 944 can also be used to change the initial orientation of the viewing camera. The camera can be moved by holding the middle mouse button while moving it and holding the right-side mouse button while moving it up or down will zoom in or out.

Each object option should be a list starting with the option name, followed by its value. The list of allowed options can be found in the [object_options], page 946 section.

**Example.** This will show a sphere falling to the ground and bouncing off without losing any energy. To start or pause the animation, press the play/pause button.

```
(%i1) p: makelist ([0,0,2.1- 9.8*t^2/2], t, 0, 0.64, 0.01)$
(%i2) p: append (p, reverse(p))$
(%i3) ball: [sphere, [radius,0.1], [thetaresolution,20],
[phiresolution,20], [position,0,0,2.1], [color,red],
[animate,position,p]]$
(%i4) ground: [cube, [xlength,2], [ylength,2], [zlength,0.2],
[position,0,0,-0.1], [color,violet]]$
(%i5) scene (ball, ground, restart)$
```
The restart option was used to make the animation restart automatically every time the last point in the position list is reached. The accepted values for the colors are the same as for the color option of plot2d.

55.3.1 Scene options

azimuth [azimuth, angle]  [Scene option]
Default value: 135
The rotation of the camera on the horizontal (x, y) plane. angle must be a real number; an angle of 0 means that the camera points in the direction of the y axis and the x axis will appear on the right.

background [background, color]  [Scene option]
Default value: black
The color of the graphics window’s background. It accepts color names or hexadecimal red-green-blue strings (see the color option of plot2d).

elevation [elevation, angle]  [Scene option]
Default value: 30
The vertical rotation of the camera. The angle must be a real number; an angle of 0 means that the camera points on the horizontal, and the default angle of 30 means that the camera is pointing 30 degrees down from the horizontal.

height [height, pixels]  [Scene option]
Default value: 500
The height, in pixels, of the graphics window. **pixels** must be a positive integer number.

**restart** [restart, value]  
Default value: false  
A true value means that animations will restart automatically when the end of the list is reached. Writing just “restart” is equivalent to [restart, true].

**tstep** [tstep, time]  
Default value: 10  
The amount of time, in mili-seconds, between iterations among consecutive animation frames. **time** must be a real number.

**width** [width, pixels]  
Default value: 500  
The width, in pixels, of the graphics window. **pixels** must be a positive integer number.

**windowname** [windowtitle, name]  
Default value: .scene  
**name** must be a string that can be used as the name of the Tk window created by Xmaxima for the **scene** graphics. The default value .scene implies that a new top level window will be created.

**windowtitle** [windowtitle, name]  
Default value: Xmaxima: scene  
**name** must be a string that will be written in the title of the window created by scene.

### 55.3.2 Scene objects

**cone** [cone, options]  
Creates a regular pyramid with height equal to 1 and a hexagonal base with vertices 0.5 units away from the axis. Options [object_height], page 947 and [object_radius], page 948 can be used to change those defaults and option [object_resolution], page 948 can be used to change the number of edges of the base; higher values will make it look like a cone. By default, the axis will be along the x axis, the middle point of the axis will be at the origin and the vertex on the positive side of the x axis; use options [object_orientation], page 947 and [object_center], page 946 to change those defaults.  
**Example.** This shows a pyramid that starts rotating around the z axis when the play button is pressed.  

\[
(\%i1) \text{scene([[cone, [orientation,0,30,0], [tstep,100],}
\text{ [animate,orientation,makelist([0,30,i],i,5,360,5)]]], restart)}
\]

**cube** [cube, options]  
A cube with edges of 1 unit and faces parallel to the xy, xz and yz planes. The lengths of the three edges can be changed with options [object_xlength], page 949,
[object_ylength], page 949 and [object_zlength], page 949, turning it into a rectangular box and the faces can be rotated with option [object_orientation], page 947.

cylinder [cylinder, options] [Scene object]
Creates a regular prism with height equal to 1 and a hexagonal base with vertices 0.5 units away from the axis. Options [object_height], page 947 and [object_radius], page 948 can be used to change those defaults and option [object_resolution], page 948 can be used to change the number of edges of the base; higher values will make it look like a cylinder. The default height can be changed with the option [object_height], page 947. By default, the axis will be along the x axis and the middle point of the axis will be at the origin; use options [object_orientation], page 947 and [object_center], page 946 to change those defaults.

sphere [sphere, options] [Scene object]
A sphere with default radius of 0.5 units and center at the origin.

55.3.3 Scene object’s options

animation [animation, property, positions] [Object option]
property should be one of the following 4 object’s properties: [object_origin], page 947, [object_scale], page 948, [object_position], page 948 or [object_orientation], page 947 and positions should be a list of points. When the play button is pressed, the object property will be changed sequentially through all the values in the list, at intervals of time given by the option [scene_tstep], page 945. The rewind button can be used to point at the start of the sequence making the animation restart after the play button is pressed again. See also [object_track], page 949.

capping [capping, number] [Object option]
Default value: 1
In a cone or a cylinder, it defines whether the base (or bases) will be shown. A value of 1 for number makes the base visible and a value of 0 makes it invisible.

center [center, point] [Object option]
Default value: [0, 0, 0]
The coordinates of the object’s geometric center, with respect to its [object_position], page 948. point can be a list with 3 real numbers, or 3 real numbers separated by commas. In a cylinder, cone or cube it will be at half its height and in a sphere at its center.

color [color, colormename] [Object option]
Default value: white
The color of the object. It accepts color names or hexadecimal red-green-blue strings (see the color option of plot2d).
**endphi** [endphi, angle]  
Default value: 180  
In a sphere phi is the angle on the vertical plane that passes through the z axis, measured from the positive part of the z axis. angle must be a number between 0 and 180 that sets the final value of phi at which the surface will end. A value smaller than 180 will eliminate a part of the sphere’s surface.  
See also [object_startphi], page 948 and [object_phiresolution], page 947.

**endtheta** [endtheta, angle]  
Default value: 360  
In a sphere theta is the angle on the horizontal plane (longitude), measured from the positive part of the x axis. angle must be a number between 0 and 360 that sets the final value of theta at which the surface will end. A value smaller than 360 will eliminate a part of the sphere’s surface.  
See also [object_starttheta], page 948 and [object_thetaresolution], page 949.

**height** [height, value]  
Default value: 1  
value must be a positive number which sets the height of a cone or a cylinder.

**linewidth** [linewidth, value]  
Default value: 1  
The width of the lines, when option [object_wireframe], page 950 is used. value must be a positive number.

**opacity** [opacity, value]  
Default value: 1  
value must be a number between 0 and 1. The lower the number, the more transparent the object will become. The default value of 1 means a completely opaque object.

**orientation** [orientation, angles]  
Default value: [0, 0, 0]  
Three angles by which the object will be rotated with respect to the three axis. angles can be a list with 3 real numbers, or 3 real numbers separated by commas. Example: [0, 0, 90] rotates the x axis of the object to the y axis of the reference frame.

**origin** [origin, point]  
Default value: [0, 0, 0]  
The coordinates of the object’s origin, with respect to which its other dimensions are defined. point can be a list with 3 real numbers, or 3 real numbers separated by commas.

**phiresolution** [phiresolution, num]  
Default value:  
The number of sub-intervals into which the phi angle interval from [object_startphi], page 948 to [object_endphi], page 947 will be divided. num must be a positive integer.  
See also [object_startphi], page 948 and [object_endphi], page 947.
points [points]  [Object option]
  Only the vertices of the triangulation used to render the surface will be shown. Example: [sphere, [points]]
  See also [object_surface], page 949 and [object_wireframe], page 950.

pointsize [pointsize, value]  [Object option]
  Default value: 1
  The size of the points, when option [object_points], page 948 is used. value must be a positive number.

position [position, point]  [Object option]
  Default value: [0, 0, 0]
  The coordinates of the object’s position. point can be a list with 3 real numbers, or 3 real numbers separated by commas.

radius [radius, value]  [Object option]
  Default value: 0.5
  The radius or a sphere or the distance from the axis to the base’s vertices in a cylinder or a cone. value must be a positive number.

resolution [resolution, number]  [Object option]
  Default value: 6
  number must be an integer greater than 2 that sets the number of edges in the base of a cone or a cylinder.

scale [scale, factors]  [Object option]
  Default value: [1, 1, 1]
  Three numbers by which the object will be scaled with respect to the three axis. factors can be a list with 3 real numbers, or 3 real numbers separated by commas. Example: [2, 0.5, 1] enlarges the object to twice its size in the x direction, reduces the dimensions in the y direction to half and leaves the z dimensions unchanged.

startphi [startphi, angle]  [Object option]
  Default value: 0
  In a sphere phi is the angle on the vertical plane that passes through the z axis, measured from the positive part of the z axis. angle must be a number between 0 and 180 that sets the initial value of phi at which the surface will start. A value bigger than 0 will eliminate a part of the sphere’s surface.
  See also [object_endphi], page 947 and [object_phiresolution], page 947.

starttheta [starttheta, angle]  [Object option]
  Default value: 0
  In a sphere theta is the angle on the horizontal plane (longitude), measured from the positive part of the x axis. angle must be a number between 0 and 360 that sets the initial value of theta at which the surface will start. A value bigger than 0 will eliminate a part of the sphere’s surface.
  See also [object_endtheta], page 947 and [object_thetaresolution], page 949.
The surfaces of the object will be rendered and the lines and points of the triangulation used to build the surface will not be shown. This is the default behavior, which can be changed using either the option [object_points], page 948 or [object_wireframe], page 950.

**thetaresolution** [thetaresolution, num]

Default value:

The number of sub-intervals into which the theta angle interval from [object_starttheta], page 948 to [object_endtheta], page 947 will be divided. num must be a positive integer.

See also [object_starttheta], page 948 and [object_endtheta], page 947.

**track** [track, positions]

positions should be a list of points. When the play button is pressed, the object position will be changed sequentially through all the points in the list, at intervals of time given by the option [scene_tstep], page 945, leaving behind a track of the object’s trajectory. The rewind button can be used to point at the start of the sequence making the animation restart after the play button is pressed again.

**Example.** This will show the trajectory of a ball thrown with speed of 5 m/s, at an angle of 45 degrees, when the air resistance can be neglected:

```lisp
(%i1) p: makelist ([0,4*t,4*t- 9.8*t^2/2], t, 0, 0.82, 0.01)$
(%i2) ball: [sphere, [radius,0.1], [color,red], [track,p]]$
(%i3) ground: [cube, [xlength,2], [ylength,4], [zlength,0.2],
    [position,0,1.5,-0.2],[color,green]]$
(%i4) scene (ball, ground)$
```

See also [object_animation], page 946.

**xlength** [xlength, length]

Default value: 1

The height of a cube in the x direction. length must be a positive number. See also [object_ylength], page 949 and [object_zlength], page 949.

**ylength** [ylength, length]

Default value: 1

The height of a cube in the y direction. length must be a positive number. See also [object_xlength], page 949 and [object_zlength], page 949.

**zlength** [zlength, length]

Default value: 1

The height of a cube in z the direction. length must be a positive number. See also [object_xlength], page 949 and [object_ylength], page 949.
wireframe [wireframe] [Object option]
Only the edges of the triangulation used to render the surface will be shown. Example:
[cube, [wireframe]]
See also [object_surface], page 949 and [object_points], page 948.
56 engineering-format

Engineering-format changes the way maxima outputs floating-point numbers to the notation engineers are used to: $a \times 10^b$ with $b$ dividable by three.

56.1 Functions and Variables for engineering-format

**engineering_format_floats**  
Default value: true  
This variable allows to temporarily switch off engineering-format.

```lisp
(%i1) load("engineering-format");
(%o1) /home/gunter/src/maxima-code/share/contrib/engineering-format.lisp
(%i2) float(sin(10)/10000);
(%o2) - 54.40211108893698e-6
(%i3) engineering_format_floats:false$
(%i4) float(sin(10)/10000);
(%o4) - 5.440211108893698e-5
```

See also `fpprintprec` and `float`.

**engineering_format_min**  
Default value: 0.0  
The minimum absolute value that isn’t automatically converted to the engineering format. See also `engineing_format_max` and `engineering_format_floats`.

```lisp
(%i1) lst: float([.05,.5,5,500,5000,500000]);
(%o1) [0.05, 0.5, 5.0, 500.0, 5000.0, 500000.0]
(%i2) load("engineering-format");
(%o2) /home/gunter/src/maxima-code/share/contrib/engineering-format.lisp
(%i3) lst;
(%o3) [50.0e-3, 500.0e-3, 5.0e+0, 500.0e+0, 5.0e+3, 500.0e+3]
(%i4) engineering_format_min:.1$
(%i5) engineering_format_max:1000$
(%i6) lst;
(%o6) [50.0e-3, 0.5, 5.0, 500.0, 5.0e+3, 500.0e+3]
```

**engineering_format_max**  
Default value: 0.0  
The maximum absolute value that isn’t automatically converted to the engineering format. See also `engineing_format_min` and `engineering_format_floats`. 
57 ezunits

57.1 Introduction to ezunits

ezunits is a package for working with dimensional quantities, including some functions for dimensional analysis. ezunits can carry out arithmetic operations on dimensional quantities and unit conversions. The built-in units include Systeme Internationale (SI) and US customary units, and other units can be declared. See also physical_constants, a collection of physical constants.

load("ezunits") loads this package. demo("ezunits") displays several examples. The convenience function known_units returns a list of the built-in and user-declared units, while display_known_unit_conversions displays the set of known conversions in an easy-to-read format.

An expression $a'b$ represents a dimensional quantity, with $a$ indicating a nondimensional quantity and $b$ indicating the dimensional units. A symbol can be used as a unit without declaring it as such; unit symbols need not have any special properties. The quantity and unit of an expression $a'b$ can be extracted by the qty and units functions, respectively.

A symbol may be declared to be a dimensional quantity, with specified quantity or specified units or both.

An expression $a'b''c$ converts from unit $b$ to unit $c$. ezunits has built-in conversions for SI base units, SI derived units, and some non-SI units. Unit conversions not already known to ezunits can be declared. The unit conversions known to ezunits are specified by the global variable known_unit_conversions, which comprises built-in and user-defined conversions. Conversions for products, quotients, and powers of units are derived from the set of known unit conversions.

As Maxima generally prefers exact numbers (integers or rationals) to inexact (float or bigfloat), so ezunits preserves exact numbers when they appear in dimensional quantities. All built-in unit conversions are expressed in terms of exact numbers; inexact numbers in declared conversions are coerced to exact.

There is no preferred system for display of units; input units are not converted to other units unless conversion is explicitly indicated. ezunits recognizes the prefixes m-, k-, M, and G- (for milli-, kilo-, mega-, and giga-) as applied to SI base units and SI derived units, but such prefixes are applied only when indicated by an explicit conversion.

Arithmetic operations on dimensional quantities are carried out by conventional rules for such operations.

- $(x'a) * (y'b)$ is equal to $(x * y)(a * b)$.
- $(x'a) + (y'a)$ is equal to $(x + y)a$.
- $(x'a)^y$ is equal to $x^{y}a^{y}$ when $y$ is nondimensional.

ezunits does not require that units in a sum have the same dimensions; such terms are not added together, and no error is reported.

ezunits includes functions for elementary dimensional analysis, namely the fundamental dimensions and fundamental units of a dimensional quantity, and computation of dimensionless quantities and natural units. The functions for dimensional analysis were adapted from similar functions in another package, written by Barton Willis.
For the purpose of dimensional analysis, a list of fundamental dimensions and an associated list of fundamental units are maintained; by default the fundamental dimensions are length, mass, time, charge, temperature, and quantity, and the fundamental units are the associated SI units, but other fundamental dimensions and units can be declared.

57.2 Introduction to physical_constants

physical_constants is a collection of physical constants, copied from CODATA 2006 recommended values (https://physics.nist.gov/cuu/Constants/). load ("physical_constants") loads this package, and loads ezunits also, if it is not already loaded.

A physical constant is represented as a symbol which has a property which is the constant value. The constant value is a dimensional quantity, as represented by ezunits. The function constvalue fetches the constant value; the constant value is not the ordinary value of the symbol, so symbols of physical constants persist in evaluated expressions until their values are fetched by constvalue.

physical_constants includes some auxiliary information, namely, a description string for each constant, an estimate of the error of its numerical value, and a property for TeX display. To identify physical constants, each symbol has the physical_constant property; propvars(physical_constant) therefore shows the list of all such symbols.

physical_constants comprises the following constants.

\%c    speed of light in vacuum
\%mu_0 magnetic constant
\%e_0  electric constant
\%Z_0  characteristic impedance of vacuum
\%G    Newtonian constant of gravitation
\%h    Planck constant
\%h_bar Planck constant
\%m_P  Planck mass
\%T_P  Planck temperature
\%l_P  Planck length
\%t_P  Planck time
\%e    elementary charge
\%Phi_0 magnetic flux quantum
\%G_0  conductance quantum
\%K_J  Josephson constant
\%R_K  von Klitzing constant
\%mu_B Bohr magneton
\%mu_N nuclear magneton
%alpha fine-structure constant
%R_inf Rydberg constant
%a_0 Bohr radius
%E_h Hartree energy
%ratio_h_me quantum of circulation
%m_e electron mass
%N_A Avogadro constant
%m_u atomic mass constant
%F Faraday constant
%R molar gas constant
%%k Boltzmann constant
%V_m molar volume of ideal gas
%n_0 Loschmidt constant
%ratio_S0_R Sackur-Tetrode constant (absolute entropy constant)
%sigma Stefan-Boltzmann constant
%c_1 first radiation constant
%c_1L first radiation constant for spectral radiance
%c_2 second radiation constant
%b Wien displacement law constant
%b_prime Wien displacement law constant


Examples:

The list of all symbols which have the physical_constant property.

(%i1) load ("physical_constants")$
(%i2) propvars (physical_constant);
(%o2) [%c, %mu_0, %e_0, %Z_0, %G, %h, %h_bar, %m_P, %T_P, %l_P,
%t_P, %%e, %phi_0, %G_0, %K_J, %R_K, %mu_B, %mu_N, %alpha,
%R_inf, %a_0, %E_h, %ratio_h_me, %m_e, %N_A, %m_u, %F, %R, %%k,
%V_m, %n_0, %ratio_S0_R, %sigma, %c_1, %c_1L, %c_2, %b, %b_prime]

Properties of the physical constant %c.

(%i1) load ("physical_constants")$
(%i2) constantp (%c);
(%o2) true
(%i3) get (%c, description);
(%o3) speed of light in vacuum
The energy equivalent of 1 pound-mass. The symbol %c persists until its value is fetched by `constvalue`.

```maxima
(%i1) load ("physical_constants")$
(%i2) m * %c^2;
     2
(%o2) %c m
(%i3) %, m = 1 ' lbm;
     2
(%o3) %c ' lbm
(%i4) constvalue (%);
     2
lbf m
(%o4) 89875517873681764 ' ------
      2
s
(%i5) E : % '' J;
Computing conversions to base units; may take a moment.
     366838848464007200
(%o5) ------------------ ' J
9
(%i6) E '' GJ;
     458548560580009
(%o6) ------------------ ' GJ
11250000
(%i7) float (%);
(%o7) 4.0759872051556356e+7 ' GJ
```

### 57.3 Functions and Variables for `ezunits`

[Operator]

The dimensional quantity operator. An expression `a\textsuperscript{b}` represents a dimensional quantity, with `a` indicating a nondimensional quantity and `b` indicating the dimensional units. A symbol can be used as a unit without declaring it as such; unit symbols need not have any special properties. The quantity and unit of an expression `a\textsuperscript{b}` can be extracted by the `qty` and `units` functions, respectively.

Arithmetic operations on dimensional quantities are carried out by conventional rules for such operations.
• \((x'^a) \ast (y'^b)\) is equal to \((x \ast y)'(a \ast b)\).

• \((x'^a) + (y'^a)\) is equal to \((x + y)'a\).

• \((x'^a)'y\) is equal to \(x'^y'a^y\) when \(y\) is nondimensional.

ezunits does not require that units in a sum have the same dimensions; such terms are not added together, and no error is reported.

load ("ezunits") enables this operator.

Examples:

SI (Systeme Internationale) units.

```
(%i1) load ("ezunits")$
(%i2) foo : 10 ' m;
    10 ' m
(%i3) qty (foo);
    10
(%i4) units (foo);
    m
(%i5) dimensions (foo);
length
```

"Customary" units.

```
(%i1) load ("ezunits")$
(%i2) bar : x ' acre;
    x ' acre
(%i3) dimensions (bar);
2
(%i3) length
(%i4) fundamental_units (bar);
2
(%i4) m
```

Units ad hoc.

```
(%i1) load ("ezunits")$
(%i2) baz : 3 ' sheep + 8 ' goat + 1 ' horse;
(%i2) 8 ' goat + 3 ' sheep + 1 ' horse
(%i3) subst ([sheep = 3*goat, horse = 10*goat], baz);
27 ' goat
(%i4) baz2 : 1000'gallon/fortnight;
    gallon
(%i4) 1000 ' ---------
    fortnight
(%i5) subst (fortnight = 14*day, baz2);
500 gallon
(%i5) --- ' ------
    7 day
```

Arithmetic operations on dimensional quantities.

```
(%i1) load ("ezunits")$
(%i2) 100 ' kg + 200 ' kg;
```
(%o2) \[ 300 \ ' \text{ kg} \]
(%i3) \[ 100 \ ' \text{ m}^3 - 100 \ ' \text{ m}^3 \]
(%o3) \[ 0 \ ' \text{ m} \]
(%i4) \[ (10 \ ' \text{ kg}) \times (17 \ ' \text{ m/s}^{-2}) \]
(%o4) \[ 170 \ ' \frac{\text{kg m}}{2} \]
(%i5) \[ \frac{x \ ' \text{ m}}{y \ ' \text{ s}} \]
(%o5) \[ \frac{x \ ' \text{ m}}{y \ ' \text{ s}} \]
(%i6) \[ (a \ ' \text{ m})^2 \]
(%o6) \[ a^2 \ ' \text{ m} \]

The unit conversion operator. An expression \( a^b^c \) converts from unit \( b \) to unit \( c \). 
\texttt{ezunits} has built-in conversions for SI base units, SI derived units, and some non-SI units. Unit conversions not already known to \texttt{ezunits} can be declared. The unit conversions known to \texttt{ezunits} are specified by the global variable \texttt{known_unit_conversions}, which comprises built-in and user-defined conversions. Conversions for products, quotients, and powers of units are derived from the set of known unit conversions.

There is no preferred system for display of units; input units are not converted to other units unless conversion is explicitly indicated. \texttt{ezunits} does not attempt to simplify units by prefixes (milli-, centi-, deci-, etc) unless such conversion is explicitly indicated.

\texttt{load ("ezunits")} enables this operator.

Examples:

The set of known unit conversions.

\texttt{(%i1)} \quad \texttt{load ("ezunits")}$
\texttt{(%i2)} \quad \texttt{display2d : false}$
\texttt{(%i3)} \quad \texttt{known_unit_conversions;}
\texttt{(%o3)} \{ \text{acre} = 4840\text{yard}^2, \text{Btu} = 1055\text{J}, \text{cfm} = \text{feet}^3/\text{minute}, \text{cm} = \text{m}/100, \text{day} = 86400\text{s}, \text{feet} = 381\text{m}/1250, \text{ft} = \text{feet}, \text{g} = \text{kg}/1000, \text{gallon} = 757\text{liter}/200, \text{GHz} = 1000000000\text{Hz}, \text{GOhm} = 1000000000\text{Ohm}, \text{GPa} = 1000000000\text{Pa}, \text{GWb} = 1000000000\text{Wb}, \text{Gg} = 1000000\text{kg}, \text{Gm} = 1000000000\text{m}, \text{Gmol} = 1000000\text{mol}, \text{Gs} = 1000000000\text{s}, \text{ha} = \text{hectare}, \text{hectare} = 100\text{m}^2, \text{hour} = 3600\text{s}, \text{Hz} = 1/\text{s}, \text{inch} = \text{feet}/12, \text{km} = 1000\text{m}, \text{kmol} = 1000\text{mol}, \text{ks} = 1000\text{s}, \text{l} = \text{liter}, \text{lbf} = \text{pound force}, \text{lbm} = \text{pound mass}, \text{liter} = \text{m}^3/1000, \text{metric ton} = \text{Mg}, \text{mg} = \text{kg}/1000000, \text{MHz} = 1000000\text{Hz}, \text{microgram} = \text{kg}/1000000000, \text{micrometer} = \text{m}/1000000, \ldots \}
micron = micrometer, microsecond = s/1000000,
mile = 5280*feet, minute = 60*s, mm = m/1000,
mmol = mol/1000, month = 2629800*s, MOhm = 1000000*Ohm,
MPa = 1000000*Pa, ms = s/1000, MWb = 1000000*Wb,
Mg = 1000*kg, Mm = 1000000*m, Mmol = 1000000000*mol,
Ms = 1000000*s, ns = s/1000000000, ounce = pound_mass/16,
oz = ounce, Ohm = s/J/C^2,
pound_force = 32*ft*pound_mass/s^2,
pound_mass = 200*kg/441, psi = pound_force/inch^2,
Pa = N/m^2, week = 604800*s, Wb = J/A, yard = 3*feet,
year = 31557600*s, C = s*A, F = C^2/J, GA = 1000000000*A,
GC = 1000000000*C, GF = 1000000000*F, GH = 1000000000*H,
GJ = 1000000000*J, GK = 1000000000*K, GN = 1000000000*N,
GS = 1000000000*S, GT = 1000000000*T, GV = 1000000000*V,
GW = 1000000000*W, Hz = J/A, J = m*N, kA = 1000*A,
kC = 1000*C, kF = 1000*F, kH = 1000*H, kHz = 1000*Hz,
kJ = 1000*J, kK = 1000*K, kN = 1000*N, kOhm = 1000*Ohm,
kPa = 1000*Pa, kS = 1000*S, kT = 1000*T, kV = 1000*V,
kW = 1000*W, kwb = 1000*Wb, mA = A/1000, mC = C/1000,
mF = F/1000, mHz = Hz/1000, mJ = J/1000,
mK = K/1000, mN = N/1000, mOhm = Ohm/1000, mPa = Pa/1000,
mS = S/1000, mT = T/1000, mV = V/1000, mW = W/1000,
mWb = Wb/100, MA = 1000000*A, MC = 1000000*C,
MF = 1000000*F, MH = 1000000*H, MJ = 1000000*J,
MK = 1000000*K, MN = 1000000*N, MS = 1000000*S,
MT = 1000000*T, MV = 1000000*V, MW = 1000000*W,
N = kg*m/s^2, R = 5*K/9, S = 1/Ohm, T = J/(m^2*A), V = J/C,
W = J/s}

Elementary unit conversions.

(%i1) load ("ezunits")$
(%i2) 1 ' ft ' ' m;
Computing conversions to base units; may take a moment.
  381
381
(%o2) ---- ' m
  1250

(%i3) %, numer;
(%o3) 0.3048 ' m

(%i4) 1 ' kg ' ' lbm;
(%o4) 441

(%i5) %, numer;
(%o5) 2.205 ' lbm

(%i6) 1 ' W ' ' Btu/hour;
720 Btu
(%o6) --- ' ---
Coercing quantities in feet and meters to one or the other.

(%i1) load ("ezunits")$
(%i2) 100 ' m + 100 ' ft;
(%o2) 100 ' m + 100 ' ft
(%i3) (100 ' m + 100 ' ft) '' ft;
Computing conversions to base units; may take a moment.

163100
(%o3) 428.0839895013123 ' ft
381
(%i4) %, numer;
(%o4) 1233.48183754752 ' ft

Dimensional analysis to find fundamental dimensions and fundamental units.

(%i1) load ("ezunits")$
(%i2) foo : 1 ' acre * ft;
(%o2) 1 ' acre ft
(%i3) dimensions (foo);
3
(%o3) length
(%i4) fundamental_units (foo);
3
(%o4) m
(%i5) foo '' m^3;
Computing conversions to base units; may take a moment.

60228605349
3
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(?o5) ------------ ' m
48828125

(?i6) %, numer;
3

(?o6) 1233.48183754752 ' m

Declared unit conversions.

(?i1) load ("ezunits")$

(?i2) declare_unit_conversion (MMBtu = 10^6*Btu, kW = 1000*W);

(?o2) done

(?i3) declare_unit_conversion (kWh = kW*hour, MWh = 1000*kWh,
   bell = 1800*s);

(?o3) done

(?i4) 1 ' kW*s ' MWh;

Computing conversions to base units; may take a moment.

1

(?o4) ------- ' MWh
3600000

(?i5) 1 ' kW/m^2 ' MMBtu/bell/ft^2;

1306449 MMBtu

(?o5) ------------ ' -----------
8242187500 2

bell ft

constvalue (x) [Function]

Shows the value and the units of one of the constants declared by package physical_ constants, which includes a list of physical constants, or of a new constant declared in package ezunits (see declare_constvalue).

Note that constant values as recognized by constvalue are separate from values declared by numerval and recognized by constantp.

Example:

(?i1) load ("physical_constants")$

(?i2) constvalue (%G);

3

m

(?o2) 6.67428 ' ----- 2

kg s

(?i3) get ('%G, 'description);

(?o3) Newtonian constant of gravitation

declare_constvalue (a, x) [Function]

Declares the value of a constant to be used in package ezunits. This function should be loaded with load ("ezunits").

Example:

(?i1) load ("ezunits")$

(?i2) declare_constvalue (FOO, 100 ' lbm / acre);
\[
(\%o2) \quad 100 \, ' \text{ lbm} \\
(\%o3) \quad 50 \, FOO \, ' \text{ acre} \\
(\%o4) \quad 5000 \, ' \text{ lbm}
\]

remove_constvalue (a)  \quad \text{[Function]}
Reverts the effect of \texttt{declare_constvalue}. This function should be loaded with \texttt{load} ("ezunits").

units (x)  \quad \text{[Function]}
Returns the units of a dimensional quantity x, or returns 1 if x is nondimensional.

x may be a literal dimensional expression \( a'b \), a symbol with declared units via \texttt{declare_units}, or an expression containing either or both of those.

This function should be loaded with \texttt{load} ("ezunits").

Example:
\[
(\%i1) \quad \text{load} ("ezunits")$
(\%i2) \quad \text{foo} : 100 \, ' \text{ kg} \\
(\%i3) \quad \text{bar} : x \, ' \text{ m/s} \\
(\%i4) \quad \text{units} (\text{foo}); \\
(\%i5) \quad \text{units} (\text{bar}); \\
(\%i6) \quad \text{units} (\text{foo} \times \text{bar}); \\
(\%i7) \quad \text{units} (\text{foo} / \text{bar}); \\
(\%i8) \quad \text{units} (\text{foo}^2);
\]

\[
(\%o2) \quad 100 \, ' \text{ --} \\
(\%o3) \quad 50 \, FOO \, ' \text{ acre} \\
(\%o4) \quad 5000 \, ' \text{ lbm} \\
(\%o5) \quad x \, ' \text{ --} \\
(\%o6) \quad \text{kg m} \\
(\%o7) \quad \text{kg s} \\
(\%o8) \quad 2 \text{ kg}
\]

\texttt{declare_units (a, u)}  \quad \text{[Function]}
 Declares that \texttt{units} should return units \( u \) for \( a \), where \( u \) is an expression. This function should be loaded with \texttt{load} ("ezunits").
Example:

(\%i1) load ("ezunits")$
(\%i2) units (aa);
(\%o2) 1
(\%i3) declare_units (aa, J);
(\%o3) J
(\%i4) units (aa);
(\%o4) J
(\%i5) units (aa^2);
(\%o5) 2
(\%i6) foo : 100 ' kg;
(\%o6) 100 ' kg
(\%i7) units (aa * foo);
(\%o7) kg J

qty (x) \hspace{1cm} \text{[Function]}
Returns the nondimensional part of a dimensional quantity \( x \), or returns \( x \) if \( x \) is nondimensional. \( x \) may be a literal dimensional expression \( a^b \), a symbol with declared quantity, or an expression containing either or both of those.
This function should be loaded with \texttt{load ("ezunits")}.

Example:

(\%i1) load ("ezunits")$
(\%i2) foo : 100 ' kg;
(\%o2) 100 ' kg
(\%i3) qty (foo);
(\%o3) 100
(\%i4) bar : v ' m/s;
(\%o4) m
(\%i5) qty (foo * bar);
(\%o5) kg m
(\%i6) qty (foo * bar);
(\%o6) 100 v

declare_qty (a, x) \hspace{1cm} \text{[Function]}
Declares that \texttt{qty} should return \( x \) for symbol \( a \), where \( x \) is a nondimensional quantity.
This function should be loaded with \texttt{load ("ezunits")}.

Example:

(\%i1) load ("ezunits")$
(\%i2) declare_qty (aa, xx);
(\%o2) xx
(\%i3) qty (aa);
unitp (x)

Returns true if x is a literal dimensional expression, a symbol declared dimensional, or an expression in which the main operator is declared dimensional. unitp returns false otherwise.

load ("ezunits") loads this function.

Examples:

unitp applied to a literal dimensional expression.

(%i1) load ("ezunits")$
(%i2) unitp (100 ' kg);
    (%o2) true

unitp applied to a symbol declared dimensional.

(%i1) load ("ezunits")$
(%i2) unitp (foo);
    (%o2) false
(%i3) declare (foo, dimensional);
    (%o3) done
(%i4) unitp (foo);
    (%o4) true

unitp applied to an expression in which the main operator is declared dimensional.

(%i1) load ("ezunits")$
(%i2) unitp (bar (x, y, z));
    (%o2) false
(%i3) declare (bar, dimensional);
    (%o3) done
(%i4) unitp (bar (x, y, z));
    (%o4) true

declare_unit_conversion (u = v, ...)

Append equations u = v, ... to the list of unit conversions known to the unit conversion operator "= . u and v are both multiplicative terms, in which any variables are units, or both literal dimensional expressions.

At present, it is necessary to express conversions such that the left-hand side of each equation is a simple unit (not a multiplicative expression) or a literal dimensional expression with the quantity equal to 1 and the unit being a simple unit. This limitation might be relaxed in future versions.

known_unit_conversions is the list of known unit conversions.
This function should be loaded with `load ("ezunits")`.

Examples:

Unit conversions expressed by equations of multiplicative terms.

```lisp
%i1) load ("ezunits")$
%i2) declare_unit_conversion (nautical_mile = 1852 * m,
   fortnight = 14 * day);
   done
%i3) 100 ' nautical_mile / fortnight '' m/s;
Computing conversions to base units; may take a moment.
   463 m
   ---- ' -
   3024 s
```

Unit conversions expressed by equations of literal dimensional expressions.

```lisp
%i1) load ("ezunits")$
%i2) declare_unit_conversion (1 ' fluid_ounce = 2 ' tablespoon);
   done
%i3) declare_unit_conversion (1 ' tablespoon = 3 ' teaspoon);
   done
%i4) 15 ' fluid_ounce '' teaspoon;
Computing conversions to base units; may take a moment.
   90 ' teaspoon
```

`declare_dimensions (a_1, d_1, ..., a_n, d_n)` [Function]

Declares `a_1`, ..., `a_n` to have dimensions `d_1`, ..., `d_n`, respectively.

Each `a_k` is a symbol or a list of symbols. If it is a list, then every symbol in `a_k` is declared to have dimension `d_k`.

`load ("ezunits")` loads these functions.

Examples:

```lisp
%i1) load ("ezunits")$
%i2) declare_dimensions ([x, y, z], length, [t, u], time);
   done
%i3) dimensions (y^2/u);
   2
   length
   ------
   time
%i4) fundamental_units (y^2/u);
   0 errors, 0 warnings
   2
   m
   --
   s
```

`remove_dimensions (a_1, ..., a_n)` [Function]

Reverts the effect of `declare_dimensions`. This function should be loaded with `load ("ezunits")`.
declare_fundamental_dimensions (d_1, d_2, d_3, ...)  
remove_fundamental_dimensions (d_1, d_2, d_3, ...)  

fundamental_dimensions

declare_fundamental_dimensions declares fundamental dimensions. Symbols d_1, d_2, d_3, ... are appended to the list of fundamental dimensions, if they are not already on the list.

remove_fundamental_dimensions reverts the effect of declare_fundamental_dimensions.

fundamental_dimensions is the list of fundamental dimensions. By default, the list comprises several physical dimensions.

load ("ezunits") loads these functions.

Examples:

(%i1) load ("ezunits") $ 
(%i2) fundamental_dimensions; 
(%o2) [length, mass, time, current, temperature, quantity] 
(%i3) declare_fundamental_dimensions (money, cattle, happiness); 
(%o3) done 
(%i4) fundamental_dimensions; 
(%o4) [length, mass, time, current, temperature, quantity, money, cattle, happiness] 
(%i5) remove_fundamental_dimensions (cattle, happiness); 
(%o5) done 
(%i6) fundamental_dimensions; 
(%o6) [length, mass, time, current, temperature, quantity, money] 

declare_fundamental_units (u_1, d_1, ..., u_n, d_n)  
remove_fundamental_units (u_1, ..., u_n) 

declare_fundamental_units declares u_1, ..., u_n to have dimensions d_1, ..., d_n, respectively. All arguments must be symbols.

After calling declare_fundamental_units, dimensions(u_k) returns d_k for each argument u_1, ..., u_n, and fundamental_units(d_k) returns u_k for each argument d_1, ..., d_n.

remove_fundamental_units reverts the effect of declare_fundamental_units.

load ("ezunits") loads these functions.

Examples:

(%i1) load ("ezunits") $ 
(%i2) declare_fundamental_dimensions (money, cattle, happiness); 
(%o2) done 
(%i3) declare_fundamental_units (dollar, money, goat, cattle, smile, happiness); 
(%o3) [dollar, goat, smile] 
(%i4) dimensions (100 ' dollar/goat/km^2); 
   money 
(%o4) -------------- 2
cattle length
(%i5) dimensions (x ' smile/kg);
    happiness
(%%o5)                               -------
    mass
(%i6) fundamental_units (money*cattle/happiness);
0 errors, 0 warnings
    dollar goat
(%%o6)                               -------
    smile

dimensions (x)                      [Function]
dimensions_as_list (x)              [Function]
dimensions returns the dimensions of the dimensional quantity x as an expression comprising products and powers of base dimensions.
dimensions_as_list returns the dimensions of the dimensional quantity x as a list, in which each element is an integer which indicates the power of the corresponding base dimension in the dimensions of x.
load ("ezunits") loads these functions.
Examples:
(%%i1) load ("ezunits")$
(%%i2) dimensions (1000 ' kg*m^2/s^3);
        2
    length mass
(%%o2)                               -------
        3
    time
(%%i3) declare_units (foo, acre*ft/hour);
    acre ft
(%%o3)                               -------
    hour
(%%i4) dimensions (foo);
            3
    length
(%%o4)                               -------
    time
(%%i1) load ("ezunits")$
(%%i2) fundamental_dimensions;
(%%o2) [length, mass, time, charge, temperature, quantity]
(%%i3) dimensions_as_list (1000 ' kg*m^2/s^3);
(%%o3) [2, 1, - 3, 0, 0, 0]
(%%i4) declare_units (foo, acre*ft/hour);
    acre ft
(%%o4)                               -------
    hour
(%%i5) dimensions_as_list (foo);
fundamental_units

function

fundamental_units(x)

fundamental_units() returns the units associated with the fundamental dimensions of x, as determined by dimensions(x).

x may be a literal dimensional expression a'b, a symbol with declared units via declare_units, or an expression containing either or both of those.

fundamental_units() returns the list of all known fundamental units, as declared by declare_fundamental_units.

load ("ezunits") loads this function.

Examples:

(%i1) load ("ezunits")$
(%i2) fundamental_units ()
(%o2) [m, kg, s, A, K, mol]
(%i3) fundamental_units (100 ' mile/hour);
   m
   -
   s
(%i4) declare_units (aa, g/foot^2);
   g
   --
   2
foot
(%i5) fundamental_units (aa);
   kg
   --
   2
m
dimensionless (L)

function

Returns a basis for the dimensionless quantities which can be formed from a list L of dimensional quantities.

load ("ezunits") loads this function.

Examples:

(%i1) load ("ezunits")$
(%i2) dimensionless ([x ' m, y ' m/s, z ' s]);
0 errors, 0 warnings
0 errors, 0 warnings
   y z
   [---]
x

Dimensionless quantities derived from fundamental physical quantities. Note that the first element on the list is proportional to the fine-structure constant.

(%i1) load ("ezunits")$
natural_unit (expr, [v_1, ..., v_n])  

Finds exponents e_1, ..., e_n such that \( \text{dimension}(expr) = \text{dimension}(v_1^{e_1} \cdot \ldots \cdot v_n^{e_n}) \).

load ("ezunits") loads this function.

Examples:
58 f90

58.1 Package f90

f90_output_line_length_max

Default value: 65

f90_output_line_length_max is the maximum number of characters of Fortran code which are output by f90 per line. Longer lines of code are divided, and printed with an ampersand & at the end of an output line and an ampersand at the beginning of the following line.

f90_output_line_length_max must be a positive integer.

Example:

```
(%i1) load ("f90")$
(%i2) foo : expand ((xxx + yyy + 7)^4);
   4  3  3  2  2
   2
   2
   2
   4
   3
   2
   4
   3
   2
   4
   3
   2
   4
   3
   2
   4

(%o2) yyy + 4 xxx yyy + 28 yyy + 6 xxx yyy + 84 xxx yyy + 294 yyy + 4 xxx yyy + 84 xxx yyy + 588 xxx yyy + 1372 yyy

(%i3) f90_output_line_length_max;
(%o3) 65
(%i4) f90 ('foo = foo);
foo = yyy**4+4*xxx*yyy**3+28*yyy**3+6*xxx**2*yyy**2+84*xxx*yyy**2+
+294*yyy**2+4*xxx**3+yyy+84*xxx**2+yyy+588*xxx*yyy+1372*yyy+xxx**2+
+4+28*xxx**2+294*xxx**2+1372*xxx+2401
(%o4) false
(%i5) f90_output_line_length_max : 40 $%
(%i6) f90 ('foo = foo);
foo = yyy**4+4*xxx*yyy**3+28*yyy**3+6*xxx**2*yyy**2+84*xxx*yyy**2+
+294*yyy**2+4*xxx**3+yyy+84*xxx**2+yyy+588*xxx*yyy+1372*yyy+xxx**2+
+4+28*xxx**2+294*xxx**2+1372*xxx+2401
(%o6) false
```

f90 (expr_1, . . ., expr_n)

Prints one or more expressions expr_1, . . ., expr_n as a Fortran 90 program. Output is printed to the standard output.

f90 prints output in the so-called "free form" input format for Fortran 90: there is no special attention to column positions. Long lines are split at a fixed width with the ampersand & continuation character; the number of output characters per line, not including ampersands, is specified by f90_output_line_length_max. f90 outputs an ampersand at the end of a split line and another at the beginning of the next line.

load("f90") loads this function. See also the function fortran.
Examples:

```maxima
(%i1) load ("f90")$
(%i2) foo : expand ((xxx + yyy + 7)^4);
   4  3  3  2  2  2
(%o2) yyy + 4 xxx yyy + 28 yyy + 6 xxx yyy + 84 xxx yyy
   2  3  2
+ 294 yyy + 4 xxx yyy + 84 xxx yyy + 588 xxx yyy + 1372 yyy
   4  3  2
+ xxx + 28 xxx + 294 xxx + 1372 xxx + 2401
(%i3) f90 ('foo = foo);
foo = yyy**4+4*xxx*yyy**3+28*yyy**2+6*xxx**2*yyy**2+84*xxx*yyy**2&
   &+294*yyy**2+4*xxx**3*yyy+84*xxx**2*yyy+588*xxx*yyy+1372*yyy+xxx**&
   &+4+28*xxx**3+294*xxx**2+1372*xxx+2401
(%o3) false
```

Multiple expressions. Capture standard output into a file via the `with_stdout` function.

```maxima
(%i1) load ("f90")$
(%i2) foo : sin (3*x + 1) - cos (7*x - 2);
(%o2) sin(3 x + 1) - cos(7 x - 2)
(%i3) with_stdout ("foo.f90",
   f90 (x=0.25, y=0.625, 'foo=foo, 'stop, 'end));
(%o3) false
(%i4) printfile ("foo.f90");
x = 0.25
y = 0.625
foo = sin(3*x+1)-cos(7*x-2)
stop
end
(%o4) foo.f90
```
59 finance

59.1 Introduction to finance

This is the Finance Package (Ver 0.1).

In all the functions, \( rate \) is the compound interest rate, \( num \) is the number of periods and must be positive and \( flow \) refers to cash flow so if you have an Output the flow is negative and positive for Inputs.

Note that before using the functions defined in this package, you have to load it writing \( \text{load("finance")}\)$.

Author: Nicolas Guarin Zapata.

59.2 Functions and Variables for finance

**days360** \((year1, month1, day1, year2, month2, day2)\) \[Function\]
Calculates the distance between 2 dates, assuming 360 days years, 30 days months.

Example:

(\%i1) \text{load("finance")}$
(\%i2) \text{days360}(2008, 12, 16, 2007, 3, 25);
(\%o2) - 621

**fv** \((rate, PV, num)\) \[Function\]
We can calculate the future value of a Present one given a certain interest rate. \( rate \) is the interest rate, \( PV \) is the present value and \( num \) is the number of periods.

Example:

(\%i1) \text{load("finance")}$
(\%i2) \text{fv}(0.12, 1000, 3);
(\%o2) 1404.928

**pv** \((rate, FV, num)\) \[Function\]
We can calculate the present value of a Future one given a certain interest rate. \( rate \) is the interest rate, \( FV \) is the future value and \( num \) is the number of periods.

Example:

(\%i1) \text{load("finance")}$
(\%i2) \text{pv}(0.12, 1000, 3);
(\%o2) 711.7802478134108

**graph_flow** \((val)\) \[Function\]
Plots the money flow in a time line, the positive values are in blue and upside; the negative ones are in red and downside. The direction of the flow is given by the sign of the value. \( val \) is a list of flow values.

Example:

(\%i1) \text{load("finance")}$
(\%i2) \text{graph_flow([-5000, -3000, 800, 1300, 1500, 2000])}$
annuity_pv (rate, PV, num)  
We can calculate the annuity knowing the present value (like an amount), it is a 
constant and periodic payment. rate is the interest rate, PV is the present value and 
um is the number of periods.

Example:
(%i1) load("finance")$
(%i2) annuity_pv(0.12,5000,10);
(%o2) 884.9208207992202

annuity_fv (rate, FV, num)  
We can calculate the annuity knowing the desired value (future value), it is a constant 
and periodic payment. rate is the interest rate, FV is the future value and num is 
the number of periods.

Example:
(%i1) load("finance")$
(%i2) annuity_fv(0.12,65000,10);
(%o2) 3703.970670389863

gEO_annuity_pv (rate, growing_rate, PV, num)  
We can calculate the annuity knowing the present value (like an amount), in a growing 
periodic payment. rate is the interest rate, growing_rate is the growing rate, PV is 
the present value and num is the number of periods.

Example:
(%i1) load("finance")$
(%i2) geo_annuity_pv(0.14,0.05,5000,10);
(%o2) 802.6888176505123

gEO_annuity_fv (rate, growing_rate, FV, num)  
We can calculate the annuity knowing the desired value (future value), in a growing 
periodic payment. rate is the interest rate, growing_rate is the growing rate, FV is 
the future value and num is the number of periods.

Example:
(%i1) load("finance")$
(%i2) geo_annuity_fv(0.14,0.05,5000,10);
(%o2) 216.5203395312695

amortization (rate, amount, num)  
Amortization table determined by a specific rate. rate is the interest rate, amount is 
the amount value, and num is the number of periods.

Example:
(%i1) load("finance")$
(%i2) amortization(0.05,56000,12)$

<table>
<thead>
<tr>
<th>&quot;n&quot;</th>
<th>&quot;Balance&quot;</th>
<th>&quot;Interest&quot;</th>
<th>&quot;Amortization&quot;</th>
<th>&quot;Payment&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>56000.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1.000</td>
<td>52481.777</td>
<td>2800.000</td>
<td>3518.223</td>
<td>6318.223</td>
</tr>
<tr>
<td>2.000</td>
<td>48787.643</td>
<td>2624.089</td>
<td>3694.134</td>
<td>6318.223</td>
</tr>
</tbody>
</table>
The amortization table determined by a specific rate and with growing payment can be calculated by `arit_amortization`. Notice that the payment is not constant, it presents an arithmetic growing, increment is then the difference between two consecutive rows in the "Payment" column. `rate` is the interest rate, `increment` is the increment, `amount` is the amount value, and `num` is the number of periods.

Example:

```plaintext
(%i1) load("finance")$
(%i2) arit_amortization(0.05,1000,56000,12)$
```

<table>
<thead>
<tr>
<th>n</th>
<th>Balance</th>
<th>Interest</th>
<th>Amortization</th>
<th>Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>56000.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1.000</td>
<td>57403.679</td>
<td>2800.000</td>
<td>-1403.679</td>
<td>1396.321</td>
</tr>
<tr>
<td>2.000</td>
<td>57877.541</td>
<td>2870.184</td>
<td>-473.863</td>
<td>2396.321</td>
</tr>
<tr>
<td>3.000</td>
<td>57375.097</td>
<td>2893.877</td>
<td>502.444</td>
<td>3396.321</td>
</tr>
<tr>
<td>4.000</td>
<td>55847.530</td>
<td>2868.755</td>
<td>1527.567</td>
<td>4396.321</td>
</tr>
<tr>
<td>5.000</td>
<td>53243.586</td>
<td>2792.377</td>
<td>2603.945</td>
<td>5396.321</td>
</tr>
<tr>
<td>6.000</td>
<td>49509.443</td>
<td>2662.179</td>
<td>3734.142</td>
<td>6396.321</td>
</tr>
<tr>
<td>7.000</td>
<td>44588.594</td>
<td>2475.472</td>
<td>4920.849</td>
<td>7396.321</td>
</tr>
<tr>
<td>8.000</td>
<td>38421.703</td>
<td>2229.430</td>
<td>6166.892</td>
<td>8396.321</td>
</tr>
<tr>
<td>9.000</td>
<td>30946.466</td>
<td>1921.085</td>
<td>7475.236</td>
<td>9396.321</td>
</tr>
<tr>
<td>10.000</td>
<td>22097.468</td>
<td>1547.323</td>
<td>8848.998</td>
<td>10396.321</td>
</tr>
<tr>
<td>11.000</td>
<td>11806.020</td>
<td>1104.873</td>
<td>10291.448</td>
<td>11396.321</td>
</tr>
<tr>
<td>12.000</td>
<td>-0.000</td>
<td>590.301</td>
<td>11806.020</td>
<td>12396.321</td>
</tr>
</tbody>
</table>

The amortization table determined by rate, amount, and number of periods can be found by `geo_amortization`. Notice that the payment is not constant, it presents a geometric growing, `growing_rate` is then the quotient between two consecutive rows in the "Payment" column. `rate` is the interest rate, `amount` is the amount value, and `num` is the number of periods.

Example:

```plaintext
(%i1) load("finance")$
(%i2) geo_amortization(0.05,0.03,56000,12)$
```

<table>
<thead>
<tr>
<th>n</th>
<th>Balance</th>
<th>Interest</th>
<th>Amortization</th>
<th>Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>56000.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1.000</td>
<td>53365.296</td>
<td>2800.000</td>
<td>2634.704</td>
<td>5434.704</td>
</tr>
</tbody>
</table>
saving (rate, amount, num)

The table that represents the values in a constant and periodic saving can be found by saving. amount represents the desired quantity and num the number of periods to save.

Example:

```maxima
(%i1) load("finance")$
(%i2) saving(0.15,12000,15)$

<table>
<thead>
<tr>
<th>n</th>
<th>Balance</th>
<th>Interest</th>
<th>Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1.000</td>
<td>252.205</td>
<td>0.000</td>
<td>252.205</td>
</tr>
<tr>
<td>2.000</td>
<td>542.240</td>
<td>37.831</td>
<td>252.205</td>
</tr>
<tr>
<td>3.000</td>
<td>875.781</td>
<td>81.336</td>
<td>252.205</td>
</tr>
<tr>
<td>4.000</td>
<td>1259.352</td>
<td>131.367</td>
<td>252.205</td>
</tr>
<tr>
<td>5.000</td>
<td>1700.460</td>
<td>188.903</td>
<td>252.205</td>
</tr>
<tr>
<td>6.000</td>
<td>2207.733</td>
<td>255.069</td>
<td>252.205</td>
</tr>
<tr>
<td>7.000</td>
<td>2791.098</td>
<td>331.160</td>
<td>252.205</td>
</tr>
<tr>
<td>8.000</td>
<td>3461.967</td>
<td>418.665</td>
<td>252.205</td>
</tr>
<tr>
<td>9.000</td>
<td>4233.467</td>
<td>519.295</td>
<td>252.205</td>
</tr>
<tr>
<td>10.000</td>
<td>5120.692</td>
<td>635.020</td>
<td>252.205</td>
</tr>
<tr>
<td>11.000</td>
<td>6141.000</td>
<td>768.104</td>
<td>252.205</td>
</tr>
<tr>
<td>12.000</td>
<td>7314.355</td>
<td>921.150</td>
<td>252.205</td>
</tr>
<tr>
<td>13.000</td>
<td>8663.713</td>
<td>1097.153</td>
<td>252.205</td>
</tr>
<tr>
<td>14.000</td>
<td>10215.474</td>
<td>1299.557</td>
<td>252.205</td>
</tr>
<tr>
<td>15.000</td>
<td>12000.000</td>
<td>1532.321</td>
<td>252.205</td>
</tr>
</tbody>
</table>
```

npv (rate, val)

Calculates the present value of a value series to evaluate the viability in a project. val is a list of varying cash flows.

Example:

```maxima
(%i1) load("finance")$
(%i2) npv(0.25,[100,500,323,124,300]);
(%o2) 714.4703999999999
```

irr (val, IO)

IRR (Internal Rate of Return) is the value of rate which makes Net Present Value zero. flowValues is a list of varying cash flows, IO is the initial investment.
Example:

(%i1) load("finance")$
(%i2) res:irr([-5000,0,800,1300,1500,2000],0)$
(%i3) rhs(res[1][1]);
(%o3) 0.03009250374237132

benefit_cost (rate,input,output) [Function]
Calculates the ratio Benefit/Cost. Benefit is the Net Present Value (NPV) of the
inputs, and Cost is the Net Present Value (NPV) of the outputs. Notice that if there
is not an input or output value in a specific period, the input/output would be a zero
for that period. rate is the interest rate, input is a list of input values, and output is
a list of output values.

Example:

(%i1) load("finance")$
(%i2) benefit_cost(0.24,[0,300,500,150],[100,320,0,180]);
(%o2) 1.427249324905784
60 fractals

60.1 Introduction to fractals
This package defines some well known fractals:

- with random IFS (Iterated Function System): the Sierpinsky triangle, a Tree and a Fern
- Complex Fractals: the Mandelbrot and Julia Sets
- the Koch snowflake sets
- Peano maps: the Sierpinski and Hilbert maps

Author: José Ramírez Labrador.

For questions, suggestions and bugs, please feel free to contact me at
pepe DOT ramirez AAATTT uca DOT es

60.2 Definitions for IFS fractals
Some fractals can be generated by iterative applications of contractive affine transformations
in a random way; see


We define a list with several contractive affine transformations, and we randomly select
the transformation in a recursive way. The probability of the choice of a transformation
must be related with the contraction ratio.

You can change the transformations and find another fractal

\texttt{sierpinskiale (n)} \hspace{1cm} \text{[Function]}
Sierpinski Triangle: 3 contractive maps; .5 contraction constant and translations; all
maps have the same contraction ratio. Argument \texttt{n} must be great enough, 10000 or
greater.

Example:
\begin{verbatim}
(%i1) load("fractals")$
(%i2) n: 10000$
(%i3) plot2d([discrete,sierpinskiale(n)], [style,dots])$
\end{verbatim}

\texttt{treefale (n)} \hspace{1cm} \text{[Function]}
3 contractive maps all with the same contraction ratio. Argument \texttt{n} must be great
enough, 10000 or greater.

Example:
\begin{verbatim}
(%i1) load("fractals")$
(%i2) n: 10000$
(%i3) plot2d([discrete,treefale(n)], [style,dots])$
\end{verbatim}

\texttt{fernfale (n)} \hspace{1cm} \text{[Function]}
4 contractive maps, the probability to choice a transformation must be related with
the contraction ratio. Argument \texttt{n} must be great enough, 10000 or greater.
Example:

\begin{verbatim}
(%i1) load("fractals")$
(%i2) n: 10000$
(%i3) plot2d([discrete,fernfale(n)], [style,dots])$
\end{verbatim}

### 60.3 Definitions for complex fractals

#### mandelbrot_set (x, y)

Mandelbrot set.

Example:

This program is time consuming because it must make a lot of operations; the computing time is also related with the number of grid points.

\begin{verbatim}
(%i1) load("fractals")$
(%i2) plot3d (mandelbrot_set, [x, -2.5, 1], [y, -1.5, 1.5],
                [gnuplot_preamble, "set view map"],
                [gnuplot_pm3d, true],
                [grid, 150, 150])$
\end{verbatim}

#### julia_set (x, y)

Julia sets.

This program is time consuming because it must make a lot of operations; the computing time is also related with the number of grid points.

Example:

\begin{verbatim}
(%i1) load("fractals")$
(%i2) plot3d (julia_set, [x, -2, 1], [y, -1.5, 1.5],
                [gnuplot_preamble, "set view map"],
                [gnuplot_pm3d, true],
                [grid, 150, 150])$
\end{verbatim}

See also \texttt{julia_parameter}.

#### julia_parameter

[Optional variable]

Default value: \%i

Complex parameter for Julia fractals. Its default value is \%i; we suggest the values 
\(-.745+\%i*1.113002, -.39054-\%i*.58679, -.15652+\%i*1.03225, -.194+\%i*.6557 \and .011031-\%i*.67037.\)

#### julia_sin (x, y)

[Function]

While function \texttt{julia_set} implements the transformation \texttt{julia_parameter+z^2}, function \texttt{julia_sin} implements \texttt{julia_parameter*sin(z)}. See source code for more details.

This program runs slowly because it calculates a lot of sines.

Example:

This program is time consuming because it must make a lot of operations; the computing time is also related with the number of grid points.

\begin{verbatim}
(%i1) load("fractals")$
\end{verbatim}
Chapter 60: fractals

(%i2) julia_parameter:1+.1*%i$
(%i3) plot3d (julia_sin, [x, -2, 2], [y, -3, 3],
    [gnuplot_preamble, "set view map"],
    [gnuplot_pm3d, true],
    [grid, 150, 150])$

See also julia_parameter.

60.4 Definitions for Koch snowflakes

snowmap (ent, nn) [Function]
Koch snowflake sets. Function snowmap plots the snow Koch map over the vertex
of an initial closed polygonal, in the complex plane. Here the orientation of the
polygon is important. Argument nn is the number of recursive applications of Koch
transformation; nn must be small (5 or 6).

Examples:
(%i1) load("fractals")$
(%i2) plot2d([discrete,
    snowmap([1,exp(%i*%pi/3),exp(-%i*%pi/3),1],4)])$
(%i3) plot2d([discrete,
    snowmap([1,exp(-%i*%pi/3),exp(%i*%pi/3),1],4)])$
(%i4) plot2d([discrete, snowmap([0,1,1+%i,%i,0],4)])$
(%i5) plot2d([discrete, snowmap([0,%i,1+%i,1,0],4)])$

60.5 Definitions for Peano maps

Continuous curves that cover an area. Warning: the number of points exponentially grows
with n.

hilbertmap (nn) [Function]
Hilbert map. Argument nn must be small (5, for example). Maxima can crash if nn
is 7 or greater.

Example:
(%i1) load("fractals")$
(%i2) plot2d([discrete,hilbertmap(6)])$

sierpinskimap (nn) [Function]
Sierpinski map. Argument nn must be small (5, for example). Maxima can crash if
nn is 7 or greater.

Example:
(%i1) load("fractals")$
(%i2) plot2d([discrete,sierpinskimap(6)])$


61 Gentran

61.1 Introduction to Gentran

Original Authors Barbara Gates and Paul Wang

Gentran is a powerful generator of foreign language code. Currently it can generate FORTRAN, 'C', and RATFOR code from Maxima language code. Gentran can translate mathematical expressions, iteration loops, conditional branching statements, data type information, function definitions, matrices and arrays, and more.

61.2 Functions for Gentran

`gentran (stmt1, stmt2, ..., stmtn, [f1, f2, ..., fm])`  
Translates each stmt into formatted code in the target language. A substantial subset of expressions and statements in the Maxima programming language can be translated directly into numerical code. The `gentran` command translates Maxima statements or procedure definitions into code in the target language (`gentranlang`: fortran, c, or ratfor). Expressions may optionally be given to Maxima for evaluation prior to translation.

`stmt1, stmt2, ..., stmtn` is a sequence of one or more statements, each of which is any Maxima user level expression, (simple, group, or block) statement, or procedure definition that can be translated into the target language.

`[f1, f2, ..., fm]` is an optional list of output files to which translated output will be written. They can be any of the following:

- `string` = the name of an output file in quotes
- `true` (no quotes) = the terminal
- `false` = the current output file(s)
- `all` = all files currently open for output by gentran

If the files are not open they will be opened; if they are open, output will be appended to them. Filenames are given as quoted strings. If the optional variable `genoutpath` (string, including the final '/') default `false` is set, it will be prepended to the output file names. If the output file list is omitted, output will be written to the current output, generally the terminal. `gentran` returns (a list of) the name(s) of file(s) to which code was written.

`gentranout (f1, f2, ..., fn)`  
Gentran maintains a list of files currently open for output by gentran commands only. `gentranout` inserts each file name represented by `f1, f2, ..., fn` into that list and opens each one for output. It also resets the current output file(s) to include all files in `f1, f2, ..., fn`. `gentranout` returns the list of files represented by `f1, f2, ..., fn`; i.e., the current output file(s) after the command has been executed.

`gentranshut (f1, f2, ..., fn)`  
`gentranshut` creates a list of file names from `f1, f2, ..., fn`, deletes each from the output file list, and closes the corresponding files. If (all of) the current output
file(s) are closed, then the current output file is reset to the terminal. gentranshut returns (a list of) the current output file(s) after the command has been executed. gentranshut(all) will close all gentran output files.

**gentranpush (f1, f2, ... , fn)**

[Function]

gentranpush pushes the file list onto the output stack. Each file in the list that is not already open for output is opened at this time. The current output file is reset to this new element on the top of the stack.

**gentranpop (f1, f2, ... , fn)**

[Function]

gentranpop deletes the top-most occurrence of the single element containing the file name(s) represented by f1, f2, ... , fn from the output stack. Files whose names have been completely removed from the output stack are closed. The current output file is reset to the (new) element on the top of the output stack. gentranpop returns the current output file(s) after this command has been executed.

**gentranin (f1, f2, ... , fn, [f1,f2, ... , fm])**

[Function]

gentranin processes mixed-language template files consisting of active parts (delimited by <<...>>) containing Maxima statements, including calls to gentran, and passive parts, assumed to contain statements in the target language (including comments), which are transcribed verbatim. Input files are processed sequentially and the results appended to the output. The presence of >> in passive parts of the file (except for in comments) is interpreted as an end-of-file and terminates processing of that file. The optional list of output files [f1,f2, ... , fm] each receive a copy of the entire output. All filespecs are quoted strings. Input files may be given as (quoted string) filenames, which will be located by Maxima file_search. The optional variable geninpath (default false) must be a list of quoted strings describing the paths to be searched for the input files. If it is set, that list replaces the standard Maxima search paths.

Active parts may contain any number of Maxima expressions and statements. They are not copied directly to the output. Instead, they are given to Maxima for evaluation. All output generated by each evaluation is sent to the output file(s). Returned values are only printed on the terminal. Active parts will most likely contain calls to gentran to generate code. This means that the result of processing a template file will be the original template file with all active parts replaced by generated code. If [f1, f2, ... , fm] is not supplied, then generated code is simply written to the current output file(s). However, if it is given, then the current output file is temporarily overridden. Generated code is written to each file represented by f1, f2, ... , fn for this command only. Files which were open prior to the call to gentranin will remain open after the call, and files which did not exist prior to the call will be created, opened, written to, and closed. The output file stack will be exactly the same both before and after the call. gentranin returns (to the terminal) the name(s) of (all) file(s) written to by this command.

**gentraninshut ()**

[Function]

A cleanup function to close input files in case where gentranin hung due to error in template.
tempvar (type)  [Function]
Generates temporary variable names by concatenating tempvarname (default 't) with sequence numbers. If type is non-false, e.g. "real*8" the corresponding type is assigned to the variable in the gentran symbol table, which may be used to generate declarations depending on the setting of the gendecs flag. It is the users responsibility to make sure temporary variable names do not conflict with the main program.

markvar (vname)  [Function]
markvar "marks" variable name vname to indicate that it currently holds a significant value.

unmarkvar (vname)  [Function]
unmarkvar "unmarks" variable name vname to indicate that it no longer holds a significant value.

markedvarp (vname)  [Function]
markedvarp tests whether the variable name vname is currently marked.

gendecs (name)  [Function]
The gendecs command can be called any time the gendecs flag is switched off to retrieve all type declarations from Gentran’s symbol table for the given subprogram name (or the "current" subprogram if false is given as its argument).

gentran_on (sw)  [Function]
Turns on the mode switch sw.

gentran_off (sw)  [Function]
Turns the given switch, sw, off.

61.3 Gentran Mode Switches

fortran  [Option variable]
ratfor  [Option variable]
c  [Option variable]

Default: off

These mode switches change the default mode of Maxima from evaluation to translation. They can be turned on and off with the gentran commands gentran_on and gentran_off. Each time a new Maxima session is started up, the system is in evaluation mode. It prints a prompt on the user’s terminal screen and waits for an expression or statement to be entered. It then proceeds to evaluate the expression, prints a new prompt, and waits for the user to enter another expression or statement. This mode can be changed to translation mode by turning on either the fortran, ratfor or c switches. After one of these switches is turned on and until it is turned off, every expression or statement entered by the user is translated into the corresponding language just as if it had been given as an argument to the gentran command. Each of the special functions that can be used from within a call to gentran can be used at the top level until the switch is turned off.
gendecs

Default: on

When the gendecs switch is turned on, gentran generates type declarations whenever possible. When gendecs is switched off, type declarations are not generated. Instead, type information is stored in gentran’s symbol table but is not retrieved in the form of declarations unless and until either the gendecs command is called or the gendecs flag is switched back. **Note:** Generated declarations may often be placed in an inappropriate place (e.g., in the middle of executable fortran code). Therefore the gendecs flag is turned off during processing of templates by **gentranin**.

61.4 Gentran Option Variables

gentranlang

Default: fortran

Selects the target numerical language. Currently, gentranlang must be fortran, ratfor, or c. Note that symbols entered in Maxima are case-sensitive. gentranlang should not be set to FORTRAN, RATFOR or C.

fortlinelen

default: 72

Maximum number of characters printed on each line of generated FORTRAN code.

minfortlinelen

Default: 40

Minimum number of characters printed on each line of generated FORTRAN code.

fortcurrind

Default: 0

Number of blank spaces printed at the beginning of each line of generated FORTRAN code (after column 6).

ratlinelen

Default: 80

Maximum number of characters printed on each line of generated Ratfor code.

clinelen

Default: 80

Maximum number of characters printed on each line of generated 'C' code.

minclinelen

Default: 40

Minimum number of characters printed on each line of generated 'C' code.

ccurind

Default: 0

Number of blank spaces printed at the beginning of each line of generated 'C' code.
tablen  [Option variable]
    Default: 4
    Number of blank spaces printed for each new level of indentation. (Automatic indentation can be turned off by setting this variable to 0.)

genfloat  [Option variable]
    Default: false
    When set to true (or any non-false value), causes integers in generated numerical code to be converted to floating point numbers, except in the following places: array subscripts, exponents, and initial, final, and step values in do-loops. An exception (for compatibility with Macsyma 2.4) is that numbers in exponentials (with base %e only) are double-floated even when genfloat is false.

dblfloat  [Option variable]
    Default: false
    If dblfloat is set to true, floating point numbers in gentran output in implementations (such as Windows Maxima under CLISP) in which float and double-float are the same will be printed as *.d0. In implementations in which float and double-float are different, floats will be coerced to double-float before being printed.

gentranseg  [Option variable]
    Default: true

maxexpprintlen  [Option variable]
    Default: 800
    When gentranseg is true (or any non-false value), causes Gentran to "segment" large expressions into subexpressions of manageable size. The segmentation facility generates a sequence of assignment statements, each of which assigns a subexpression to an automatically generated temporary variable name. This sequence is generated in such a way that temporary variables are re-used as soon as possible, thereby keeping the number of automatically generated variables to a minimum. The maximum allowable expression size can be controlled by setting the maxexpprintlen variable to the maximum number of characters allowed in an expression printed in the target numerical language (excluding spaces and other whitespace characters automatically printed by the formatter). When the segmentation routine generates temporary variables, it places type declarations in the symbol table for those variables if possible. It uses the following rules to determine their type:

1. If the type of the variable to which the large expression is being assigned is already known (i.e., has been declared by the user via a TYPE form), then the temporary variables will be declared to be of that same type. 2. If the global variable tempvartype has a non-false value, then the temporary variables are declared to be of that type. 3. Otherwise, the variables are not declared unless implicit has been set to true.

gentranopt  [Option variable]
    Default: false
    When set to true (or any non-false value), causes Gentran to replace each block of straightline code by an optimized sequence of assignments obtained from the Maxima
optimize command. (The optimize command takes an expression and replaces common subexpressions by temporary variable names. It returns the resulting assignment statement, preceded by common-subexpression-to-temporary-variable assignments.

\textbf{tempvarname} \hspace{1cm} [Option variable]
Default: 't
Name used as the prefix when generating temporary variable names.

\textbf{optimvarname} \hspace{1cm} [Option variable]
default: 'u
is the preface used to generate temporary file names produced by the optimizer when \texttt{gentranopt} is \texttt{true}. When both \texttt{gentranseg} and \texttt{gentranopt} are true, the optimizer generates temporary file names using \texttt{optimvarname} while the segmentation routine uses \texttt{tempvarname} preventing conflict.

\textbf{tempvarnum} \hspace{1cm} [Option variable]
Default: 0
Number appended onto tempvarname to create a temporary variable name. If the temporary variable name resulting from appending tempvarnum onto the end of tempvarname has already been generated and still holds a useful value or has a different type than requested, then the number is incremented until one is found that was not previously generated or does not still hold a significant value or a different type.

\textbf{tempvartype} \hspace{1cm} [Option variable]
Default: false
Target language variable type (e.g., INTEGER, REAL*8, FLOAT, etc.) used as a default for automatically generated variables whose type cannot be determined otherwise. If tempvartype is false, then generated temporary variables whose type cannot be determined are not automatically declared.

\textbf{implicit} \hspace{1cm} [Option variable]
Default: false
If implicit is set to \texttt{true} temporary variables are assigned their implicit type according to Fortran rules based on the initial letter of the name. If gendecs is on, this results in printed type declarations.

\textbf{gentranparser} \hspace{1cm} [Option variable]
Default: false
If gentranparser is set to \texttt{true} Maxima forms input to gentran will be parsed and an error will be produced if an expression cannot be translated. Otherwise, untranslatable expressions may generate anomalous output, sometimes containing explicit calls to Maxima functions.

\textbf{genstmtno} \hspace{1cm} [Option variable]
Default: 25000
Number used when a statement number must be generated. Note: it is the user's responsibility to make sure this number will not clash with statement numbers in template files.
genstmtincr  [Option variable]
Default: 1
number by which genstmtno is incremented each time a new statement number is
generated.

usefortcomplex  [Option variable]
Default: false
If usefortcomplex is true, real numbers in expressions declared to be complex by
type(complex, . . .) will be printed in Fortran complex number format (realpart, 0.0).
This is a purely syntactic device and does not carry out any complex calculations.

61.5 Gentran Evaluation Forms

The following special functions can be included in Maxima statements which are to be
translated by the gentran command to indicate that they are to be partially or fully eval-
uated by Maxima before being translated into numerical code. Note that these functions
have the described effect only when supplied in arguments to the gentran command.

eval (exp)  [Function]
Where exp is any Maxima expression or statement which, after evaluation by Maxima,
results in an expression that can be translated by gentran into the target language.
When eval is called from an argument that is to be translated, it tells gentran to give
the expression to Maxima for evaluation first, and then to translate the result of that
evaluation.

rsetq (var, exp)  [Function]
Where var is any Maxima variable, matrix or array element, and exp is any Max-
ima expression which, after evaluation by Maxima results in an expression that can
be translated by Gentran into the target language. This is equivalent to VAR :
EVAL(EXP) :

lsetq (var, exp)  [Function]
Where var is any Maxima user level matrix or array element with indices which, after
evaluation by Maxima, will result in expressions that can be translated by Gentran,
and exp is any Maxima user level expression that can be translated into the target
language. This is equivalent to var[eval(s1), eval(s2), ...]: exp where s1, s2,
... are indices.

lrsetq (var, exp)  [Function]
Where var is any Maxima matrix or array element with indices which, after eval-
uation by Maxima, will result in expressions that can be translated by Gentran; and
exp is any user level expression which, after evaluation, will result in an expression
that can be translated by Gentran into the target language. This is equivalent to
var[eval(s1), eval(s2), ...]: eval(exp);

type (type, v1 . . . vn)  [Function]
Places information in the gentran symbol table to assign type to variables v1 . . . vn.
This may result in type declarations printed by gentran depending on the setting of
gendecs. type must be called from within gentran and does not evaluate its arguments
unless eval() is used.
literal (arg1, arg2, ..., argn)

where arg1, arg2, ..., argn is an argument list containing one or more arg’s, each of which either is, or evaluates to, an atom. The atoms tab and cr have special meanings. arg’s are not evaluated unless given as arguments to eval. This function call is replaced by the character sequence resulting from concatenation of the given atoms. Double quotes are stripped from all string type arg’s, and each occurrence of the reserved atom tab or cr is replaced by a tab to the current level of indentation, or an end-of-line character.
62 **ggf**

### 62.1 Functions and Variables for ggf

**GGFINFINITY**

[Option variable]
Default value: 3
This is an option variable for function ggf.
When computing the continued fraction of the generating function, a partial quotient having a degree (strictly) greater than **GGFINFINITY** will be discarded and the current convergent will be considered as the exact value of the generating function; most often the degree of all partial quotients will be 0 or 1; if you use a greater value, then you should give enough terms in order to make the computation accurate enough.
See also ggf.

**GGFCFMAX**

[Option variable]
Default value: 3
This is an option variable for function ggf.
When computing the continued fraction of the generating function, if no good result has been found (see the **GGFINFINITY** flag) after having computed **GGFCFMAX** partial quotients, the generating function will be considered as not being a fraction of two polynomials and the function will exit. Put freely a greater value for more complicated generating functions.
See also ggf.

**ggf** (1) [Function]
Compute the generating function (if it is a fraction of two polynomials) of a sequence, its first terms being given. *l* is a list of numbers.
The solution is returned as a fraction of two polynomials. If no solution has been found, it returns with **done**.
This function is controlled by global variables **GGFINFINITY** and **GGFCFMAX**. See also **GGFINFINITY** and **GGFCFMAX**.
To use this function write first `load("ggf")`.

```
(%i1) load("ggf")$
(%i2) makelist(fib(n),n,0,10);
(%o2) [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55]
(%i3) ggf(%);

(%o3) x
    - -----------
       2
       x + x - 1

(%i4) taylor(%o3,x,0,10);

(%o4)/T/ x + x + 2 x + 3 x + 5 x + 8 x + 13 x + 21 x + 34 x + 55 x
```
(%i5) makelist(2*fib(n+1)-fib(n),n,0,10);
(%o5) [2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123]
(%i6) ggf(%);
    x - 2
(%o6) --
   2
   x + x - 1
(%i7) taylor(%),x,0,10;
(%o7)/T/ 2 + x + 3 x + 4 x + 7 x + 11 x + 18 x + 29 x + 47 x + 76 x
         + 123 x + . . .

As these examples show, the generating function does create a function whose Taylor series has coefficients that are the elements of the original list.
63 graphs

63.1 Introduction to graphs

The graphs package provides graph and digraph data structure for Maxima. Graphs and
digraphs are simple (have no multiple edges nor loops), although digraphs can have a
directed edge from u to v and a directed edge from v to u.

Internally graphs are represented by adjacency lists and implemented as a lisp structures.
Vertices are identified by their ids (an id is an integer). Edges/arcs are represented by lists
of length 2. Labels can be assigned to vertices of graphs/digraphs and weights can be
assigned to edges/arcs of graphs/digraphs.

There is a draw_graph function for drawing graphs. Graphs are drawn using a force
based vertex positioning algorithm. draw_graph can also use graphviz programs available
from https://www.graphviz.org. draw_graph is based on the maxima draw package.

To use the graphs package, first load it with load("graphs").

63.2 Functions and Variables for graphs

63.2.1 Building graphs

create_graph

create_graph (v_list, e_list)
create_graph (n, e_list)
create_graph (v_list, e_list, directed)

Creates a new graph on the set of vertices v_list and with edges e_list.

v_list is a list of vertices [v1, v2, ..., vn] or a list of vertices together with vertex
labels [[v1, l1], [v2, l2], ..., [vn, ln]]. A vertex may be any integer, and
v_list may contain vertices in any order. A label may be any Maxima expression, and
two or more vertices may have the same label.

n is the number of vertices. Vertices will be identified by integers from 0 to n-1.

e_list is a list of edges [e1, e2, ..., em] or a list of edges together with edge-weights
[[e1, w1], ..., [em, wm]].

If directed is not false, a directed graph will be returned.

Example 1: create a cycle on 3 vertices:

(\%i1) load ("graphs")$
(\%i2) g : create_graph([1,2,3], [[1,2], [2,3], [1,3]])$
(\%i3) print_graph(g)$

Graph on 3 vertices with 3 edges.

Adjacencies:
3 : 1 2
2 : 3 1
1 : 3 2

Example 2: create a cycle on 3 vertices with edge weights:

(\%i1) load ("graphs")$
(%i2) g : create_graph([1, 2, 3], [[1, 2], 1.0], [[1, 3], 3.0])$
(%i3) print_graph(g)$
Graph on 3 vertices with 3 edges.
Adjacencies:
  3 : 1 2
  2 : 3 1
  1 : 3 2

Example 3: create a directed graph:
(%i1) load ("graphs")$
(%i2) d : create_graph([1, 2, 3, 4], [[1, 3], [1, 4], [2, 3], [2, 4]], 'directed = true)$
(%i3) print_graph(d)$
Digraph on 4 vertices with 4 arcs.
Adjacencies:
  4 :
  3 :
  2 : 4 3
  1 : 4 3

\textbf{copy\_graph (g)} \hspace{1cm} \textbf{[Function]}\hspace{1cm}

Returns a copy of the graph \textit{g}.

\textbf{circulant\_graph (n, d)} \hspace{1cm} \textbf{[Function]}\hspace{1cm}

Returns the circulant graph with parameters \textit{n} and \textit{d}.

Example:
(%i1) load ("graphs")$
(%i2) g : circulant_graph(10, [1, 3])$
(%i3) print_graph(g)$
Graph on 10 vertices with 20 edges.
Adjacencies:
  9 : 2 6 0 8
  8 : 1 5 9 7
  7 : 0 4 8 6
  6 : 9 3 7 5
  5 : 8 2 6 4
  4 : 7 1 5 3
  3 : 6 0 4 2
  2 : 9 5 3 1
  1 : 8 4 2 0
  0 : 7 3 9 1
clebsch_graph ()
  Returns the Clebsch graph.

complement_graph (g)
  Returns the complement of the graph \( g \).

complete_bipartite_graph (n, m)
  Returns the complete bipartite graph on \( n+m \) vertices.

complete_graph (n)
  Returns the complete graph on \( n \) vertices.

cycle_digraph (n)
  Returns the directed cycle on \( n \) vertices.

cycle_graph (n)
  Returns the cycle on \( n \) vertices.

cuboctahedron_graph (n)
  Returns the cuboctahedron graph.

cube_graph (n)
  Returns the \( n \)-dimensional cube.

dodecahedron_graph ()
  Returns the dodecahedron graph.

empty_graph (n)
  Returns the empty graph on \( n \) vertices.

flower_snark (n)
  Returns the flower graph on \( 4n \) vertices.

  Example:
  \[
  \%(i1) \text{load ("graphs")}$
  \%(i2) f5 : \text{flower_snark(5)}$
  \%(i3) \text{chromatic_index(f5)}$
  \%(o3) 4$
  
from_adjacency_matrix (A)
  Returns the graph represented by its adjacency matrix \( A \).

frucht_graph ()
  Returns the Frucht graph.

graph_product (g1, g1)
  Returns the direct product of graphs \( g1 \) and \( g2 \).

  Example:
  \[
  \%(i1) \text{load ("graphs")}$
  \%(i2) \text{grid : graph_product(path_graph(3), path_graph(4))}$
  \%(i3) \text{draw_graph(grid)}$
  

graph_union (g1, g1)  

Returns the union (sum) of graphs g1 and g2.

grid_graph (n, m)  

Returns the \( n \times m \) grid.

great_rhombicosidodecahedron_graph ()  

Returns the great rhombicosidodecahedron graph.

great_rhombicuboctahedron_graph ()  

Returns the great rhombicuboctahedron graph.

grotzch_graph ()  

Returns the Grotzch graph.

heawood_graph ()  

Returns the Heawood graph.

icosahedron_graph ()  

Returns the icosahedron graph.

icosidodecahedron_graph ()  

Returns the icosidodecahedron graph.

induced_subgraph (V, g)  

Returns the graph induced on the subset V of vertices of the graph g.

Example:

```
(%i1) load ("graphs")$
(%i2) p : petersen_graph();$
(%i3) V : [0,1,2,3,4];$
(%i4) g : induced_subgraph(V, p);$
(%i5) print_graph(g);$
Graph on 5 vertices with 5 edges.
Adjacencies:
  4 : 3 0
  3 : 2 4
  2 : 1 3
  1 : 0 2
  0 : 1 4
```

line_graph (g)  

Returns the line graph of the graph g.

make_graph  

make_graph (vrt, f)  

make_graph (vrt, f, oriented)  

Creates a graph using a predicate function f.

vrt is a list or set of vertices, or an integer.

When vrt is a list or set, its elements may be any integers, and, if a list, may be listed in any order.
When \(\text{vrt}\) is an integer, vertices of the graph will be integers from 1 to \(\text{vrt}\).

\(f\) is a predicate function. Two vertices \(a\) and \(b\) will be connected if \(f(a, b) = \text{true}\).

If \textit{directed} is not \textit{false}, then the graph will be directed.

**Example 1:**

```lisp
(%i1) load("graphs")
(%i2) g : make_graph(powerset({1,2,3,4,5}, 2), disjointp)
(%i3) is_isomorphic(g, petersen_graph());
(%o3) true
(%i4) get_vertex_label(1, g);
(%o4) {1, 2}
```

**Example 2:**

```lisp
(%i1) load("graphs")
(%i2) f(i, j) := is (mod(j, i)=0)
(%i3) g : make_graph(20, f, directed=true)
(%i4) out_neighbors(4, g);
(%o4) [8, 12, 16, 20]
(%i5) in_neighbors(18, g);
(%o5) [1, 2, 3, 6, 9]
```

**mycielski_graph** \((g)\)  
Returns the mycielskian graph of the graph \(g\).

**new_graph** ()  
Returns the graph with no vertices and no edges.

**path_digraph** \((n)\)  
Returns the directed path on \(n\) vertices.

**path_graph** \((n)\)  
Returns the path on \(n\) vertices.

**petersen_graph** ()  
Returns the Petersen graph \(P\{n,d\}\). The default values for \(n\) and \(d\) are \(n=5\) and \(d=2\).

**random_bipartite_graph** \((a, b, p)\)  
Returns a random bipartite graph on \(a+b\) vertices. Each edge is present with probability \(p\).

**random_digraph** \((n, p)\)  
Returns a random directed graph on \(n\) vertices. Each arc is present with probability \(p\).

**random_regular_graph** ()  
Returns a random \(d\)-regular graph on \(n\) vertices. The default value for \(d\) is \(d=3\).
random_graph (n, p)  
Returns a random graph on n vertices. Each edge is present with probability p.

random_graph1 (n, m)  
Returns a random graph on n vertices and random m edges.

random_network (n, p, w)  
Returns a random network on n vertices. Each arc is present with probability p and has a weight in the range [0,w]. The function returns a list [network, source, sink].

Example:

(%i1) load ("graphs")$
(%i2) [net, s, t] : random_network(50, 0.2, 10.0);
(%o2) [DIGRAPH, 50, 51]
(%i3) max_flow(net, s, t)$
(%i4) first(%);
(%o4) 27.65981397932507

random_tournament (n)  
Returns a random tournament on n vertices.

random_tree (n)  
Returns a random tree on n vertices.

small_rhombicosidodecahedron_graph ()  
Returns the small rhombicosidodecahedron graph.

small_rhombicuboctahedron_graph ()  
Returns the small rhombicuboctahedron graph.

snub_cube_graph ()  
Returns the snub cube graph.

snub_dodecahedron_graph ()  
Returns the snub dodecahedron graph.

truncated_cube_graph ()  
Returns the truncated cube graph.

truncated_dodecahedron_graph ()  
Returns the truncated dodecahedron graph.

truncated_icosahehedron_graph ()  
Returns the truncated icosahedron graph.

truncated_tetrahedron_graph ()  
Returns the truncated tetrahedron graph.

tutte_graph ()  
Returns the Tutte graph.
underlying_graph (g)

    Returns the underlying graph of the directed graph g.

wheel_graph (n)

    Returns the wheel graph on n+1 vertices.

63.2.2 Graph properties

adjacency_matrix (gr)

    Returns the adjacency matrix of the graph gr.
    Example:
    (%i1) load ("graphs")$
    (%i2) c5 : cycle_graph(4)$
    (%i3) adjacency_matrix(c5);

    [ 0 1 0 1 ]
    [           ]
    [ 1 0 1 0 ]
    [   ]
    [ 0 1 0 1 ]
    [ ]
    [ 1 0 1 0 ]

    (%o3)

average_degree (gr)

    Returns the average degree of vertices in the graph gr.
    Example:
    (%i1) load ("graphs")$
    (%i2) average_degree(grotzch_graph());

    40
    (%o2) --
         11

biconnected_components (gr)

    Returns the (vertex sets of) 2-connected components of the graph gr.
    Example:
    (%i1) load ("graphs")$
    (%i2) g : create_graph(
        [1,2,3,4,5,6,7],
        [1,2],[2,3],[2,4],[3,4],
        [4,5],[5,6],[4,6],[6,7])$
    (%i3) biconnected_components(g);

    (%o3) [[6, 7], [4, 5, 6], [1, 2], [2, 3, 4]]

bipartition (gr)

    Returns a bipartition of the vertices of the graph gr or an empty list if gr is not bipartite.
Example:

```
(%i1) load ("graphs")$
(%i2) h : heawood_graph()$
(%i3) [A,B]:bipartition(h);
(%o3) [[8, 12, 6, 10, 0, 2, 4], [13, 5, 11, 7, 9, 1, 3]]
(%i4) draw_graph(h, show_vertices=A, program=circular)$
```

**chromatic_index (gr)**

[Function]

Returns the chromatic index of the graph gr.

Example:

```
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) chromatic_index(p);
(%o3) 4
```

**chromatic_number (gr)**

[Function]

Returns the chromatic number of the graph gr.

Example:

```
(%i1) load ("graphs")$
(%i2) chromatic_number(cycle_graph(5));
(%o2) 3
(%i3) chromatic_number(cycle_graph(6));
(%o3) 2
```

**clear_edge_weight (e, gr)**

[Function]

Removes the weight of the edge e in the graph gr.

Example:

```
(%i1) load ("graphs")$
(%i2) g : create_graph(3, [[[0,1], 1.5], [[1,2], 1.3]])$
(%i3) get_edge_weight([0,1], g);
(%o3) 1.5
(%i4) clear_edge_weight([0,1], g)$
(%i5) get_edge_weight([0,1], g);
(%o5) 1
```

**clear_vertex_label (v, gr)**

[Function]

Removes the label of the vertex v in the graph gr.

Example:

```
(%i1) load ("graphs")$
(%i2) g : create_graph([[[0,"Zero"], [1, "One"]], [[0,1]]])$
(%i3) get_vertex_label(0, g);
(%o3) Zero
(%i4) clear_vertex_label(0, g); done
(%i5) get_vertex_label(0, g);
(%o5) false
```
connected_components (gr) [Function]
Returns the (vertex sets of) connected components of the graph gr.

Example:
(\%i1) load ("graphs")$
(\%i2) g: graph_union(cycle_graph(5), path_graph(4))$
(\%i3) connected_components(g);
(\%o3) [[1, 2, 3, 4, 0], [8, 7, 6, 5]]

diameter (gr) [Function]
Returns the diameter of the graph gr.

Example:
(\%i1) load ("graphs")$
(\%i2) diameter(dodecahedron_graph());
(\%o2) 5

degree_sequence (gr) [Function]
Returns the list of vertex degrees of the graph gr.

Example:
(\%i1) load ("graphs")$
(\%i2) p : petersen_graph()$
(\%i3) [ch_index, col] : edge_coloring(p);
(\%o3) [4, [[0, 5], 3], [[5, 7], 1], [[0, 1], 1], [[1, 6], 2],
[[6, 8], 1], [[1, 2], 3], [[2, 7], 4], [[7, 9], 2], [[2, 3], 2],
[[3, 8], 3], [[5, 8], 2], [[3, 4], 1], [[4, 9], 4], [[6, 9], 3],
[[0, 4], 2]]
(\%i4) assoc([0,1], col);
(\%o4) 1
(\%i5) assoc([0,5], col);
(\%o5) 3

edge_coloring (gr) [Function]
Returns an optimal coloring of the edges of the graph gr.

The function returns the chromatic index and a list representing the coloring of the edges of gr.

Example:
(\%i1) load ("graphs")$
(\%i2) p : petersen_graph()$
(\%i3) [ch_index, col] : edge_coloring(p);
(\%o3) [4, [[0, 5], 3], [[5, 7], 1], [[0, 1], 1], [[1, 6], 2],
[[6, 8], 1], [[1, 2], 3], [[2, 7], 4], [[7, 9], 2], [[2, 3], 2],
[[3, 8], 3], [[5, 8], 2], [[3, 4], 1], [[4, 9], 4], [[6, 9], 3],
[[0, 4], 2]]
(\%i4) assoc([0,1], col);
(\%o4) 1
(\%i5) assoc([0,5], col);
(\%o5) 3

edge_connectivity (gr) [Function]
Returns the edge-connectivity of the graph gr.

See also min_edge_cut.

edges (gr) [Function]
Returns the list of edges (arcs) in a (directed) graph gr.
Example:

(%i1) load ("graphs")$
(%i2) edges(complete_graph(4));
(%o2) [[2, 3], [1, 3], [1, 2], [0, 3], [0, 2], [0, 1]]

get_edge_weight
get_edge_weight (e, gr)
get_edge_weight (e, gr, ifnot)
Returns the weight of the edge e in the graph gr.
If there is no weight assigned to the edge, the function returns 1. If the edge is not present in the graph, the function signals an error or returns the optional argument ifnot.

Example:

(%i1) load ("graphs")$
(%i2) c5 : cycle_graph(5)$
(%i3) get_edge_weight([1,2], c5);
(%o3) 1
(%i4) set_edge_weight([1,2], 2.0, c5);
(%o4) done
(%i5) get_edge_weight([1,2], c5);
(%o5) 2.0

get_vertex_label (v, gr)
Returns the label of the vertex v in the graph gr.
If no label is assigned to vertex v, get_vertex_label returns false.

Example:

(%i1) load ("graphs")$
(%i2) g: create_graph([[0, "Zero"], [1, "One"], [2, "Other"], [3, "Other"]], [])$
(%i3) get_vertex_label(0, g);
(%o3) Zero
(%i4) get_vertex_label(2, g);
(%o4) false

get_unique_vertex_by_label (l, gr)
Returns the unique vertex which has the label l in graph gr.
If there is no such vertex, get_unique_vertex_by_label returns false.
If there are two or more vertices with label l, get_unique_vertex_by_label reports an error.

Example:

(%i1) load ("graphs")$
(%i2) g: create_graph ([[0, "Zero"], [1, "One"], [2, "Other"], [3, "Other"]], [])$
(%i3) get_unique_vertex_by_label ("Zero", g);
(%o3) 0
(%i4) get_unique_vertex_by_label ("Two", g);
(%o4) false
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(%i5) errcatch (get_unique_vertex_by_label ("Other", g));
get_unique_vertex_by_label: two or more vertices have the same label "Other"
(%o5) []

get_all_vertices_by_label (l, gr) [Function]
Returns all vertices, if any, which have the label l in graph gr.
If there are no such vertices, get_all_vertices_by_label returns an empty list [].
Example:
(%i1) load ("graphs")$
(%i2) g: create_graph ([[0, "Zero"], [1, "One"], [2, "Other"], [3, "Other"]], [])$
(%i3) get_all_vertices_by_label ("Zero", g);
(%o3) [0]
(%i4) get_all_vertices_by_label ("Two", g);
(%o4) []
(%i5) get_all_vertices_by_label ("Other", g);
(%o5) [2, 3]

graph_charpoly (gr, x) [Function]
Returns the characteristic polynomial (in variable x) of the graph gr.
Example:
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) graph_charpoly(p, x), factor;
5 4
(x - 3) (x - 1) (x + 2)
(%o3)

graph_center (gr) [Function]
Returns the center of the graph gr.
Example:
(%i1) load ("graphs")$
(%i2) g : grid_graph(5,5)$
(%i3) graph_center(g);
(%o3) [12]

graph_eigenvalues (gr) [Function]
Returns the eigenvalues of the graph gr. The function returns eigenvalues in the same format as maxima eigenvalues function.
Example:
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) graph_eigenvalues(p);
(%o3) [[3, - 2, 1], [1, 4, 5]]

graph_periphery (gr) [Function]
Returns the periphery of the graph gr.
Example:
(%i1) load ("graphs")$
(%i2) g : grid_graph(5,5)$
(%i3) graph_periphery(g);  
   [24, 20, 4, 0]

**graph_size (gr)**  
Returns the number of edges in the graph gr.

Example:

```lisp
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) graph_size(p);  
   15
```

**graph_order (gr)**  
Returns the number of vertices in the graph gr.

Example:

```lisp
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) graph_order(p);  
   10
```

**girth (gr)**  
Returns the length of the shortest cycle in gr.

Example:

```lisp
(%i1) load ("graphs")$
(%i2) g : heawood_graph()$
(%i3) girth(g);  
   6
```

**hamilton_cycle (gr)**  
Returns the Hamilton cycle of the graph gr or an empty list if gr is not hamiltonian.

Example:

```lisp
(%i1) load ("graphs")$
(%i2) c : cube_graph(3)$
(%i3) hc : hamilton_cycle(c);  
   [7, 3, 2, 6, 4, 0, 1, 5, 7]
(%i4) draw_graph(c, show_edges=vertices_to_cycle(hc))$
```

**hamilton_path (gr)**  
Returns the Hamilton path of the graph gr or an empty list if gr does not have a Hamilton path.

Example:

```lisp
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) hp : hamilton_path(p);  
   [0, 5, 7, 2, 1, 6, 8, 3, 4, 9]
(%i4) draw_graph(p, show_edges=vertices_to_path(hp))$
```
**isomorphism** \((gr1, gr2)\)  
Returns a an isomorphism between graphs/digraphs \(gr1\) and \(gr2\). If \(gr1\) and \(gr2\) are not isomorphic, it returns an empty list.

Example:
```lisp
(%i1) load ("graphs")$
(%i2) clk5:complement_graph(line_graph(complete_graph(5)))$
(%i3) isomorphism(clk5, petersen_graph());
(%o3) [9 -> 0, 2 -> 1, 6 -> 2, 5 -> 3, 0 -> 4, 1 -> 5, 3 -> 6, 4 -> 7, 7 -> 8, 8 -> 9]
```

**in_neighbors** \((v, gr)\)  
Returns the list of in-neighbors of the vertex \(v\) in the directed graph \(gr\).

Example:
```lisp
(%i1) load ("graphs")$
(%i2) p : path_digraph(3)$
(%i3) in_neighbors(2, p);
(%o3) [1]
(%i4) out_neighbors(2, p);
(%o4) []
```

**is_biconnected** \((gr)\)  
Returns true if \(gr\) is 2-connected and false otherwise.

Example:
```lisp
(%i1) load ("graphs")$
(%i2) is_biconnected(cycle_graph(5));
(%o2) true
(%i3) is_biconnected(path_graph(5));
(%o3) false
```

**is_bipartite** \((gr)\)  
Returns true if \(gr\) is bipartite (2-colorable) and false otherwise.

Example:
```lisp
(%i1) load ("graphs")$
(%i2) is_bipartite(petersen_graph());
(%o2) false
(%i3) is_bipartite(heawood_graph());
(%o3) true
```

**is_connected** \((gr)\)  
Returns true if the graph \(gr\) is connected and false otherwise.

Example:
```lisp
(%i1) load ("graphs")$
(%i2) is_connected(graph_union(cycle_graph(4), path_graph(3)));
(%o2) false
```
is_digraph (gr)

Returns true if gr is a directed graph and false otherwise.

Example:

(%i1) load ("graphs")$
(%i2) is_digraph(path_graph(5));
(%o2) false
(%i3) is_digraph(path_digraph(5));
(%o3) true

is_edge_in_graph (e, gr)

Returns true if e is an edge (arc) in the (directed) graph g and false otherwise.

Example:

(%i1) load ("graphs")$
(%i2) c4 : cycle_graph(4)$
(%i3) is_edge_in_graph([2,3], c4);
(%o3) true
(%i4) is_edge_in_graph([3,2], c4);
(%o4) true
(%i5) is_edge_in_graph([2,4], c4);
(%o5) false
(%i6) is_edge_in_graph([3,2], cycle_digraph(4));
(%o6) false

is_graph (gr)

Returns true if gr is a graph and false otherwise.

Example:

(%i1) load ("graphs")$
(%i2) is_graph(path_graph(5));
(%o2) true
(%i3) is_graph(path_digraph(5));
(%o3) false

is_graph_or_digraph (gr)

Returns true if gr is a graph or a directed graph and false otherwise.

Example:

(%i1) load ("graphs")$
(%i2) is_graph_or_digraph(path_graph(5));
(%o2) true
(%i3) is_graph_or_digraph(path_digraph(5));
(%o3) true

is_isomorphic (gr1, gr2)

Returns true if graphs/digraphs gr1 and gr2 are isomorphic and false otherwise.

See also isomorphism.

Example:

(%i1) load ("graphs")$
is_planar \( (gr) \) [Function]

Returns \texttt{true} if \( gr \) is a planar graph and \texttt{false} otherwise.

The algorithm used is the Demoucron’s algorithm, which is a quadratic time algorithn.

Example:

\begin{verbatim}
(%i1) load ("graphs")
(%i2) is_planar(dodecahedron_graph());
   (%o2) true
(%i3) is_planar(petersen_graph());
   (%o3) false
(%i4) is_planar(petersen_graph(10,2));
   (%o4) true
\end{verbatim}

is_sconnected \( (gr) \) [Function]

Returns \texttt{true} if the directed graph \( gr \) is strongly connected and \texttt{false} otherwise.

Example:

\begin{verbatim}
(%i1) load ("graphs")
(%i2) is_sconnected(cycle_digraph(5));
   (%o2) true
(%i3) is_sconnected(path_digraph(5));
   (%o3) false
\end{verbatim}

is_vertex_in_graph \( (v, gr) \) [Function]

Returns \texttt{true} if \( v \) is a vertex in the graph \( g \) and \texttt{false} otherwise.

Example:

\begin{verbatim}
(%i1) load ("graphs")
(%i2) c4 : cycle_graph(4)$
(%i3) is_vertex_in_graph(0, c4);
   (%o3) true
(%i4) is_vertex_in_graph(6, c4);
   (%o4) false
\end{verbatim}

is_tree \( (gr) \) [Function]

Returns \texttt{true} if \( gr \) is a tree and \texttt{false} otherwise.

Example:

\begin{verbatim}
(%i1) load ("graphs")
(%i2) is_tree(random_tree(4));
   (%o2) true
(%i3) is_tree(graph_union(random_tree(4), random_tree(5)));
   (%o3) false
\end{verbatim}
laplacian_matrix (gr)  
Returns the laplacian matrix of the graph gr.

Example:
(%i1) load ("graphs")$
(%i2) laplacian_matrix(cycle_graph(5));
(%o2)
\[
\begin{bmatrix}
2 & -1 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
-1 & 0 & 0 & -1 & 2 \\
\end{bmatrix}
\]

max_clique (gr)  
Returns a maximum clique of the graph gr.

Example:
(%i1) load ("graphs")$
(%i2) g : random_graph(100, 0.5)$
(%i3) max_clique(g);
(%o3) [6, 12, 31, 36, 52, 59, 62, 63, 80]

max_degree (gr)  
Returns the maximal degree of vertices of the graph gr and a vertex of maximal degree.

Example:
(%i1) load ("graphs")$
(%i2) g : random_graph(100, 0.02)$
(%i3) max_degree(g);
(%o3) [6, 79]
(%i4) vertex_degree(95, g);
(%o4) 2

max_flow (net, s, t)  
Returns a maximum flow through the network net with the source s and the sink t.
The function returns the value of the maximal flow and a list representing the weights of the arcs in the optimal flow.

Example:
(%i1) load ("graphs")$
(%i2) net : create_graph(  
    [1,2,3,4,5,6],  
    [[[1,2], 1.0],  
      [[1,3], 0.3],  
      [[2,4], 0.2],  
      [[2,5], 0.3],  
    )
[[3,4], 0.1],
[[3,5], 0.1],
[[4,6], 1.0],
[[5,6], 1.0]],
directed=true)$
(%i3) [flow_value, flow] : max_flow(net, 1, 6);
(%o3) [0.7, [[[1, 2], 0.5], [[1, 3], 0.2], [[2, 4], 0.2],
[[2, 5], 0.3], [[3, 4], 0.1], [[3, 5], 0.1], [[4, 6], 0.3],
[[5, 6], 0.4]]]
(%i4) fl : 0$
(%i5) for u in out_neighbors(1, net)
do fl : fl + assoc([1, u], flow)$
(%i6) fl;
(%o6) 0.7

max_independent_set (gr) [Function]
Returns a maximum independent set of the graph gr.
Example:
(%i1) load ("graphs")$
(%i2) d : dodecahedron_graph()$
(%i3) mi : max_independent_set(d);
(%o3) [0, 3, 5, 9, 10, 11, 18, 19]
(%i4) draw_graph(d, show_vertices=mi)$

max_matching (gr) [Function]
Returns a maximum matching of the graph gr.
Example:
(%i1) load ("graphs")$
(%i2) d : dodecahedron_graph()$
(%i3) m : max_matching(d);
(%o3) [[[5, 7], [8, 9], [6, 10], [14, 19], [13, 18], [12, 17],
[11, 16], [0, 15], [3, 4], [1, 2]]
(%i4) draw_graph(d, show_edges=m)$

min_degree (gr) [Function]
Returns the minimum degree of vertices of the graph gr and a vertex of minimum
degree.
Example:
(%i1) load ("graphs")$
(%i2) g : random_graph(100, 0.1)$
(%i3) min_degree(g);
(%o3) [3, 49]
(%i4) vertex_degree(21, g);
(%o4) 9

min_edge_cut (gr) [Function]
Returns the minimum edge cut in the graph gr.
See also edge_connectivity.
min_vertex_cover (gr)  
Returns the minimum vertex cover of the graph gr.

min_vertex_cut (gr)  
Returns the minimum vertex cut in the graph gr.
See also vertex_connectivity.

minimum_spanning_tree (gr)  
Returns the minimum spanning tree of the graph gr.

Example:
(%i1) load ("graphs")$
(%i2) g : graph_product(path_graph(10), path_graph(10))$
(%i3) t : minimum_spanning_tree(g)$
(%i4) draw_graph(g, show_edges=edges(t))$

neighbors (v, gr)  
Returns the list of neighbors of the vertex v in the graph gr.

Example:
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) neighbors(3, p);
(%o3) [4, 8, 2]

odd_girth (gr)  
Returns the length of the shortest odd cycle in the graph gr.

Example:
(%i1) load ("graphs")$
(%i2) g : graph_product(cycle_graph(4), cycle_graph(7))$
(%i3) girth(g);
(%o3) 4
(%i4) odd_girth(g);
(%o4) 7

out_neighbors (v, gr)  
Returns the list of out-neighbors of the vertex v in the directed graph gr.

Example:
(%i1) load ("graphs")$
(%i2) p : path_digraph(3)$
(%i3) in_neighbors(2, p);
(%o3) [1]
(%i4) out_neighbors(2, p);
(%o4) []

planar_embedding (gr)  
Returns the list of facial walks in a planar embedding of gr and false if gr is not a planar graph.
The graph gr must be biconnected.
The algorithm used is the Demoucron's algorithm, which is a quadratic time algorithm.

Example:

(%i1) load ("graphs")$
(%i2) planar_embedding(grid_graph(3,3));
(%o2) [[3, 6, 7, 8, 5, 2, 1, 0], [4, 3, 0, 1], [3, 4, 7, 6],
     [8, 7, 4, 5], [1, 2, 5, 4]]

print_graph (gr) [Function]
Prints some information about the graph gr.

Example:

(%i1) load ("graphs")$
(%i2) c5 : cycle_graph(5)$
(%i3) print_graph(c5)$
Graph on 5 vertices with 5 edges.
Adjacencies:
4 : 0 3
3 : 4 2
2 : 3 1
1 : 2 0
0 : 4 1
(%i4) dc5 : cycle_digraph(5)$
(%i5) print_graph(dc5)$
Digraph on 5 vertices with 5 arcs.
Adjacencies:
4 : 0
3 : 4
2 : 3
1 : 2
0 : 1
(%i6) out_neighbors(0, dc5);
(%o6) [1]

radius (gr) [Function]
Returns the radius of the graph gr.

Example:

(%i1) load ("graphs")$
(%i2) radius(dodecahedron_graph());
(%o2) 5

set_edge_weight (e, w, gr) [Function]
Assigns the weight w to the edge e in the graph gr.

Example:

(%i1) load ("graphs")$
(%i2) g : create_graph([1, 2], [[[1,2], 1.2]]);
(%i3) get_edge_weight([1,2], g);
(%o3) 1.2
(%i4) set_edge_weight([1,2], 2.1, g);  
(%o4) done
(%i5) get_edge_weight([1,2], g);  
(%o5) 2.1

**set_vertex_label** \((v, l, gr)\)

Assigns the label \(l\) to the vertex \(v\) in the graph \(gr\).

A label may be any Maxima expression, and two or more vertices may have the same label.

Example:

```maxima
(%i1) load ("graphs")$
(%i2) g : create_graph([[1, "One"], [2, "Two"]], [[1, 2]])$
(%i3) get_vertex_label(1, g);
(%o3) One
(%i4) set_vertex_label(1, "oNE", g);
(%o4) done
(%i5) get_vertex_label(1, g);
(%o5) oNE
```

**shortest_path** \((u, v, gr)\)

Returns the shortest path from \(u\) to \(v\) in the graph \(gr\).

Example:

```maxima
(%i1) load ("graphs")$
(%i2) d : dodecahedron_graph()$
(%i3) path : shortest_path(0, 7, d);
(%o3) [0, 1, 19, 13, 7]
(%i4) draw_graph(d, show_edges=vertices_to_path(path))$
```

**shortest_weighted_path** \((u, v, gr)\)

Returns the length of the shortest weighted path and the shortest weighted path from \(u\) to \(v\) in the graph \(gr\).

The length of a weighted path is the sum of edge weights of edges in the path. If an edge has no weight, then it has a default weight 1.

Example:

```maxima
(%i1) load ("graphs")$
(%i2) g: petersen_graph(20, 2)$
(%i3) for e in edges(g) do set_edge_weight(e, random(1.0), g)$
(%i4) shortest_weighted_path(0, 10, g);  
(%o4) [2.575143920268482, [0, 20, 38, 36, 34, 32, 30, 10]]
```

**strong_components** \((gr)\)

Returns the strong components of a directed graph \(gr\).
Example:

```lisp
(%i1) load ("graphs")$
(%i2) t : random_tournament(4)$
(%i3) strong_components(t);
(%o3) [[1], [0], [2], [3]]
(%i4) vertex_out_degree(3, t);
(%o4) 3
```

`topological_sort (dag)` [Function]

Returns a topological sorting of the vertices of a directed graph `dag` or an empty list if `dag` is not a directed acyclic graph.

Example:

```lisp
(%i1) load ("graphs")$
(%i2) g:create_graph(
   [1, 2, 3, 4, 5],
   [[1, 2], [2, 5], [5, 3],
    [5, 4], [3, 4], [1, 3]
   ],
   directed=true)$
(%i3) topological_sort(g);
(%o3) [1, 2, 5, 3, 4]
```

`vertex_connectivity (g)` [Function]

Returns the vertex connectivity of the graph `g`.

See also `min_vertex_cut`.

`vertex_degree (v, gr)` [Function]

Returns the degree of the vertex `v` in the graph `gr`.

`vertex_distance (u, v, gr)` [Function]

Returns the length of the shortest path between `u` and `v` in the (directed) graph `gr`.

Example:

```lisp
(%i1) load ("graphs")$
(%i2) d : dodecahedron_graph()$
(%i3) vertex_distance(0, 7, d);
(%o3) 4
(%i4) shortest_path(0, 7, d);
(%o4) [0, 1, 19, 13, 7]
```

`vertex_eccentricity (v, gr)` [Function]

Returns the eccentricity of the vertex `v` in the graph `gr`.

Example:

```lisp
(%i1) load ("graphs")$
(%i2) g:cycle_graph(7)$
(%i3) vertex_eccentricity(0, g);
(%o3) 3
```
vertex_in_degree (v, gr)          [Function]
Returns the in-degree of the vertex \( v \) in the directed graph \( gr \).

Example:

\[
\text{(%i1) load ("graphs")}\$
\text{(%i2) p5 : path_digraph(5)}$
\text{(%i3) print_graph(p5)}$
\]
Digraph on 5 vertices with 4 arcs.
Adjacencies:
  4 :  3 : 4
  2 :  3
  1 :  2
  0 :  1
\text{(%i4) vertex_in_degree(4, p5);} \\
\text{(%o4) 1} \\
\text{(%i5) in_neighbors(4, p5);} \\
\text{(%o5) [3]}

vertex_out_degree (v, gr)          [Function]
Returns the out-degree of the vertex \( v \) in the directed graph \( gr \).

Example:

\[
\text{(%i1) load ("graphs")}\$
\text{(%i2) t : random_tournament(10)}$
\text{(%i3) vertex_out_degree(0, t);} \\
\text{(%o3) 2} \\
\text{(%i4) out_neighbors(0, t);} \\
\text{(%o4) [7, 1]}

vertices (gr)          [Function]
Returns the list of vertices in the graph \( gr \).

Example:

\[
\text{(%i1) load ("graphs")}\$
\text{(%i2) vertices(complete_graph(4));} \\
\text{(%o2) [3, 2, 1, 0]}

vertex_coloring (gr)          [Function]
Returns an optimal coloring of the vertices of the graph \( gr \).
The function returns the chromatic number and a list representing the coloring of the vertices of \( gr \).

Example:

\[
\text{(%i1) load ("graphs")}\$
\text{(%i2) p:petersen_graph()}$
\text{(%i3) vertex_coloring(p);} \\
\text{(%o3) [3, [[0, 2], [1, 3], [2, 2], [3, 3], [4, 1], [5, 3], [6, 1], [7, 1], [8, 2], [9, 2]]]}

wiener_index (gr)
  Returns the Wiener index of the graph gr.
  Example:
  
  (%i2) wiener_index(dodecahedron_graph());
  (%o2) 500

63.2.3 Modifying graphs

add_edge (e, gr)
  Adds the edge e to the graph gr.
  Example:
  
  (%i1) load ("graphs")$
  (%i2) p : path_graph(4)$
  (%i3) neighbors(0, p);
  (%o3) [1]
  (%i4) add_edge([0,3], p);
  (%o4) done
  (%i5) neighbors(0, p);
  (%o5) [3, 1]

add_edges (e_list, gr)
  Adds all edges in the list e_list to the graph gr.
  Example:
  
  (%i1) load ("graphs")$
  (%i2) g : empty_graph(3)$
  (%i3) add_edges([[0,1],[1,2]], g)$
  (%i4) print_graph(g)$
  Graph on 3 vertices with 2 edges.
  Adjacencies:
  2 : 1
  1 : 2 0
  0 : 1

add_vertex (v, gr)
  Adds the vertex v to the graph gr.
  Example:
  
  (%i1) load ("graphs")$
  (%i2) g : path_graph(2)$
  (%i3) add_vertex(2, g)$
  (%i4) print_graph(g)$
  Graph on 3 vertices with 1 edges.
  Adjacencies:
  2 :
  1 : 0
  0 : 1
add_vertices (v_list, gr)  
Adds all vertices in the list v_list to the graph gr. A vertex may be any integer, and v_list may contain vertices in any order.

connect_vertices (v_list, u_list, gr)  
Connects all vertices from the list v_list with the vertices in the list u_list in the graph gr.

v_list and u_list can be single vertices or lists of vertices.

Example:

(%i1) load ("graphs")$
(%i2) g : empty_graph(4)$
(%i3) connect_vertices(0, [1,2,3], g)$
(%i4) print_graph(g)$
Graph on 4 vertices with 3 edges.
Adjacencies:
  3 : 0
  2 : 0
  1 : 0
  0 : 3 2 1

contract_edge (e, gr)  
Contracts the edge e in the graph gr.

Example:

(%i1) load ("graphs")$
(%i2) g : create_graph(
    8, [[0,3],[1,3],[2,3],[3,4],[4,5],[4,6],[4,7]])$
(%i3) print_graph(g)$
Graph on 8 vertices with 7 edges.
Adjacencies:
  7 : 4
  6 : 4
  5 : 4
  4 : 7 6 5 3
  3 : 4 2 1 0
  2 : 3
  1 : 3
  0 : 3
(%i4) contract_edge([3,4], g)$
(%i5) print_graph(g)$
Graph on 7 vertices with 6 edges.
Adjacencies:
  7 : 3
  6 : 3
  5 : 3
  3 : 5 6 7 2 1 0
  2 : 3
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1 : 3
0 : 3

remove_edge(e, gr) [Function]
Removes the edge e from the graph gr.
Example:
(%i1) load("graphs")$
(%i2) c3 : cycle_graph(3)$
(%i3) remove_edge([0,1], c3)$
(%i4) print_graph(c3)$
Graph on 3 vertices with 2 edges.
Adjacencies:
  2 : 0 1
  1 : 2
  0 : 2

remove_vertex(v, gr) [Function]
Removes the vertex v from the graph gr.

63.2.4 Reading and writing to files
dimacs_export [Function]
dimacs_export(gr, fl)
dimacs_export(gr, fl, comment1, ..., commentn)
Exports the graph into the file fl in the DIMACS format. Optional comments will be added to the top of the file.
dimacs_import(fl) [Function]
Returns the graph from file fl in the DIMACS format.

graph6_decode(str) [Function]
Returns the graph encoded in the graph6 format in the string str.
graph6_encode(gr) [Function]
Returns a string which encodes the graph gr in the graph6 format.
graph6_export(gr_list, fl) [Function]
Exports graphs in the list gr_list to the file fl in the graph6 format.
graph6_import(fl) [Function]
Returns a list of graphs from the file fl in the graph6 format.

sparse6_decode(str) [Function]
Returns the graph encoded in the sparse6 format in the string str.
sparse6_encode(gr) [Function]
Returns a string which encodes the graph gr in the sparse6 format.
sparse6_export(gr_list, fl) [Function]
Exports graphs in the list gr_list to the file fl in the sparse6 format.
sparse6_import(fl) [Function]
Returns a list of graphs from the file fl in the sparse6 format.
63.2.5 Visualization

draw_graph

[Function]

draw_graph (graph)
draw_graph (graph, option1, ..., optionk)

Draws the graph using the Chapter 53 [draw-pkg], page 815, package.

The algorithm used to position vertices is specified by the optional argument program.
The default value is program=spring_embedding. draw_graph can also use the
graphviz programs for positioning vertices, but graphviz must be installed separately.

Example 1:

```
(%i1) load ("graphs")
(%i2) g:grid_graph(10,10)
(%i3) m:max_matching(g)
(%i4) draw_graph(g,
spring_embedding_depth=100,
show_edges=m, edge_type=dots,
vertex_size=0)
```

Example 2:

```
(%i1) load ("graphs")
(%i2) g:create_graph(16,
[0,1],[1,3],[2,3],[0,2],[3,4],[2,4],
[5,6],[6,4],[4,7],[6,7],[7,8],[7,10],[7,11],
[8,10],[11,10],[8,9],[11,12],[9,15],[12,13],
[10,14],[15,14],[13,14])
(%i3) t:minimum_spanning_tree(g)
(%i4) draw_graph(g,
show_edges=edges(t),
show_edge_width=4,
show_edge_color=green,
vertex_type=filled_square,
vertex_size=2)
```

Example 3:

```
(%i1) load ("graphs")
(%i2) g:create_graph(16,
[0,1],[1,3],[2,3],[0,2],[3,4],[2,4],
[5,6],[6,4],[4,7],[6,7],[7,8],[7,10],[7,11],
[8,10],[11,10],[8,9],[11,12],[9,15],[12,13],
[10,14],[15,14],[13,14])
(%i3) mi : max_independent_set(g)
```
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 (%i4) draw_graph(
    g,
    show_vertices=mi,
    show_vertex_type=filled_up_triangle,
    show_vertex_size=2,
    edge_color=cyan,
    edge_width=3,
    show_id=true,
    text_color=brown
  )$

Example 4:

 (%i1) load ("graphs")$
 (%i2) net : create_graph(
    [0,1,2,3,4,5],
    [
      [[0,1], 3], [[0,2], 2],
      [[1,3], 1], [[1,4], 3],
      [[2,3], 2], [[2,4], 2],
      [[4,5], 2], [[3,5], 2]
    ],
    directed=true
  )$
 (%i3) draw_graph(
    net,
    show_weight=true,
    vertex_size=0,
    show_vertices=[0,5],
    show_vertex_type=filled_square,
    head_length=0.2,
    head_angle=10,
    edge_color="dark-green",
    text_color=blue
  )$

Example 5:

 (%i1) load("graphs")$
 (%i2) g: petersen_graph(20, 2);
 (%o2) GRAPH
 (%i3) draw_graph(g, redraw=true, program=planar_embedding);
 (%o3) done

Example 6:

 (%i1) load("graphs")$
 (%i2) t: tutte_graph();
 (%o2) GRAPH
 (%i3) draw_graph(t, redraw=true,
    fixed_vertices=[1,2,3,4,5,6,7,8,9]);
 (%o3) done
**draw_graph_program**  
[Option variable]  
Default value: **spring_embedding**  
The default value for the program used to position vertices in **draw_graph** program.

**show_id**  
[draw_graph option]  
Default value: **false**  
If **true** then ids of the vertices are displayed.

**show_label**  
[draw_graph option]  
Default value: **false**  
If **true** then labels of the vertices are displayed.

**label_alignment**  
[draw_graph option]  
Default value: **center**  
Determines how to align the labels/ids of the vertices. Can be **left**, **center** or **right**.

**show_weight**  
[draw_graph option]  
Default value: **false**  
If **true** then weights of the edges are displayed.

**vertex_type**  
[draw_graph option]  
Default value: **circle**  
Defines how vertices are displayed. See the **point_type** option for the **draw** package for possible values.

**vertex_size**  
[draw_graph option]  
The size of vertices.

**vertex_color**  
[draw_graph option]  
The color used for displaying vertices.

**show_vertices**  
[draw_graph option]  
Default value: **[]**  
Display selected vertices in the using a different color.

**show_vertex_type**  
[draw_graph option]  
Defines how vertices specified in **show_vertices** are displayed. See the **point_type** option for the **draw** package for possible values.

**show_vertex_size**  
[draw_graph option]  
The size of vertices in **show_vertices**.

**show_vertex_color**  
[draw_graph option]  
The color used for displaying vertices in the **show_vertices** list.

**vertex_partition**  
[draw_graph option]  
Default value: **[]**  
A partition [[[v1,v2,...],...,[vk,...,vn]] of the vertices of the graph. The vertices of each list in the partition will be drawn in a different color.
**vertex_coloring** [draw_graph option]
Specifications coloring of the vertices. The coloring \( \text{col} \) must be specified in the format as returned by \text{vertex_coloring}.

**edge_color** [draw_graph option]
The color used for displaying edges.

**edge_width** [draw_graph option]
The width of edges.

**edge_type** [draw_graph option]
Defines how edges are displayed. See the \text{line_type} option for the \text{draw} package.

**show_edges** [draw_graph option]
Display edges specified in the list \( e_{\text{list}} \) using a different color.

**show_edge_color** [draw_graph option]
The color used for displaying edges in the \text{show_edges} list.

**show_edge_width** [draw_graph option]
The width of edges in \text{show_edges}.

**show_edge_type** [draw_graph option]
Defines how edges in \text{show_edges} are displayed. See the \text{line_type} option for the \text{draw} package.

**edge_partition** [draw_graph option]
A partition \([e_1, e_2, ...], ..., [e_k, ..., e_m]\) of edges of the graph. The edges of each list in the partition will be drawn using a different color.

**edge_coloring** [draw_graph option]
The coloring of edges. The coloring must be specified in the format as returned by the function \text{edge_coloring}.

**redraw** [draw_graph option]
Default value: \text{false}
If \text{true}, vertex positions are recomputed even if the positions have been saved from a previous drawing of the graph.

**head_angle** [draw_graph option]
Default value: 15
The angle for the arrows displayed on arcs (in directed graphs).

**head_length** [draw_graph option]
Default value: 0.1
The length for the arrows displayed on arcs (in directed graphs).

**spring_embedding_depth** [draw_graph option]
Default value: 50
The number of iterations in the spring embedding graph drawing algorithm.
terminal  [draw_graph option]
The terminal used for drawing (see the terminal option in the draw package).

file_name  [draw_graph option]
The filename of the drawing if terminal is not screen.

program  [draw_graph option]
Defines the program used for positioning vertices of the graph. Can be one of the graphviz programs (dot, neato, twopi, circ, fdp), circular, spring_embedding or planar_embedding. planar_embedding is only available for 2-connected planar graphs. When program=spring_embedding, a set of vertices with fixed position can be specified with the fixed_vertices option.

fixed_vertices  [draw_graph option]
Specifies a list of vertices which will have positions fixed along a regular polygon. Can be used when program=spring_embedding.

vertices_to_path (v_list)  [Function]
Converts a list v_list of vertices to a list of edges of the path defined by v_list.

vertices_to_cycle (v_list)  [Function]
Converts a list v_list of vertices to a list of edges of the cycle defined by v_list.
64 grobner

64.1 Introduction to grobner

grobner is a package for working with Groebner bases in Maxima.
To use the following functions you must load the grobner.lisp package.

\[ \text{load("grobner")}; \]
A demo can be started by

\[ \text{demo("grobner.demo")}; \]
or
\[ \text{batch("grobner.demo")} \]

Some of the calculation in the demo will take a lot of time therefore the output
grobner-demo.output of the demo can be found in the same directory as the demo file.

64.1.1 Notes on the grobner package

The package was written by
Marek Rychlik

http://alamos.math.arizona.edu

and is released 2002-05-24 under the terms of the General Public License(GPL) (see file
grobner.lisp). This documentation was extracted from the files
README, grobner.lisp, grobner.demo, grobner-demo.output

by Günter Nowak. Suggestions for improvement of the documentation can be discussed
at the maxima-mailing-list maxima@math.utexas.edu. The code is a little bit out of date now. Modern implementation use the fast \( F_4 \) algorithm described in

A new efficient algorithm for computing Gröbner bases (F4)
Jean-Charles Faugère
LIP6/CNRS Université Paris VI
January 20, 1999

64.1.2 Implementations of admissible monomial orders in grobner

- **lex**
  pure lexicographic, default order for monomial comparisons
- **grlex**
  total degree order, ties broken by lexicographic
- **grevlex**
  total degree, ties broken by reverse lexicographic
- **invlex**
  inverse lexicographic order
64.2 Functions and Variables for grobner

64.2.1 Global switches for grobner

poly_monomial_order [Option variable]
  Default value: lex
  This global switch controls which monomial order is used in polynomial and Groebner
  Bases calculations. If not set, lex will be used.

poly_coefficient_ring [Option variable]
  Default value: expression_ring
  This switch indicates the coefficient ring of the polynomials that will be used in
grobner calculations. If not set, maxima’s general expression ring will be used. This
variable may be set to ring_of_integers if desired.

poly_primary_elimination_order [Option variable]
  Default value: false
  Name of the default order for eliminated variables in elimination-based functions. If
not set, lex will be used.

poly_secondary_elimination_order [Option variable]
  Default value: false
  Name of the default order for kept variables in elimination-based functions. If not
set, lex will be used.

poly_elimination_order [Option variable]
  Default value: false
  Name of the default elimination order used in elimination calculations. If set, it
overrides the settings in variables poly_primary_elimination_order and poly_
secondary_elimination_order. The user must ensure that this is a true elimination
order valid for the number of eliminated variables.

poly_return_term_list [Option variable]
  Default value: false
  If set to true, all functions in this package will return each polynomial as a list of
terms in the current monomial order rather than a maxima general expression.

poly_grobner_debug [Option variable]
  Default value: false
  If set to true, produce debugging and tracing output.

poly_grobner_algorithm [Option variable]
  Default value: buchberger
  Possible values:
  
  • buchberger
  • parallel_buchberger
  • gebauer_moeller

  The name of the algorithm used to find the Groebner Bases.
poly_top_reduction_only  
  [Option variable]
  Default value: false
  If not false, use top reduction only whenever possible. Top reduction means that division algorithm stops after the first reduction.

64.2.2 Simple operators in grobner

poly_add, poly_subtract, poly_multiply and poly_expt are the arithmetical operations on polynomials. These are performed using the internal representation, but the results are converted back to the maxima general form.

poly_add (poly1, poly2, varlist)  
  Adds two polynomials poly1 and poly2.

  (%i1) poly_add(z+x^2*y,x-z,[x,y,z]);
  2
  (%o1) x y + x

poly_subtract (poly1, poly2, varlist)  
  Subtracts a polynomial poly2 from poly1.

  (%i1) poly_subtract(z+x^2*y,x-z,[x,y,z]);
  2
  (%o1) 2 z + x y - x

poly_multiply (poly1, poly2, varlist)  
  Returns the product of polynomials poly1 and poly2.

  (%i2) poly_multiply(z+x^2*y,x-z,[x,y,z])-(z+x^2*y)*(x-z),expand;
  (%o1) 0

poly_s_polynomial (poly1, poly2, varlist)  
  Returns the syzygy polynomial (S-polynomial) of two polynomials poly1 and poly2.

poly_primitive_part (poly1, varlist)  
  Returns the polynomial poly divided by the GCD of its coefficients.

  (%i11) poly_primitive_part(35*y+21*x,[x,y]);
  5
  (%o1) y + 3 x

poly_normalize (poly, varlist)  
  Returns the polynomial poly divided by the leading coefficient. It assumes that the division is possible, which may not always be the case in rings which are not fields.

64.2.3 Other functions in grobner

poly_expand (poly, varlist)  
  This function parses polynomials to internal form and back. It is equivalent to expand(poly) if poly parses correctly to a polynomial. If the representation is not compatible with a polynomial in variables varlist, the result is an error. It can be
used to test whether an expression correctly parses to the internal representation. The following examples illustrate that indexed and transcendental function variables are allowed.

```
(%i1) poly_expand((x-y)*(y+x),[x,y]);
   2   2
(%o1)       x - y
(%i2) poly_expand((y+x)^2,[x,y]);
   2
(%o2)      y + 2 x y + x
(%i3) poly_expand((y+x)^5,[x,y]);
   5  4  2  3  3  2  4  5
(%o3) y + 5 x y + 10 x y + 10 x y + 5 x y + x
(%i4) poly_expand(-1-x*exp(y)+x^2/sqrt(y),[x]);
 2
(%o4) - x %e + -------- - 1
     sqrt(y)
(%i5) poly_expand(-1-sin(x)^2+sin(x),[sin(x)]);
   2
(%o5) - sin(x) + sin(x) - 1
```

`poly_expt (poly, number, varlist)`

exponentitates `poly` by a positive integer `number`. If `number` is not a positive integer number an error will be raised.

```
(%i1) poly_expt(x-y,3,[x,y])-(x-y)^3,expand;
(%o1) 0
```

`poly_content (poly, varlist)`

`poly_content` extracts the GCD of its coefficients

```
(%i1) poly_content(35*y+21*x,[x,y]);
(%o1) 7
```

`poly_pseudo_divide (poly, polylist, varlist)`

Pseudo-divide a polynomial `poly` by the list of `n` polynomials `polylist`. Return multiple values. The first value is a list of quotients `a`. The second value is the remainder `r`. The third argument is a scalar coefficient `c`, such that `c * poly` can be divided by `polylist` within the ring of coefficients, which is not necessarily a field. Finally, the fourth value is an integer count of the number of reductions performed. The resulting objects satisfy the equation:

\[ c \cdot poly = \sum_{i=1}^{n} (a_i \cdot polylist_i) + r \]
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**poly_exact_divide** (poly1, poly2, varlist)  
Divide a polynomial poly1 by another polynomial poly2. Assumes that exact division with no remainder is possible. Returns the quotient.

**poly_normal_form** (poly, polylist, varlist)  
poly_normal_form finds the normal form of a polynomial poly with respect to a set of polynomials polylist.

**poly_buchberger_criterion** (polylist, varlist)  
Returns true if polylist is a Groebner basis with respect to the current term order, by using the Buchberger criterion: for every two polynomials h1 and h2 in polylist the S-polynomial S(h1, h2) reduces to 0 modulo polylist.

**poly_buchberger** (polylist_fl varlist)  
poly_buchberger performs the Buchberger algorithm on a list of polynomials and returns the resulting Groebner basis.

### 64.2.4 Standard postprocessing of Groebner Bases

The **k-th elimination ideal** I_k of an ideal I over K[x_1,...,x_1] is I ∩ K[x_{k+1},...,x_n].

The **colon ideal** I : J is the ideal {h|∀w ∈ J : wh ∈ I}.

The ideal I : p^∞ is the ideal {h|∃n ∈ N : p^n h ∈ I}.

The ideal I : J^∞ is the ideal {h|∃n ∈ N, ∃p ∈ J : p^n h ∈ I}.

The **radical ideal** √I is the ideal {h|∃n ∈ N : h^n ∈ I}.

**poly_reduction** (polylist, varlist)  
poly_reduction reduces a list of polynomials polylist, so that each polynomial is fully reduced with respect to the other polynomials.

**poly_minimization** (polylist, varlist)  
Returns a sublist of the polynomial list polylist spanning the same monomial ideal as polylist but minimal, i.e. no leading monomial of a polynomial in the sublist divides the leading monomial of another polynomial.

**poly_normalize_list** (polylist, varlist)  
poly_normalize_list applies poly_normalize to each polynomial in the list. That means it divides every polynomial in a list polylist by its leading coefficient.

**poly_grobner** (polylist, varlist)  
Returns a Groebner basis of the ideal span by the polynomials polylist. Affected by the global flags.

**poly_reduced_grobner** (polylist, varlist)  
Returns a reduced Groebner basis of the ideal span by the polynomials polylist. Affected by the global flags.

**poly_depends_p** (poly, var, varlist)  
poly_depends tests whether a polynomial depends on a variable var.
poly_elimination_ideal (polylist, number, varlist)  [Function]
poly_elimination_ideal returns the grobner basis of the number-th elimination ideal of an ideal specified as a list of generating polynomials (not necessarily Groebner basis).

poly_colon_ideal (polylist1, polylist2, varlist)  [Function]
Returns the reduced Groebner basis of the colon ideal
\[ I(polylist1) : I(polylist2) \]
where polylist1 and polylist2 are two lists of polynomials.

poly_ideal_intersection (polylist1, polylist2, varlist)  [Function]
poly_ideal_intersection returns the intersection of two ideals.

poly_lcm (poly1, poly2, varlist)  [Function]
Returns the lowest common multiple of poly1 and poly2.

poly_gcd (poly1, poly2, varlist)  [Function]
Returns the greatest common divisor of poly1 and poly2.

    See also ezgcd, gcd, gcdex, and gcdivide.

Example:

    (%i1) p1:6*x^3+19*x^2+19*x+6;
    3  2
    (%o1) 6 x + 19 x + 19 x + 6
    (%i2) p2:6*x^5+13*x^4+12*x^3+13*x^2+6*x;
    5  4  3  2
    (%o2) 6 x + 13 x + 12 x + 13 x + 6 x
    (%i3) poly_gcd(p1, p2, [x]);
    2
    (%o3) 6 x + 13 x + 6

poly_grobner_equal (polylist1, polylist2, varlist)  [Function]
poly_grobner_equal tests whether two Groebner Bases generate the same ideal. Returns true if two lists of polynomials polylist1 and polylist2, assumed to be Groebner Bases, generate the same ideal, and false otherwise. This is equivalent to checking that every polynomial of the first basis reduces to 0 modulo the second basis and vice versa. Note that in the example below the first list is not a Groebner basis, and thus the result is false.

    (%i1) poly_grobner_equal([y+x,x-y],[x,y],[x,y]);
    (%o1) false

poly_grobner_subsetp (polylist1, polylist2, varlist)  [Function]
poly_grobner_subsetp tests whether an ideal generated by polylist1 is contained in the ideal generated by polylist2. For this test to always succeed, polylist2 must be a Groebner basis.

poly_grobner_member (poly, polylist, varlist)  [Function]
Returns true if a polynomial poly belongs to the ideal generated by the polynomial list polylist, which is assumed to be a Groebner basis. Returns false otherwise.
poly_grobner_member tests whether a polynomial belongs to an ideal generated by a list of polynomials, which is assumed to be a Groebner basis. Equivalent to normal_form being 0.

\textbf{poly_ideal_saturation1 (polylist, poly, varlist)} \quad \text{[Function]}
\begin{align*}
\text{Returns the reduced Groebner basis of the saturation of the ideal } & \\
\quad & I(polylist) : poly^\infty
\end{align*}

Geometrically, over an algebraically closed field, this is the set of polynomials in the ideal generated by polylist which do not identically vanish on the variety of poly.

\textbf{poly_ideal_saturation (polylist1, polylist2, varlist)} \quad \text{[Function]}
\begin{align*}
\text{Returns the reduced Groebner basis of the saturation of the ideal } & \\
\quad & I(polylist1) : I(polylist2)^\infty
\end{align*}

Geometrically, over an algebraically closed field, this is the set of polynomials in the ideal generated by polylist1 which do not identically vanish on the variety of polylist2.

\textbf{poly_ideal_polysaturation1 (polylist1, polylist2, varlist)} \quad \text{[Function]}
\begin{align*}
\text{polylist2 is a list of n polynomials } & [poly1,\ldots,polyn]. \text{ Returns the reduced Groebner basis of the ideal } \\
\quad & I(polylist1) : poly1^\infty : \ldots : polyn^\infty
\end{align*}

obtained by a sequence of successive saturations in the polynomials of the polynomial list polylist2 of the ideal generated by the polynomial list polylist1.

\textbf{poly_ideal_polysaturation (polylist, polylistlist, varlist)} \quad \text{[Function]}
\begin{align*}
\text{polylistlist is a list of n list of polynomials } & [polylist1,\ldots,polylistn]. \text{ Returns the reduced Groebner basis of the saturation of the ideal } \\
\quad & I(polylist) : I(polylist1)^\infty : \ldots : I(polylist_n)^\infty
\end{align*}

\textbf{poly_saturation_extension (poly, polylist, varlist1, varlist2)} \quad \text{[Function]}
poly_saturation_extension implements the famous Rabinowitz trick.

\textbf{poly_polysaturation_extension (poly, polylist, varlist1, varlist2)} \quad \text{[Function]}

65 hompack

65.1 Introduction to hompack

Hompack is a Common Lisp translation (via f2cl) of the Fortran library HOMPACK, as obtained from Netlib.

65.2 Functions and Variables for hompack

hompack_polsys (eqnlist, varlist [iflg1, epsbig, epssml, numrr])

Finds the roots of the system of polynomials in the variables varlist in the system of equations in eqnlist. The number of equations must match number of variables. Each equation must be a polynomial with variables in varlist. The coefficients must be real numbers.

The optional keyword arguments provide some control over the algorithm.

epsbig is the local error tolerance allowed by the path tracker, defaulting to 1e-4.

epssml is the accuracy desired for the final solution, defaulting to 1d-14.

numrr is the number of multiples of 1000 steps that will be tried before abandoning a path, defaulting to 10.

iflg1 defaulting to 0, controls the algorithm as follows:

0  If the problem is to be solved without calling polsys’ scaling routine, sclgnp, and without using the projective transformation.

1  If scaling but no projective transformation is to be used.

10 If no scaling but projective transformation is to be used.

11 If both scaling and projective transformation are to be used.

hompack_polsys returns a list. The elements of the list are:

retcode Indicates whether the solution is valid or not.

0  Solution found without problems

1  Solution succeeded but iflg2 indicates some issues with a root. (That is, iflg2 is not all ones.)

-1 NN, the declared dimension of the number of terms in the polynomials, is too small. (This should not happen.)

-2 MMAXT, the declared dimension for the internal coefficient and degree arrays, is too small. (This should not happen.)

-3 TTOTDG, the total degree of the equations, is too small. (This should not happen.)

-4 LENWK, the length of the internal real work array, is too small. (This should not happen.)
LENIWK, the length of the internal integer work array, is too small. (This should not happen.)

iflg1 is not 0 or 1, or 10 or 11. (This should not happen; an error should be thrown before polsys is called.)

roots The roots of the system of equations. This is in the same format as solve would return.

iflg2 A list containing information on how the path for the m'th root terminated:
1 Normal return
2 Specified error tolerance cannot be met. Increase epsbig and epssml and rerun.
3 Maximum number of steps exceeded. To track the path further, increase numrr and rerun the path. However, the path may be diverging, if the lambda value is near 1 and the roots values are large.
4 Jacobian matrix does not have full rank. The algorithm has failed (the zero curve of the homotopy map cannot be followed any further).
5 The tracking algorithm has lost the zero curve of the homotopy map and is not making progress. The error tolerances epsbig and epssml were too lenient. The problem should be restarted with smaller error tolerances.
6 The normal flow newton iteration in stepnf or rootnf failed to converge. The error tolerance epsbig may be too stringent.
7 Illegal input parameters, a fatal error.

lambda A list of the final lambda value for the m-th root, where lambda is the continuation parameter.

arclen A list of the arc length of the m-th root.

nfe A list of the number of jacobian matrix evaluations required to track the m-th root.

Here are some examples of using hompack_polsys.

(%i1) load(hompack)$
(%i2) hompack_polsys([x1^2-1, x2^2-2],[x1,x2]);
(%o2) [0,

[[x1 = -1.354505666901954e-16*%i)-0.9999999999999999,
  x2 = 3.52147935979316e-16*%i-1.414213562373095],
 [x1 = 1.0-5.536432658639868e-18*%i,
  x2 = (-4.213674137126362e-17*%i)-1.414213562373095],
 [x1 = (-9.475939894034927e-16*%i+1.414213562373095],
  x2 = 2.669654624736742e-16*%i+1.414213562373095],
The analytical solution can be obtained with solve:

\[
\text{solve([}x_1^2-1, x_2^2-2\text{],}\ [x_1, x_2])
\]

\[
\begin{align*}
\{&x_1 = 1, x_2 = -\sqrt{2}\}, \\
&\{x_1 = 1, x_2 = \sqrt{2}\}, \\
&\{x_1 = -1, x_2 = -\sqrt{2}\}, \\
&\{x_1 = -1, x_2 = \sqrt{2}\}
\end{align*}
\]

We see that \texttt{hompack\_polsys} returned the correct answer except that the roots are in a different order and there is a small imaginary part.

Another example, with corresponding solution from solve:

\[
\text{solve([}x_1^2+2*x_2^2+x_1*x_2 - 5, 2*x_1^2+2*x_2^2 + x_2-4\text{],}\ [x_1, x_2])
\]

\[
\begin{align*}
\{&x_1 = 0.08945540336850383, x_2 = 1.557667481081721-4.109128293921921e-17*\text{i}\}, \\
&\{x_1 = 0.5920619420732697-1.942890293094024e-16*\text{i}, x_2 = 6.938893903907228e-17*\text{i}+1.383859154368197\}, \\
&\{x_1 = 1.201557352500749, x_2 = -1.66727025803531, x_2 = -0.9106198942815954\}
\end{align*}
\]

Note that \texttt{hompack\_polsys} can sometimes be very slow. Perhaps \texttt{solve} can be used. Or perhaps \texttt{eliminate} can be used to convert the system of polynomials into one polynomial for which \texttt{allroots} can find all the roots.
66 impdiff

66.1 Functions and Variables for impdiff

\texttt{implicit\_derivative} (\texttt{f,indvarlist,orderlist,depvar}) \texttt{[Function]}

This subroutine computes implicit derivatives of multivariable functions. \texttt{f} is an array function, the indexes are the derivative degree in the \texttt{indvarlist} order; \texttt{indvarlist} is the independent variable list; \texttt{orderlist} is the order desired; and \texttt{depvar} is the dependent variable.

To use this function write first \texttt{load("impdiff")}. 
67 interpol

67.1 Introduction to interpol

Package `interpol` defines the Lagrangian, the linear and the cubic splines methods for polynomial interpolation.

For comments, bugs or suggestions, please contact me at 'mario AT edu DOT xunta DOT es'.

67.2 Functions and Variables for interpol

`lagrange`  

`lagrange(points)`  
`lagrange(points, option)`  

Computes the polynomial interpolation by the Lagrangian method. Argument `points` must be either:

- a two column matrix, `p:matrix([2,4],[5,6],[9,3])`;
- a list of pairs, `p: [[2,4],[5,6],[9,3]]`,
- a list of numbers, `p: [4,6,3]`, in which case the abscissas will be assigned automatically to 1, 2, 3, etc.

In the first two cases the pairs are ordered with respect to the first coordinate before making computations.

With the `option` argument it is possible to select the name for the independent variable, which is 'x by default; to define another one, write something like `varname='z`.

Note that when working with high degree polynomials, floating point evaluations are unstable.

See also `linearinterpol`, `cspline`, and `ratinterpol`.

Examples:

```lisp
(%i1) load("interpol")
(%i2) p:[[7,2],[8,2],[1,5],[3,2],[6,7]]
(%i3) lagrange(p);
(x - 7) (x - 6) (x - 3) (x - 1)  
35
(x - 8) (x - 6) (x - 3) (x - 1)
- --------------------------
  12
7 (x - 8) (x - 7) (x - 3) (x - 1)
+ --------------------------
  30
(x - 8) (x - 7) (x - 6) (x - 1)
- --------------------------
  60
(x - 8) (x - 7) (x - 6) (x - 3)
```
\[
\text{(i4) } f(x) := \frac{(x - 7) (x - 6) (x - 3) (x - 1)}{(x - 8) (x - 6) (x - 3) (x - 1) - \frac{7}{12} \frac{(x - 8) (x - 7) (x - 3) (x - 1)}{(x - 8) (x - 7) (x - 6) (x - 1) - \frac{60}{30}} + \frac{84}{30} \frac{(x - 8) (x - 7) (x - 6) (x - 3)}{(x - 8) (x - 7) (x - 6) (x - 3) + \frac{84}{30}}
\]

\[
\text{(i5) } \text{/* Evaluate the polynomial at some points */}
\]
\[
\text{expand(map(f,}\{2.3,5/7,\%pi\});}
\]
\[
\text{\{ - 1.567535, \[\frac{919062}{84035}, \frac{73}{420}, \frac{\%pi}{210}, \frac{701}{420}, \frac{\%pi}{210}, \frac{8957}{420}, \frac{\%pi}{210}, \frac{5288}{105}, \frac{\%pi}{5}\}\}}
\]

\[
\text{(i6) } \%,\text{numer;}
\]
\[
\text{(i7) } \text{load(}\text{draw});\text{ /* load draw package */}
\]
\[
\text{(i8) } \text{/* Plot the polynomial together with points */}
\]
\[
\text{draw2d(}
\text{color = red,}
\text{key = "Lagrange polynomial",}
\text{explicit(f(x),x,0,10),}
\text{point_size = 3,}
\text{color = blue,}
\text{key = "Sample points",}
\text{points(p))}$
\]
\[
\text{(i9) } \text{/* Change variable name */}
\]
\[
\text{lagrange(p, varname=w);}
\]
\[
\text{\{ - 1.567535, \[\frac{60}{30}, \frac{5288}{105}, \frac{\%pi}{5}\}\}}
\]
The characteristic or indicator function on the half-open interval \([a, b)\), that is, including \(a\) and excluding \(b\).

When \(x \geq a\) and \(x < b\) evaluates to \texttt{true} or \texttt{false}, \texttt{charfun2} returns 1 or 0, respectively.

Otherwise, \texttt{charfun2} returns a partially-evaluated result in terms of \texttt{charfun}.

Package \texttt{interpol} loads this function.

See also \texttt{charfun}.

Examples:

When \(x \geq a\) and \(x < b\) evaluates to \texttt{true} or \texttt{false}, \texttt{charfun2} returns 1 or 0, respectively.

\begin{verbatim}
(%i1) load ("interpol") $
(%i2) charfun2 (5, 0, 100);
(%o2) 1
(%i3) charfun2 (-5, 0, 100);
(%o3) 0
\end{verbatim}

Otherwise, \texttt{charfun2} returns a partially-evaluated result in terms of \texttt{charfun}.

\begin{verbatim}
(%i1) load ("interpol") $
(%i2) charfun2 (t, 0, 100);
(%o2) charfun((0 <= t) and (t < 100))
(%i3) charfun2 (5, u, v);
(%o3) charfun((u <= 5) and (5 < v))
(%i4) assume (v > u, u > 5);
(%o4) [v > u, u > 5]
(%i5) charfun2 (5, u, v);
(%o5) 0
\end{verbatim}

\texttt{linearinterpol} \hspace{1em} [Function]

\texttt{linearinterpol} (\texttt{points})

\texttt{linearinterpol} (\texttt{points}, \texttt{option})

Computes the polynomial interpolation by the linear method. Argument \texttt{points} must be either:

- a two column matrix, \texttt{p:matrix([2,4],[5,6],[9,3])},
- a list of pairs, \texttt{p: [[2,4],[5,6],[9,3]]}. 

\begin{verbatim}

\end{verbatim}
• a list of numbers, \( p: [4,6,3] \), in which case the abscissas will be assigned automatically to 1, 2, 3, etc.

In the first two cases the pairs are ordered with respect to the first coordinate before making computations.

With the option argument it is possible to select the name for the independent variable, which is '\( x \) by default; to define another one, write something like \( \text{varname} = 'z \).

See also \texttt{lagrange}, \texttt{cspline}, and \texttt{ratinterpol}.

Examples:

\begin{verbatim}
(%i1) load("interpol")$
(%i2) p: matrix([7,2],[8,3],[1,5],[3,2],[6,7])$
(%i3) linearinterpol(p);
  13  3 x
 2  2
+ (x - 5) charfun2(x, 7, inf) + (37 - 5 x) charfun2(x, 6, 7)
  5 x
+ (--- - 3) charfun2(x, 3, 6)
  3
(%i4) f(x):=''%;
  13  3 x
(%o4) f(x) := (-- - ---) charfun2(x, minf, 3)
 2  2
+ (x - 5) charfun2(x, 7, inf) + (37 - 5 x) charfun2(x, 6, 7)
  5 x
+ (--- - 3) charfun2(x, 3, 6)
  3
(%i5) /* Evaluate the polynomial at some points */
map(f,[7.3,25/7,%pi]);
  62  5 %pi
(%o5) [2.3, --, ----- - 3]
  21  3
(%i6) %,numer;
(%o6) [2.3, 2.952380952380953, 2.235987755982989]
(%i7) load("draw")$ /* load draw package */
(%i8) /* Plot the polynomial together with points */
\end{verbatim}
$\text{linearinterpol}(p, \text{varname='s')};$

\begin{align*}
13 & 3 s \\
2 & 2 \\
+ (s-5) \text{charfun2}(s, 7, \text{inf}) + (37-5 s) \text{charfun2}(s, 6, 7) \\
5 & s \\
+ (-3) \text{charfun2}(s, 3, 6)
\end{align*}

$\text{cspline}$ [Function]

$\text{cspline}(\text{points})$

$\text{cspline}(\text{points, option1, option2, ...})$

Computes the polynomial interpolation by the cubic splines method. Argument \text{points} must be either:

- a two column matrix, \text{p}: \text{matrix([2,4],[5,6],[9,3])},
- a list of pairs, \text{p}: \text{ [[2,4],[5,6],[9,3]] },
- a list of numbers, \text{p}: [4,6,3], in which case the abscissas will be assigned automatically to 1, 2, 3, etc.

In the first two cases the pairs are ordered with respect to the first coordinate before making computations.

There are three options to fit specific needs:

- \text{'d1}, default 'unknown, is the first derivative at \text{x}_1; if it is 'unknown, the second derivative at \text{x}_1 is made equal to 0 (natural cubic spline); if it is equal to a number, the second derivative is calculated based on this number.
- \text{'dn}, default 'unknown, is the first derivative at \text{x}_n; if it is 'unknown, the second derivative at \text{x}_n is made equal to 0 (natural cubic spline); if it is equal to a number, the second derivative is calculated based on this number.
- \text{'varname}, default 'x, is the name of the independent variable.

See also \text{lagrange}, \text{linearinterpol}, and \text{ratinterpol}.

Examples:

$\%(i1) \text{load("interpol")}$

$\%(i2) \text{p: [[7,2],[8,2],[1,5],[3,2],[6,7]]}$

$\%(i3) \text{// Unknown first derivatives at the extremes}$

$\text{is equivalent to natural cubic splines} */$

$\text{cspline(p);}$

\begin{align*}
3 & 2 \\
1159 x & 1159 x & 6091 x & 8283 \\
3288 & 1096 & 3288 & 1096 \\
3 & 2 \\
2587 x & 5174 x & 494117 x & 108928 \\
+ (-3) & + (-3) & + (-3) & + (-3) \text{charfun2}(x, 7, \text{inf})
\end{align*}

$\%(o3)$

$\text{charfun2}(x, \text{minf}, 3)$

$\text{charfun2}(x, \text{7, inf})$
(%i4) f(x):=''
(%i5) /* Some evaluations */
  map(f,[2.3,5/7,%pi]), numer;
(%o5) [1.991460766423356, 5.823200187269903, 2.227405312429507]
(%i6) load("draw")$ /* load draw package */
(%i7) /* Plotting interpolating function */
  draw2d(
    color = red,
    key = "Cubic splines",
    explicit(f(x),x,0,10),
    point_size = 3,
    color = blue,
    key = "Sample points",
    points(p))$
(%i8) /* New call, but giving values at the derivatives */
  cspline(p,d1=0,dn=0);
(%o8) (------- - -------- + ------- + ----) charfun2(x, minf, 3)
     2256  2256  2256  752
+ (- ------- + -------- - ------- + ------) charfun2(x, 7, inf)
     564  564  141  141
+ (------ - -------- + ------- - -----) charfun2(x, 6, 7)
     188  564  141  47
+ (- ------ + -------- - ------ + -----) charfun2(x, 3, 6)
     5076  188  141  47
(%i8) /* Defining new interpolating function */
  g(x):=''
(%i9) /* Plotting both functions together */
  draw2d(
    color = black,
    key = "Cubic splines (default)",
    explicit(f(x),x,0,10),
Chapter 67: interpol

```
color      = red,
key        = "Cubic splines (d1=0,dn=0)",
explicit(g(x),x,0,10),
point_size = 3,
color      = blue,
key        = "Sample points",
points(p))$
```

ratinterpol

ratinterpol (points, numdeg)
ratinterpol (points, numdeg, option1)

Generates a rational interpolator for data given by points and the degree of the numerator being equal to numdeg; the degree of the denominator is calculated automatically. Argument points must be either:
- a two column matrix, p:matrix([2,4],[5,6],[9,3]),
- a list of pairs, p: [[2,4],[5,6],[9,3]],
- a list of numbers, p: [4,6,3], in which case the abscissas will be assigned automatically to 1, 2, 3, etc.

In the first two cases the pairs are ordered with respect to the first coordinate before making computations.

There is one option to fit specific needs:
- 'varname, default 'x, is the name of the independent variable.

See also lagrange, linearinterpol, cspline, minpack_1squares, and Chapter 69 [lbfgs-pkg], page 1055.

Examples:

```
(%i1) load("interpol")$
(%i2) load("draw")$
(%i3) p:[][7.2,2.5],[8.5,2.1],[1.6,5.1],[3.4,2.4],[6.7,7.9]]$
(%i4) for k:0 thru length(p)-1 do
   draw2d(
      explicit(ratinterpol(p,k),x,0,9),
      point_size = 3,
      points(p),
      title = concat("Degree of numerator = ",k),
      yrange=[0,10])$
```
68 lapack

68.1 Introduction to lapack

lapack is a Common Lisp translation (via the program f2cl) of the Fortran library LAPACK, as obtained from the SLATEC project.

68.2 Functions and Variables for lapack

dgeev

<table>
<thead>
<tr>
<th>Function</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>dgeev (A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>dgeev (A, right_p, left_p)</td>
</tr>
</tbody>
</table>

Computes the eigenvalues and, optionally, the eigenvectors of a matrix A. All elements of A must be integer or floating point numbers. A must be square (same number of rows and columns). A might or might not be symmetric.

dgeev(A) computes only the eigenvalues of A. dgeev(A, right_p, left_p) computes the eigenvalues of A and the right eigenvectors when right_p = true and the left eigenvectors when left_p = true.

A list of three items is returned. The first item is a list of the eigenvalues. The second item is false or the matrix of right eigenvectors. The third item is false or the matrix of left eigenvectors.

The right eigenvector v(j) (the j-th column of the right eigenvector matrix) satisfies

A.v(j) = lambda(j).v(j)

where lambda(j) is the corresponding eigenvalue. The left eigenvector u(j) (the j-th column of the left eigenvector matrix) satisfies

u(j) * H.A = lambda(j).u(j) * H

where u(j) * H denotes the conjugate transpose of u(j). The Maxima function ctranspose computes the conjugate transpose.

The computed eigenvectors are normalized to have Euclidean norm equal to 1, and largest component has imaginary part equal to zero.

Example:

(%i1) load ("lapack")$
(%i2) fpprintprec : 6;
(%o2) 6
(%i3) M : matrix ([9.5, 1.75], [3.25, 10.45]);
(%o3) 
[ 9.5 1.75 ]
[ 3.25 10.45 ]
(%i4) dgeev (M);
(%o4) [[7.54331, 12.4067], false, false]
(%i5) [L, v, u] : dgeev (M, true, true);
(%o5) [[7.54331, 12.4067],
[ [ - .666642 - .515792 ]
[ - .745378 - .856714 ]
]
[ - .856714  - .745378 ]
[                   ]
[ .515792  - .666642 ]

(%i6) D : apply (diag_matrix, L);
    [ 7.54331  0 ]
(%o6)      [                ]
[   0   12.4067 ]

(%i7) M . v - v . D;
    [ 0.0  - 8.88178E-16 ]
(%o7)      [                ]
[ - 8.88178E-16  0.0 ]

(%i8) transpose (u) . M - D . transpose (u);
    [ 0.0  - 4.44089E-16 ]
(%o8)      [                ]
[ 0.0  0.0 ]

\[ \text{Function \ dgeqrf} \ (A) \]
Computes the QR decomposition of the matrix \(A\). All elements of \(A\) must be integer or floating point numbers. \(A\) may or may not have the same number of rows and columns.

A list of two items is returned. The first item is the matrix \(Q\), which is a square, orthonormal matrix which has the same number of rows as \(A\). The second item is the matrix \(R\), which is the same size as \(A\), and which has all elements equal to zero below the diagonal. The product \(Q \cdot R\), where "\cdot" is the noncommutative multiplication operator, is equal to \(A\) (ignoring floating point round-off errors).

(%i1) load ("lapack")$
(%i2) fpprintprec : 6$
(%i3) M : matrix ([1, -3.2, 8], [-11, 2.7, 5.9])$
(%i4) [q, r] : dgeqrf (M);
\[ [ - .0905357 .995893 ] \]
\[ [ .995893 .0905357 ] \]
\[ [ - 11.0454 2.97863 5.15148 ] \]
\[ [ 0 - 2.94241 8.50131 ] \]

(%i5) q . r - M;
\[ [ 0.0 - 1.33227E-15 8.88178E-16 ] \]

(%i6) mat_norm (%, 1);
(%o6) 3.10862E-15

\[ \text{Function \ dgesv} \ (A, \ b) \]
Computes the solution \(x\) of the linear equation \(Ax = b\), where \(A\) is a square matrix, and \(b\) is a matrix of the same number of rows as \(A\) and any number of columns. The return value \(x\) is the same size as \(b\).
The elements of $A$ and $b$ must evaluate to real floating point numbers via float; thus elements may be any numeric type, symbolic numerical constants, or expressions which evaluate to floats. The elements of $x$ are always floating point numbers. All arithmetic is carried out as floating point operations.

`dgesv` computes the solution via the LU decomposition of $A$.

Examples:

`dgesv` computes the solution of the linear equation $Ax = b$.

```lisp
(%i1) A : matrix ([1, -2.5], [0.375, 5]);
(%o1) [ 1 - 2.5 ]
      [     ]
      [ 0.375 5 ]
(%i2) b : matrix ([1.75], [-0.625]);
(%o2) [ 1.75 ]
      [     ]
      [- 0.625 ]
(%i3) x : dgesv (A, b);
       [ 1.210526315789474 ]
    (%o3) [ ]
       [ - 0.215789473684211 ]
(%i4) dlange (inf_norm, b - A.x);
    (%o4) 0.0
```

$b$ is a matrix with the same number of rows as $A$ and any number of columns. $x$ is the same size as $b$.

```lisp
(%i1) A : matrix ([1, -0.15], [1.82, 2]);
(%o1) [ 1 - 0.15 ]
      [     ]
      [ 1.82 2 ]
(%i2) b : matrix ([3.7, 1, 8], [-2.3, 5, -3.9]);
(%o2) [ 3.7 1 8 ]
      [     ]
      [- 2.3 5 - 3.9 ]
(%i3) x : dgesv (A, b);
       [ 3.103827540695117 1.20985481742191 6.781786185657722 ]
    (%o3) [ ]
       [ -3.974483062032557 1.399032116146062 -8.121425428948527 ]
(%i4) dlange (inf_norm, b - A.x);
    (%o4) 1.1102230246251565E-15
```

The elements of $A$ and $b$ must evaluate to real floating point numbers.

```lisp
(%i1) A : matrix ([5, -%pi], [1b0, 11/17]);
(%o1) [ 5 - %pi ]
      [     ]
      [ 1.0b0 -- ]
      [     ]
(%i2) b : matrix ([%e], [sin(1)]);
```
dgesvd

dgesvd (A)
dgesvd (A, left_p, right_p)

Computes the singular value decomposition (SVD) of a matrix A, comprising the
singular values and, optionally, the left and right singular vectors. All elements of A
must be integer or floating point numbers. A might or might not be square (same
number of rows and columns).

Let m be the number of rows, and n the number of columns of A. The singular value
decomposition of A comprises three matrices, U, Sigma, and V^T, such that
A = U.Sigma.V^T

where U is an m-by-m unitary matrix, Sigma is an m-by-n diagonal matrix, and V^T
is an n-by-n unitary matrix.

Let sigma[i] be a diagonal element of Sigma, that is, Sigma[i, i] = sigma[i]. The ele-
ments sigma[i] are the so-called singular values of A; these are real and nonnegative,
and returned in descending order. The first min(m, n) columns of U and V are the
left and right singular vectors of A. Note that dgesvd returns the transpose of V, not
V itself.

dgesvd(A) computes only the singular values of A. dgesvd(A, left_p, right_p) com-
putes the singular values of A and the left singular vectors when left_p = true
and the right singular vectors when right_p = true.

A list of three items is returned. The first item is a list of the singular values. The
second item is false or the matrix of left singular vectors. The third item is false
or the matrix of right singular vectors.

Example:

(%i1) load ("lapack");
(%i2) fpprintprec : 6;
(%o2) 6
(%i3) M: matrix([1, 2, 3], [3.5, 0.5, 8], [-1, 2, -3], [4, 9, 7]);
(%o3)
[ 1 2 3 ]
[   ]
[ 3.5 0.5 8 ]
[   ]
[ -1 2 -3 ]
[   ]
[ 4 9 7 ]

(%i4) x : dgesv (A, b);
(%o4) [ 0.690375643155986 ]
[   ]
[ 0.233510982552952 ]

(%i5) dlags (inf_norm, b - A . x);
(%o5) 2.220446049250313E-16

[Function]

dgesvd
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%%(i4) dgesvd (M);
%%(o4)   [[14.4744, 6.38637, .452547], false, false]
%%(i5) [sigma, U, VT] : dgesvd (M, true, true);
%%(o5)   [[14.4744, 6.38637, .452547],
[ - .256731 .00816168 .959029 + .119523 ]
[ - .526456 .672116 - .206236 - .478091 ]
[ .107997 - .532278 - .0708315 - 0.83666 ]
[ - .803287 - .514659 - .180867 .239046 ]
[ - .374486 - .538209 - .755044 ]
[ .130623 - .836799 0.5317 ]
[ - .917986 .100488 .383672 ]
%%(i6) m : length (U);
%%(o6)   4
%%(i7) n : length (VT);
%%(o7)   3
%%(i8) Sigma:
      genmatrix(lambda ([i, j], if i=j then sigma[i] else 0),
               m, n);
      [ 14.4744 0 0 ]
      [ ]
      [ 0 6.38637 0 ]
%%(o8)   [ ]
      [ 0 0 .452547 ]
      [ ]
      [ 0 0 0 ]
%%(i9) U . Sigma . VT - M;
      [ 1.11022E-15 0.0 1.77636E-15 ]
      [ ]
      [ 1.33227E-15 1.66533E-15 0.0 ]
%%(o9)   [ ]
      [ - 4.44089E-16 - 8.88178E-16 4.44089E-16 ]
      [ ]
      [ 8.88178E-16 1.77636E-15 8.88178E-16 ]
%%(i10) transpose (U) . U;
      [ 1.0 5.55112E-17 2.498E-16 2.77556E-17 ]
      [ ]
      [ 5.55112E-17 1.0 5.55112E-17 4.16334E-17 ]
%%(o10) [ ]
      [ 2.498E-16 5.55112E-17 1.0 + 2.08167E-16 ]
      [ ]
      [ 2.77556E-17 4.16334E-17 - 2.08167E-16 1.0 ]
%%(i11) VT . transpose (VT);
(\%o11) [ 0.0 1.0 5.55112E-17 ]
[ - 5.55112E-17 5.55112E-17 1.0 ]

\textbf{dlange} (\textit{norm}, \textit{A})
\textbf{zlange} (\textit{norm}, \textit{A})

Computes a norm or norm-like function of the matrix \textit{A}. If \textit{A} is a real matrix, use \textit{dlange}. For a matrix with complex elements, use \textit{zlange}.

\textit{norm} specifies the kind of norm to be computed:

\textbf{max} \quad \text{Compute} \quad \max(\text{abs}(A(i,j))) \quad \text{where} \quad i \quad \text{and} \quad j \quad \text{range over the rows and columns, respectively, of} \quad A. \quad \text{Note that this function is not a proper matrix norm.}

\textbf{one\_norm} \quad \text{Compute the} \quad L[1] \quad \text{norm of} \quad A, \quad \text{that is, the maximum of the sum of the absolute value of elements in each column.}

\textbf{inf\_norm} \quad \text{Compute the} \quad L[\infty] \quad \text{norm of} \quad A, \quad \text{that is, the maximum of the sum of the absolute value of elements in each row.}

\textbf{frobenius} \quad \text{Compute the Frobenius norm of} \quad A, \quad \text{that is, the square root of the sum of squares of the matrix elements.}

\textbf{dgemm} \quad [\text{Function}]

\textbf{dgemm} (\textit{A, B})
\textbf{dgemm} (\textit{A, B, options})

Compute the product of two matrices and optionally add the product to a third matrix.

In the simplest form, \textbf{dgemm}(\textit{A, B}) \text{ computes the product of the two real matrices,} \textit{A} \text{ and} \textit{B}. 

In the second form, \textbf{dgemm} \text{ computes} \quad \alpha \ast A \ast B + \beta \ast C \quad \text{where} \quad A, \quad B, \quad C \quad \text{are real matrices of the appropriate sizes and} \quad \alpha \text{ and} \quad \beta \quad \text{are real numbers.} \quad \text{Optionally,} \quad A \text{ and/or} \quad B \quad \text{can be transposed before computing the product.} \quad \text{The extra parameters are specified by optional keyword arguments: The keyword arguments are optional and may be specified in any order. They all take the form key=val. The keyword arguments are:}

\textbf{C} \quad \text{The matrix} \quad C \quad \text{that should be added. The default is} \quad \text{false}, \quad \text{which means no matrix is added.}

\textbf{alpha} \quad \text{The product of} \quad A \text{ and} \quad B \quad \text{is multiplied by this value. The default is} \quad 1.

\textbf{beta} \quad \text{If a matrix} \quad C \quad \text{is given, this value multiplies} \quad C \quad \text{before it is added. The default value is} \quad 0, \quad \text{which implies that} \quad C \quad \text{is not added, even if} \quad C \quad \text{is given. Hence, be sure to specify a non-zero value for} \quad \text{beta.}

\textbf{transpose\_a} \quad \text{If} \quad \text{true}, \quad \text{the transpose of} \quad A \quad \text{is used instead of} \quad A \quad \text{for the product. The default is} \quad \text{false.}
**transpose_b**

If *true*, the transpose of \( B \) is used instead of \( B \) for the product. The default is *false*.

```latex
\texttt{(\%i1) load ("lapack")$}
(\%i2) A : matrix([[1,2,3],[4,5,6],[7,8,9]]);
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}
(\%o2)

test \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}
(\%i3) B : matrix([[\textbf{-}1,\textbf{-}2,\textbf{-}3],[\textbf{-}4,\textbf{-}5,\textbf{-}6],[\textbf{-}7,\textbf{-}8,\textbf{-}9]]);
\begin{bmatrix}
\textbf{-}1 & \textbf{-}2 & \textbf{-}3 \\
\textbf{-}4 & \textbf{-}5 & \textbf{-}6 \\
\textbf{-}7 & \textbf{-}8 & \textbf{-}9
\end{bmatrix}
(\%o3)

test \begin{bmatrix}
\textbf{-}1 & \textbf{-}2 & \textbf{-}3 \\
\textbf{-}4 & \textbf{-}5 & \textbf{-}6 \\
\textbf{-}7 & \textbf{-}8 & \textbf{-}9
\end{bmatrix}
(\%i4) C : matrix([[3,2,1],[6,5,4],[9,8,7]]);
\begin{bmatrix}
3 & 2 & 1 \\
6 & 5 & 4 \\
9 & 8 & 7
\end{bmatrix}
(\%o4)

test \begin{bmatrix}
3 & 2 & 1 \\
6 & 5 & 4 \\
9 & 8 & 7
\end{bmatrix}
(\%i5) dgemm(A,B);
\begin{bmatrix}
\text{\textbf{-}30} & \text{\textbf{-}36} & \text{\textbf{-}42} \\
\text{} & \text{} & \text{}
\end{bmatrix}
(\%o5)

test \begin{bmatrix}
\text{\textbf{-}30} & \text{\textbf{-}36} & \text{\textbf{-}42} \\
\text{} & \text{} & \text{}
\end{bmatrix}
(\%i6) transpose(A) . B;
\begin{bmatrix}
\text{\textbf{-}66} & \text{\textbf{-}78} & \text{\textbf{-}90} \\
\text{} & \text{} & \text{}
\end{bmatrix}
(\%o6)

test \begin{bmatrix}
\text{\textbf{-}66} & \text{\textbf{-}78} & \text{\textbf{-}90} \\
\text{} & \text{} & \text{}
\end{bmatrix}
(\%i7) dgemm(A,B,transpose_a=true);
\begin{bmatrix}
\text{-}102 & \text{-}126 & \text{-}150 \\
\text{} & \text{} & \text{}
\end{bmatrix}
(\%o7)

test \begin{bmatrix}
\text{-}102 & \text{-}126 & \text{-}150 \\
\text{} & \text{} & \text{}
\end{bmatrix}
(\%i8) dgemm(A,B,transpose_a=true);
\begin{bmatrix}
\text{-}102 & \text{-}126 & \text{-}150 \\
\text{} & \text{} & \text{}
\end{bmatrix}
(\%o8)

test \begin{bmatrix}
\text{-}102 & \text{-}126 & \text{-}150 \\
\text{} & \text{} & \text{}
\end{bmatrix}
```

```
(%i9) dgemm(A,B,c=C,beta=1);
[ - 27.0  - 34.0  - 41.0 ]
[ ]
(%o9) [ - 60.0  - 76.0  - 92.0 ]
[ ]
[ - 93.0  - 118.0  - 143.0 ]
(%i10) A . B + C;
[ - 27  - 34  - 41 ]
[ ]
(%o10) [ - 60  - 76  - 92 ]
[ ]
[ - 93  - 118  - 143 ]
(%i11) dgemm(A,B,c=C,beta=1, alpha=-1);
[ 33.0  38.0  43.0 ]
[ ]
(%o11) [ 72.0  86.0  100.0 ]
[ ]
[ 111.0  134.0  157.0 ]
(%i12) -A . B + C;
[ 33  38  43 ]
[ ]
(%o12) [ 72  86  100 ]
[ ]
[ 111  134  157 ]

zgeev

zgeev (A)
zgeev (A, right_p, left_p)
Like dgeev, but the matrix A is complex.

zheev

zheev (A)
zheev (A, eigvec_p)
Like dgeev, but the matrix A is assumed to be a square complex Hermitian matrix. If eigvec_p is true, then the eigenvectors of the matrix are also computed.

No check is made that the matrix A is, in fact, Hermitian.

A list of two items is returned, as in dgeev: a list of eigenvalues, and false or the matrix of the eigenvectors.

An example of computing the eigenvalues and then eigenvalues and eigenvectors of an Hermitian matrix.

(%i11) load("lapack")$
(%i12) A: matrix(
[9.14 +%i*0.00 ,  -4.37  -%i*9.22 ,  -1.98  -%i*1.72 ,  -8.96  -%i*9.50],
[-4.37  +%i*9.22 ,  -3.35   +%i*0.00 ,   2.25   +%i*9.51 ,   2.57  +%i*2.40],
[-1.98  +%i*1.72 ,   2.25   +%i*9.51 ,  -4.82  -%i*0.00 ,  -3.24  +%i*2.04],
[-8.96  +%i*9.50 ,   2.57  -%i*2.40 ,  -3.24  -%i*2.04 ,   8.44  +%i*0.00]);
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(%o2) 
[ 9.14  (- 9.22 %i) - 4.37  (- 1.72 %i) - 1.98  (- 9.5 %i) - 8.96 ]
[ ]
[ 9.22 %i + 4.37  - 3.35  2.25  - 9.51 %i  2.4 %i + 2.57 ]
[ ]
[ 1.72 %i + 1.98  9.51 %i + 2.25  - 4.82  2.04 %i - 3.24 ]
[ ]
[ 9.5 %i - 8.96  2.57  - 2.4 %i  (- 2.04 %i) - 3.24  8.44 ]

(%i3) zheev(A);
(%o3) [[- 16.00474647209473, - 6.764970154793324, 6.665711453507098, 25.51400517338097], false]

(%i4) E:zheev(A,true)$
(%i5) E[1];
(%o5) [- 16.00474647209474, - 6.764970154793325, 6.665711453507101, 25.51400517338096]

(%i6) E[2];
[ 0.2674650533172745 %i + 0.2175453586665017 ]
[ ]
[ 0.00269673086619885 %i + 0.6968836773391712 ]

(%o6) Col 1 = [
[ (- 0.6082406376714117 %i) - 0.01210614292697931 ]
[ ]
[ 0.1593081858095037 ]
[ 0.2644937470667444 %i + 0.4773693349937472 ]
[ ]
[ (- 0.2852389036031621 %i) - 0.1414362742011673 ]

Col 2 = [
[ 0.2654607680986639 %i + 0.4467818117184174 ]
[ ]
[ 0.5750762708542709 ]
[ 0.2810649767305922 %i - 0.1335263928245182 ]
[ ]
[ 0.2866310132869556 %i - 0.4536971347853274 ]

Col 3 = [
[ (- 0.293368432754295 %i) - 0.4954972425541057 ]
[ ]
[ 0.5325337537576771 ]
[ (- 0.5737316575503476 %i) - 0.3966146799427706 ]
[ ]
[ 0.01826502619021457 %i + 0.3530557704387017 ]

Col 4 = [
[ 0.1673700900085425 %i + 0.01476684746229564 ]
[ ]
[ 0.6002632636961784 ]

25.51400517338097, false]
69 lbfgs

69.1 Introduction to lbfgs

lbfgs is an implementation of the L-BFGS algorithm [1] to solve unconstrained minimization problems via a limited-memory quasi-Newton (BFGS) algorithm. It is called a limited-memory method because a low-rank approximation of the Hessian matrix inverse is stored instead of the entire Hessian inverse. The program was originally written in Fortran [2] by Jorge Nocedal, incorporating some functions originally written by Jorge J. Moré and David J. Thuente, and translated into Lisp automatically via the program f2cl. The Maxima package lbfgs comprises the translated code plus an interface function which manages some details.

References:

69.2 Functions and Variables for lbfgs

lbfgs [Function]

\[
\text{lbfgs}(\text{FOM}, \ X, \ X0, \ \text{epsilon}, \ \text{iprint})
\]

\[
\text{lbfgs}([\text{FOM}, \ \text{grad}] \ X, \ X0, \ \text{epsilon}, \ \text{iprint})
\]

Finds an approximate solution of the unconstrained minimization of the figure of merit FOM over the list of variables X, starting from initial estimates X0, such that
\[
\text{norm}(\text{grad}(\text{FOM})) < \text{epsilon} \ast \text{max}(1, \text{norm}(X)).
\]

grad, if present, is the gradient of FOM with respect to the variables X. grad may be a list or a function that returns a list, with one element for each element of X. If not present, the gradient is computed automatically by symbolic differentiation. If FOM is a function, the gradient grad must be supplied by the user.

The algorithm applied is a limited-memory quasi-Newton (BFGS) algorithm [1]. It is called a limited-memory method because a low-rank approximation of the Hessian matrix inverse is stored instead of the entire Hessian inverse. Each iteration of the algorithm is a line search, that is, a search along a ray in the variables X, with the search direction computed from the approximate Hessian inverse. The FOM is always decreased by a successful line search. Usually (but not always) the norm of the gradient of FOM also decreases.

iprint controls progress messages printed by lbfgs.

iprint[1]

\[
iprint[1] \text{ controls the frequency of progress messages.}
\]

\[
iprint[1] < 0
\]

No progress messages.

\[
iprint[1] = 0
\]

Messages at the first and last iterations.
iprint[1] > 0
Print a message every iprint[1] iterations.

iprint[2]

iprint[2] controls the verbosity of progress messages.

iprint[2] = 0
Print out iteration count, number of evaluations of FOM, value of FOM, norm of the gradient of FOM, and step length.

iprint[2] = 1
Same as iprint[2] = 0, plus X0 and the gradient of FOM evaluated at X0.

iprint[2] = 2
Same as iprint[2] = 1, plus values of X at each iteration.

iprint[2] = 3
Same as iprint[2] = 2, plus the gradient of FOM at each iteration.

The columns printed by lbfgs are the following.

I  Number of iterations. It is incremented for each line search.

NFN Number of evaluations of the figure of merit.

FUNC Value of the figure of merit at the end of the most recent line search.

GNORM Norm of the gradient of the figure of merit at the end of the most recent line search.

STEEPLNGTH
An internal parameter of the search algorithm.

Additional information concerning details of the algorithm are found in the comments of the original Fortran code [2].

See also lbfgs_nfeval_max and lbfgs_ncorrections.

References:

Examples:
The same FOM as computed by FGCIMATE in the program sdrive.f in the LBFBS package from Netlib. Note that the variables in question are subscripted variables. The FOM has an exact minimum equal to zero at u[k] = 1 for k = 1,..., 8.

(\%i1) load ("lbfgs")$
(\%i2) t1[j] := 1 - u[j];
(\%o2) \quad \quad \quad \quad t1 := 1 - u
      \quad \quad \quad \quad j
(\%i3) t2[j] := 10*(u[j + 1] - u[j]^2);
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(%o3) \[ t2 := 10 \left( u_j - u_{j+1} \right) \]

(%i4) \( n : 8; \)

(%o4) 8

(%i5) \( \text{FOM} := \text{sum} \left( t1[2*j - 1]^2 + t2[2*j - 1]^2, j, 1, n/2 \right); \)

(%o5) \[ 100 \left( u_8 - u_7 \right) + \left( u_7 - u_6 \right) + 100 \left( u_6 - u_5 \right) + \left( u_5 - u_4 \right) + 100 \left( u_4 - u_3 \right) + \left( u_3 - u_2 \right) + 100 \left( u_2 - u_1 \right) + \left( u_1 - u_0 \right) \]

(%i6) \( \text{lbfgs} \left( \text{FOM}, \left[u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\right], [-1.2, 1, -1.2, 1, -1.2, 1, -1.2, 1], 1e-3, [1, 0] \right); \)

*************************************************
N= 8 NUMBER OF CORRECTIONS=25
INITIAL VALUES
F= 9.680000000000000D+01 GNORM= 4.65735755084533D+02

I  NFN  FUNC  GNORM  STEPLENGTH
1  3 1.651479526340304D+01 4.324359291335977D+00 7.9261539349631D-04
2  4 1.650209316638371D+01 3.575788161060007D+00 1.000000000000000D+00
3  5 1.645461701312851D+01 6.20869903061577D+00 1.000000000000000D+00
4  6 1.636673012755888D+01 1.175589920979489D+00 1.000000000000000D+00
5  7 1.612153014409201D+01 2.292797147151288D+00 1.000000000000000D+00
6  8 1.569118407390620D+01 3.687447158775571D+00 1.000000000000000D+00
7  9 1.510361958399842D+01 4.501931728123680D+00 1.000000000000000D+00
8  10 1.391077857754294D+01 4.526061463810632D+00 1.000000000000000D+00
9  11 1.165625686278198D+01 2.748348965356917D+00 1.000000000000000D+00
10 12 9.859422687859137D+00 2.114949742316444D+00 1.000000000000000D+00
11 13 7.815442521732281D+00 6.107623257665560D+00 1.000000000000000D+00
12 14 7.346380905773160D+00 2.165281166714631D+00 1.000000000000000D+00
13 15 6.30460634066370D+00 1.401220851762050D+00 1.000000000000000D+00
14 16 5.238763939851439D+00 1.702473787613255D+00 1.000000000000000D+00
15 17 3.75401679040670D+00 7.981845727704576D+00 1.000000000000000D+00
16 18 3.001238402309352D+00 3.925482944716691D+00 2.33329631296807D+00
17 19 2.794390709718290D+00 8.243329982546473D+00 2.503577283782332D+00
18 20 2.563783526187590D+00 1.035413426521790D+00 1.000000000000000D+00
19 21 2.019429976377856D+00 1.065187312346769D+00 1.000000000000000D+00
20 22 1.428003167670903D+00 2.475962450826961D+00 1.000000000000000D+00
21 23 1.19787426861340D+00 8.441707983493810D+00 4.303451060808756D+00
22 24 9.023848941942773D-01 1.113189216635162D+01 1.000000000000000D+00
23 25 5.508226405863770D-01 2.380830600326308D+00 1.000000000000000D+00
24 26 3.902893258815567D-01 5.625595816844221D+00 4.834988416524465D+00
25 27 3.207542206990315D-01 1.149444645416472D+01 1.000000000000000D+00
26 28 1.874468266362791D-01 3.632482152880997D+00 1.000000000000000D+00
27 29 9.575763380706598D-2 4.816497446154354D+00 1.000000000000000D+00
THE MINIMIZATION TERMINATED WITHOUT DETECTING ERRORS.

IFLAG = 0
(%o6) [u = 1.000005339816132, u = 1.000009942840108,
1 2
u = 1.000005339816132, u = 1.000009942840108,
3 4
u = 1.000005339816132, u = 1.000009942840108,
5 6
u = 1.000005339816132, u = 1.000009942840108]
7 8

A regression problem. The FOM is the mean square difference between the predicted value $F(X[i])$ and the observed value $Y[i]$. The function $F$ is a bounded monotone function (a so-called "sigmoidal" function). In this example, `lbfgs` computes approximate values for the parameters of $F$ and `plot2d` displays a comparison of $F$ with the observed data.

(%i1) load ("lbfgs")$
(%i2) FOM : '((1/length(X))*sum((F(X[i]) - Y[i])^2, i, 1,
length(X)));
    2
(%o2) -----------------------------------
        length(X)

(%i3) X : [1, 2, 3, 4, 5];
(%o3) [1, 2, 3, 4, 5]
(%i4) Y : [0, 0.5, 1, 1.25, 1.5];
(%o4) [0, 0.5, 1, 1.25, 1.5]
(%i5) F(x) := A/(1 + exp(-B*(x - C)));
(%o5) F(x) := ----------------------
            1 + exp((- B) (x - C))

(%i6) ''FOM;
(%o6) ((------------------- - 1.5) + (------------------- - 1.25)
           - B (5 - C)            - B (4 - C)
               %e + 1             %e + 1

Gradient of FOM is specified (instead of computing it automatically). Both the FOM and its gradient are passed as functions to lbfgs.

\( F(a, b, c) := (a - 5)^2 + (b - 3)^4 + (c - 2)^6 \)

\[
\begin{align*}
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} &= 2A + 2A - B (3 - C) - B (2 - C) \\
\frac{\partial}{\partial x} + 1 &= \frac{\partial}{\partial y} + 1 \\
\frac{\partial}{\partial z} + 1 \neq \frac{\partial}{\partial z} + 1 \end{align*}
\]

\[
\begin{align*}
\frac{A}{5} &= \frac{B}{1 - C} + 2 \\
\frac{\partial F}{\partial x} + 1 &= \frac{\partial F}{\partial y} + 1 \\
\frac{\partial F}{\partial z} + 1 &= \frac{\partial F}{\partial z} + 1
\end{align*}
\]
****************************

<table>
<thead>
<tr>
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<th>NFN</th>
<th>FUNC</th>
<th>GNORM</th>
<th>STEPLENGTH</th>
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<td>1.000000000000000D+00</td>
</tr>
</tbody>
</table>

THE MINIMIZATION TERMINATED WITHOUT DETECTING ERRORS.
IFLAG = 0

(%o4) [a = 5.000086823042934, b = 3.052395429705181,
c = 1.927980629919583]

**lbfgs_nfeval_max**

[Variable]
Default value: 100

**lbfgs_nfeval_max** is the maximum number of evaluations of the figure of merit (FOM) in **lbfgs**. When **lbfgs_nfeval_max** is reached, **lbfgs** returns the result of the last successful line search.

**lbfgs_ncorrections**

[Variable]
Default value: 25

**lbfgs_ncorrections** is the number of corrections applied to the approximate inverse Hessian matrix which is maintained by **lbfgs**.
70 lindstedt

70.1 Functions and Variables for lindstedt

Lindstedt (eq,pvar,torder,ic) [Function]
This is a first pass at a Lindstedt code. It can solve problems with initial
conditions entered, which can be arbitrary constants, (just not %k1 and
%k2) where the initial conditions on the perturbation equations are
\( z[i] = 0, z'[i] = 0 \) for \( i > 0 \). \( ic \) is the list of initial conditions.

Problems occur when initial conditions are not given, as the constants in the perturba-
tion equations are the same as the zero order equation solution. Also, problems occur
when the initial conditions for the perturbation equations are not \( z[i] = 0, z'[i] = 0 \)
for \( i > 0 \), such as the Van der Pol equation.

Example:

(\%i1) load("makeOrders")$

(\%i2) load("lindstedt")$

(\%i3) Lindstedt('diff(x,t,2)+x-(e*x^3)/6,e,2,[1,0]);

\[
\begin{align*}
2 e (\cos(5 T) - 24 \cos(3 T) + 23 \cos(T)) \\
36864 e (\cos(3 T) - \cos(T)) \\
- \frac{1}{192} + \cos(T),
\end{align*}
\]

\[
\begin{align*}
2 \frac{7 e e}{3072} T = (- \frac{1}{16} \frac{1}{16} + 1) t]
\end{align*}
\]

To use this function write first \texttt{load("makeOrders")} and \texttt{load("lindstedt")}. 
71 linearalgebra

71.1 Introduction to linearalgebra

linearalgebra is a collection of functions for linear algebra.

Example:

(%i1) M : matrix ([1, 2], [1, 2]);
(%o1)

(%i2) nullspace (M);

(%o2)

(%i3) columnspace (M);

(%o3)

(%i4) ptriangularize (M - z*ident(2), z);

(%o4)

(%i5) M : matrix ([1, 2, 3], [4, 5, 6], [7, 8, 9]) - z*ident(3);

(%i6) MM : ptriangularize (M, z);

(%i7) algebraic : true;

(%o7) true

(%i8) tellrat (MM [3, 3]);
(%o8) \[z - 15z - 18z\]
(%i9) MM : ratsimp (MM);
[ 4  5 - z  6  ]
[              ]
[              ]
[ 2            ]
(%o9) \[ 66 7z - 102z - 132 \]
[ 49 49 0  0  0 ]
(%i10) nullspace (MM);
[ 1  2  3  4  5  6  7  8  9  10  11  12  13  14  15  16 ]
(%o10) \[ \] span([ 8 ])
[  ]
[ 2  2  5  6  9  10  13  14  12  11  8  7  4  3  2  1  ]
(%i11) M : matrix ([1, 2, 3, 4], [5, 6, 7, 8], [9, 10, 11, 12], [13, 14, 15, 16]);
(%o11) \[ \]
(%i12) columnspace (M);
[ 3 ] [ 4 ] [ 7 ] [ 8 ] [ 11 ] [ 12 ] [  ] [  ]
(%o12) \[ \] span([ 0 ] [ 1 ] [ 1 ] [ 0 ] [ 0 ] [ - 1 ] [ 1 ] [ 0 ]
[ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ]
(%i13) apply ('orthogonal_complement, args (nullspace (transpose (M))));
(%o13) \[ \] span([ 0 ] [ 1 ] [ 1 ] [ 0 ] [ 2 ] [ - 1 ] [ ] [ ]
[ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ]
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### 71.2 Functions and Variables for linearalgebra

**addmatrices** \( f, M_1, \ldots, M_n \)

Using the function \( f \) as the addition function, return the sum of the matrices \( M_1, \ldots, M_n \). The function \( f \) must accept any number of arguments (a Maxima nary function).

Examples:

\[
\begin{align*}
\text{(i1) } & m1 : \text{matrix([1,2],[3,4])} \\
\text{(i2) } & m2 : \text{matrix([7,8],[9,10])} \\
\text{(i3) } & \text{addmatrices('max,m1,m2)}; \\
\text{(o3) } & \text{matrix([7,8],[9,10])} \\
\text{(i4) } & \text{addmatrices('max,m1,m2,5*m1)}; \\
\text{(o4) } & \text{matrix([7,10],[15,20])}
\end{align*}
\]

**blockmatrixp** \( M \)

Return true if and only if \( M \) is a matrix and every entry of \( M \) is a matrix.

**columnop** \( M, i, j, \theta \)

If \( M \) is a matrix, return the matrix that results from doing the column operation \( C_{\text{i}} \leftarrow C_{\text{i}} - \theta \cdot C_{\text{j}} \). If \( M \) doesn’t have a row \( i \) or \( j \), signal an error.

**columnswap** \( M, i, j \)

If \( M \) is a matrix, swap columns \( i \) and \( j \). If \( M \) doesn’t have a column \( i \) or \( j \), signal an error.

**columnspace** \( M \)

If \( M \) is a matrix, return span \( \{v_1, \ldots, v_n\} \), where the set \( \{v_1, \ldots, v_n\} \) is a basis for the column space of \( M \). The span of the empty set is \( \{0\} \). Thus, when the column space has only one member, return \( \text{span()} \).

**cholesky**

**cholesky** \( M \)

**cholesky** \( M, \text{field} \)

Return the Cholesky factorization of the matrix selfadjoint (or hermitian) matrix \( M \). The second argument defaults to ‘generalring.’ For a description of the possible values for \( \text{field} \), see \text{lufactor}.

**ctranspose** \( M \)

Return the complex conjugate transpose of the matrix \( M \). The function \text{ctranspose} uses \text{matrix_element_transpose} to transpose each matrix element.

**diag_matrix** \( d_1, d_2, \ldots, d_n \)

Return a diagonal matrix with diagonal entries \( d_1, d_2, \ldots, d_n \). When the diagonal entries are matrices, the zero entries of the returned matrix are zero matrices of the appropriate size; for example:

\[
\begin{align*}
\text{(i1) } & \text{diag_matrix(diag_matrix(1,2),diag_matrix(3,4))};
\end{align*}
\]
\[
\begin{bmatrix}
  1 & 0 \\
  2 & 0 \\
  0 & 4 \\
\end{bmatrix}
\]

(%i1)
\[
\begin{bmatrix}
  0 & 0 \\
  3 & 0 \\
  4 & 0 \\
\end{bmatrix}
\]

(%i2) diag_matrix(p,q);
\[
\begin{bmatrix}
p & 0 \\
0 & q \\
\end{bmatrix}
\]

\textbf{dotproduct}\ (u, v)

Return the dotproduct of vectors \( u \) and \( v \). This is the same as \textbf{conjugate} (\textbf{transpose}\ (u)) \cdot v. The arguments \( u \) and \( v \) must be column vectors.

eigens\_by\_jacobi

\textbf{eigens\_by\_jacobi}\ (A)
\textbf{eigens\_by\_jacobi}\ (A, \textit{field\_type})

Computes the eigenvalues and eigenvectors of \( A \) by the method of Jacobi rotations. \( A \) must be a symmetric matrix (but it need not be positive definite nor positive semidefinite). \textit{field\_type} indicates the computational field, either \textit{floatfield} or \textit{bigfloatfield}. If \textit{field\_type} is not specified, it defaults to \textit{floatfield}.

The elements of \( A \) must be numbers or expressions which evaluate to numbers via \textit{float} or \textit{bfloat} (depending on \textit{field\_type}).

Examples:

(%i1) S: matrix([1/sqrt(2), 1/sqrt(2)],[1/sqrt(2), 1/sqrt(2)]);
\[
\begin{bmatrix}
  \frac{1}{\sqrt{2}} & 1 \\
  \frac{1}{\sqrt{2}} & 1 \\
\end{bmatrix}
\]

(%o1)
\[
\begin{bmatrix}
  \sqrt{2} & \frac{\sqrt{2}}{\sqrt{2}} \\
  \sqrt{2} & \frac{\sqrt{2}}{\sqrt{2}} \\
\end{bmatrix}
\]

(%i2) L : matrix ([sqrt(3), 0], [0, sqrt(5)]);
\[
\begin{bmatrix}
  \sqrt{3} & 0 \\
  0 & \sqrt{5} \\
\end{bmatrix}
\]

(%o2)
\[
\begin{bmatrix}
  \sqrt{3} & 0 \\
  0 & \sqrt{5} \\
\end{bmatrix}
\]

(%i3) M : S . L . transpose (S);
\[
\begin{bmatrix}
  \sqrt{5} & \sqrt{3} & \sqrt{5} & \sqrt{3} \\
  \sqrt{3} & \sqrt{5} & \sqrt{3} & \sqrt{5} \\
  \sqrt{5} & \sqrt{3} & \sqrt{5} & \sqrt{3} \\
  \sqrt{3} & \sqrt{5} & \sqrt{3} & \sqrt{5} \\
\end{bmatrix}
\]

(%o3)
\[
\begin{bmatrix}
  2 & 2 & 2 & 2 \\
  2 & 2 & 2 & 2 \\
\end{bmatrix}
\]
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(%i4) eigens_by_jacobi (M);
The largest percent change was 0.1454972243679
The largest percent change was 0.0
number of sweeps: 2
number of rotations: 1
(%o4) [ [1.732050807568877, 2.23606797749979],
     [ 0.70710678118655 0.70710678118655 ]
     [ ]
     [ - 0.70710678118655 0.70710678118655 ]

(%i5) float ([[sqrt(3), sqrt(5)], S]);
(%o5) [ [1.732050807568877, 2.23606797749979],
     [ 0.70710678118655 0.70710678118655 ]
     [ ]
     [ - 0.70710678118655 0.70710678118655 ]

(%i6) eigens_by_jacobi (M, bigfloatfield);
The largest percent change was 1.454972243679028b-1
The largest percent change was 0.0b0
number of sweeps: 2
number of rotations: 1
(%o6) [ [1.732050807568877b0, 2.23606797749979b0],
     [ 7.071067811865475b-1 7.071067811865475b-1 ]
     [ ]
     [ - 7.071067811865475b-1 7.071067811865475b-1 ]

get_lu_factors (x)  [Function]
When x = lu_factor (A), then get_lu_factors returns a list of the form [P, L, U], where P is a permutation matrix, L is lower triangular with ones on the diagonal, and U is upper triangular, and A = P L U.

hankel  [Function]
hankel (col)
hankel (col, row)
Return a Hankel matrix H. The first column of H is col; except for the first entry, the last row of H is row. The default for row is the zero vector with the same length as col.

hessian (f, x)  [Function]
Returns the Hessian matrix of f with respect to the list of variables x. The (i, j)-th element of the Hessian matrix is diff(f, x[i], 1, x[j], 1).

Examples:
  (%i1) hessian (x * sin (y), [x, y]);
     [ 0 cos(y) ]
     [ ]
     [ cos(y) - x sin(y) ]
  (%o1)

  (%i2) depends (F, [a, b]);
  (%o2) [F(a, b)]
  (%i3) hessian (F, [a, b]);
hilbert_matrix (n)  
Return the n by n Hilbert matrix. When n isn’t a positive integer, signal an error.

identfor 
identfor (M)  
identfor (M, fld)
Return an identity matrix that has the same shape as the matrix M. The diagonal entries of the identity matrix are the multiplicative identity of the field fld; the default for fld is generalring.
The first argument M should be a square matrix or a non-matrix. When M is a matrix, each entry of M can be a square matrix – thus M can be a blocked Maxima matrix. The matrix can be blocked to any (finite) depth.
See also zerofor

invert_by_lu (M, (rng generalring))  
Invert a matrix M by using the LU factorization. The LU factorization is done using the ring rng.

jacobian (f, x)  
Returns the Jacobian matrix of the list of functions f with respect to the list of variables x. The (i, j)-th element of the Jacobian matrix is \( \text{diff}(f[i], x[j]) \).
Examples:

```
(%i1) jacobian ([sin (u - v), sin (u * v)], [u, v]);
(%o1)                 [                      ]
[ cos(v - u) - cos(v - u) ]
[                      ]
[ v cos(u v) u cos(u v) ]

(%i2) depends ([F, G], [y, z]);

(%o2)                      [F(y, z), G(y, z)]

(%i3) jacobian ([F, G], [y, z]);

(%o3)                      [                      ]
[ dF  dF ]
[ -- -- ]
[ dy  dz ]
```
kronecker_product \((A, B)\)

Return the Kronecker product of the matrices \(A\) and \(B\).

listp

\[\text{listp} (e, p)\]

\[\text{listp} (e)\]

Given an optional argument \(p\), return true if \(e\) is a Maxima list and \(p\) evaluates to true for every list element. When listp is not given the optional argument, return true if \(e\) is a Maxima list. In all other cases, return false.

locate_matrix_entry \((M, r_1, c_1, r_2, c_2, f, \text{rel})\)

The first argument must be a matrix; the arguments \(r_1\) through \(c_2\) determine a sub-matrix of \(M\) that consists of rows \(r_1\) through \(r_2\) and columns \(c_1\) through \(c_2\).

Find an entry in the sub-matrix \(M\) that satisfies some property. Three cases:

1. rel = 'bool and \(f\) a predicate:
   Scan the sub-matrix from left to right then top to bottom, and return the index of the first entry that satisfies the predicate \(f\). If no matrix entry satisfies \(f\), return false.

2. rel = 'max and \(f\) real-valued:
   Scan the sub-matrix looking for an entry that maximizes \(f\). Return the index of a maximizing entry.

3. rel = 'min and \(f\) real-valued:
   Scan the sub-matrix looking for an entry that minimizes \(f\). Return the index of a minimizing entry.

lu_backsub \((M, b)\)

When \(M = \text{lu_factor} (A, \text{field})\), then \(\text{lu_backsub} (M, b)\) solves the linear system \(A \times = b\).

The \(n\) by \(m\) matrix \(b\), with \(n\) the number of rows of the matrix \(A\), contains one right hand side per column. If there is only one right hand side then \(b\) must be a \(n\) by 1 matrix.

Each column of the matrix \(x=\text{lu_backsub} (M, b)\) is the solution corresponding to the respective column of \(b\).

Examples:

\[
(\%i1) \ A : \text{matrix ([[1 - z, 3], [3, 8 - z]]);}
\]

\[
(\%o1) \begin{bmatrix}
1 - z & 3 \\
3 & 8 - z
\end{bmatrix}
\]

\[
(\%i2) \ M : \text{lu_factor} (A, \text{generalring});
\]

\[
(\%o2) \begin{bmatrix}
1 - z & 3 \\
3 & 8 - z
\end{bmatrix}
\]

\[
(\%i3) \ b : \text{matrix([a],[c]);}
\]

\[
(\%o3) \begin{bmatrix}
a
\end{bmatrix}
\]
(%i3) [ c ]
(%i4) x : lu_backsub(M, b);

[ 3 a
 3 (c - ----)
 1 - z
 a - --------------
 9
 (- z) - ---- + 8
 1 - z
 ------------------
(%o4) [ 1 - z

(%o5) [ 0 ]
(%o6) [ a d ]
(%o7) [ c f ]
(%i8) ratsimp(A . x - B);
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lu_factor (M, field)

Return a list of the form \([LU, \text{perm}, \text{fld}]\), or \([LU, \text{perm}, \text{fld}, \text{lower-cnd upper-cnd}]\), where

1. The matrix \(LU\) contains the factorization of \(M\) in a packed form. Packed form means three things: First, the rows of \(LU\) are permuted according to the list \(\text{perm}\). If, for example, \(\text{perm}\) is the list \([3, 2, 1]\), the actual first row of the \(LU\) factorization is the third row of the matrix \(LU\). Second, the lower triangular factor of \(M\) is the lower triangular part of \(LU\) with the diagonal entries replaced by all ones. Third, the upper triangular factor of \(M\) is the upper triangular part of \(LU\).

2. When the field is either \(\text{floatfield}\) or \(\text{complexfield}\), the numbers \(\text{lower-cnd}\) and \(\text{upper-cnd}\) are lower and upper bounds for the infinity norm condition number of \(M\). For all fields, the condition number might not be estimated; for such fields, \(\text{lu_factor}\) returns a two item list. Both the lower and upper bounds can differ from their true values by arbitrarily large factors. (See also \(\text{mat_cond}\).)

The argument \(M\) must be a square matrix.

The optional argument \(\text{fld}\) must be a symbol that determines a ring or field. The pre-defined fields and rings are:

(a) \(\text{generalring}\) – the ring of Maxima expressions,
(b) \(\text{floatfield}\) – the field of floating point numbers of the type double,
(c) \(\text{complexfield}\) – the field of complex floating point numbers of the type double,
(d) \(\text{crering}\) – the ring of Maxima CRE expressions,
(e) \(\text{rationalfield}\) – the field of rational numbers,
(f) \(\text{runningerror}\) – track the all floating point rounding errors,
(g) \(\text{noncommutingring}\) – the ring of Maxima expressions where multiplication is the non-commutative dot operator.

When the field is \(\text{floatfield}, \text{complexfield},\) or \(\text{runningerror}\), the algorithm uses partial pivoting; for all other fields, rows are switched only when needed to avoid a zero pivot.

Floating point addition arithmetic isn’t associative, so the meaning of ‘field’ differs from the mathematical definition.

A member of the field \(\text{runningerror}\) is a two member Maxima list of the form \([x, n]\), where \(x\) is a floating point number and \(n\) is an integer. The relative difference between the ‘true’ value of \(x\) and \(x\) is approximately bounded by the machine epsilon times \(n\). The running error bound drops some terms that of the order the square of the machine epsilon.

There is no user-interface for defining a new field. A user that is familiar with Common Lisp should be able to define a new field. To do this, a user must define functions for the arithmetic operations and functions for converting from the field representation to Maxima and back. Additionally, for ordered fields (where partial pivoting will be used), a user must define functions for the magnitude and for comparing field
members. After that all that remains is to define a Common Lisp structure \texttt{mring}. The file \texttt{mring} has many examples.

To compute the factorization, the first task is to convert each matrix entry to a member of the indicated field. When conversion isn’t possible, the factorization halts with an error message. Members of the field needn’t be Maxima expressions. Members of the \texttt{complexfield}, for example, are Common Lisp complex numbers. Thus after computing the factorization, the matrix entries must be converted to Maxima expressions.

See also \texttt{get_lu_factors}.

Examples:

\begin{verbatim}
(%i1) w[i,j] := random (1.0) + %i * random (1.0);
(%o1) w := random(1.) + %i random(1.)

(%i2) showtime : true$
Evaluation took 0.00 seconds (0.00 elapsed)

(%i3) M : genmatrix (w, 100, 100)$
Evaluation took 7.40 seconds (8.23 elapsed)

(%i4) lu_factor (M, complexfield)$
Evaluation took 28.71 seconds (35.00 elapsed)

(%i5) lu_factor (M, generalring)$
Evaluation took 109.24 seconds (152.10 elapsed)

(%i6) showtime : false$

(%i7) M : matrix ([1 - z, 3], [3, 8 - z]);
   [ 1 - z  3 ]
   [           ]
   [ 3   8 - z ]

(%i8) lu_factor (M, generalring);
   [ 1 - z  3 ]
   [           ]
   [ 3   9 ]
   [ 1, 2], [1 - z, 2 - z + 8]

(%i9) get_lu_factors (%);
   [ 1 0 ] [ 1 - z  3 ]
   [ 1 0 ] [           ]
   [ 0 1 ] [ 1 - z  3 ]
   [ 0 1 ] [ 1 - z, 2 - z + 8]

(%i10) %[1] . %[2] . %[3];
   [ 1 - z  3 ]
   [           ]
   [ 3   8 - z ]
\end{verbatim}
mat_cond

mat_cond (M, 1)
mat_cond (M, inf)

Return the $p$-norm matrix condition number of the matrix $m$. The allowed values for $p$ are 1 and inf. This function uses the LU factorization to invert the matrix $m$. Thus the running time for mat_cond is proportional to the cube of the matrix size; lu_factor determines lower and upper bounds for the infinity norm condition number in time proportional to the square of the matrix size.

mat_norm

mat_norm (M, 1)
mat_norm (M, inf)
mat_norm (M, frobenius)

Return the matrix $p$-norm of the matrix $M$. The allowed values for $p$ are 1, inf, and frobenius (the Frobenius matrix norm). The matrix $M$ should be an unblocked matrix.

matrixp

matrixp (e, p)
matrixp (e)

Given an optional argument $p$, return true if $e$ is a matrix and $p$ evaluates to true for every matrix element. When matrixp is not given an optional argument, return true if $e$ is a matrix. In all other cases, return false.

See also blockmatrixp

matrix_size (M)

Return a two member list that gives the number of rows and columns, respectively of the matrix $M$.

mat_fullunblocker (M)

If $M$ is a block matrix, unblock the matrix to all levels. If $M$ is a matrix, return $M$; otherwise, signal an error.

mat_trace (M)

Return the trace of the matrix $M$. If $M$ isn’t a matrix, return a noun form. When $M$ is a block matrix, mat_trace(M) returns the same value as does mat_trace(mat_unblocker(m)).

mat_unblocker (M)

If $M$ is a block matrix, unblock $M$ one level. If $M$ is a matrix, mat_unblocker (M) returns $M$; otherwise, signal an error.

Thus if each entry of $M$ is matrix, mat_unblocker (M) returns an unblocked matrix, but if each entry of $M$ is a block matrix, mat_unblocker (M) returns a block matrix with one less level of blocking.

If you use block matrices, most likely you’ll want to set matrix_element_mult to "." and matrix_element_transpose to 'transpose. See also mat_fullunblocker.

Example:

(%i1) A : matrix ([[1, 2], [3, 4]]);
(%i1) [1 2]
    [3 4]
(%o1)

(%i2) B : matrix([[7, 8], [9, 10]]);
    [ 7 8 ]
    [ ]
(%o2) [ ]
    [ 9 10 ]

(%i3) matrix([[A, B]]);
    [ [ 1 2 ] [ 7 8 ] ]
    [ [ ] [ ] ]
    [ [ 3 4 ] [ 9 10 ] ]
(%o3)

(%i4) mat_unblocker (%);
    [ 1 2 7 8 ]
    [ ]
    [ 3 4 9 10 ]

nullspace (M)

If M is a matrix, return span (v_1, ..., v_n), where the set \{v_1, ..., v_n\} is a basis for the nullspace of M. The span of the empty set is \{0\}. Thus, when the nullspace has only one member, return span ().

nullity (M)

If M is a matrix, return the dimension of the nullspace of M.

orthogonal_complement (v_1, ..., v_n)

Return span (u_1, ..., u_m), where the set \{u_1, ..., u_m\} is a basis for the orthogonal complement of the set \{v_1, ..., v_n\}.

Each vector v_1 through v_n must be a column vector.

polytocompanion (p, x)

If p is a polynomial in x, return the companion matrix of p. For a monic polynomial p of degree n, we have p = (-1)^n charpoly (polytocompanion (p, x)).

When p isn't a polynomial in x, signal an error.

ptriangularize (M, v)

If M is a matrix with each entry a polynomial in v, return a matrix M2 such that
(1) M2 is upper triangular,
(2) M2 = E_n \ldots E_1 M, where E_1 through E_n are elementary matrices whose entries are polynomials in v,
(3) \mid \text{det} (M) \mid = \mid \text{det} (M2) \mid,

Note: This function doesn't check that every entry is a polynomial in v.

rowop (M, i, j, theta)

If M is a matrix, return the matrix that results from doing the row operation R_i \leftarrow R_i - theta * R_j. If M doesn't have a row i or j, signal an error.
**linalg_rank** *(M)*

Return the rank of the matrix *M*. This function is equivalent to function `rank`, but it uses a different algorithm: it finds the column space of the matrix and counts its elements, since the rank of a matrix is the dimension of its column space.

```
(%i1) linalg_rank(matrix([1,2],[2,4]));
(%o1) 1
(%i2) linalg_rank(matrix([1,b],[c,d]));
(%o2) 2
```

**rowswap** *(M, i, j)*

If *M* is a matrix, swap rows *i* and *j*. If *M* doesn’t have a row *i* or *j*, signal an error.

**toeplitz**

```
toeplitz (col)
toeplitz (col, row)
```

Return a Toeplitz matrix *T*. The first first column of *T* is *col*; except for the first entry, the first row of *T* is row. The default for row is complex conjugate of *col*. Example:

```
(%i1) toeplitz([1,2,3],[x,y,z]);
     [ 1 y z ]
     [      ]
     [ 2 1 y ]
     [      ]
     [ 3 2 1 ]
(%o1)
(%i2) toeplitz([1,1+%i]);
     [ 1 1 - %I ]
     [        ]
     [ %I + 1 1 ]
(%o2)
```

**vandermonde_matrix** *(x_1, ..., x_n)*

Return a *n* by *n* matrix whose *i*-th row is \([1, x_i, x_i^2, \ldots, x_i^{(n-1)}]\).

**zerofor**

```
zerofor (M)
zerofor (M, fld)
```

Return a zero matrix that has the same shape as the matrix *M*. Every entry of the zero matrix is the additive identity of the field *fld*; the default for *fld* is `generalring`. The first argument *M* should be a square matrix or a non-matrix. When *M* is a matrix, each entry of *M* can be a square matrix – thus *M* can be a blocked Maxima matrix. The matrix can be blocked to any (finite) depth.

See also `identfor`

**zeromatrixp** *(M)*

If *M* is not a block matrix, return `true` if \((\text{equal} (e, 0))\) is true for each element *e* of the matrix *M*. If *M* is a block matrix, return `true` if `zeromatrixp` evaluates to `true` for each element of *e*. 
72 lsquares

72.1 Introduction to lsquares

lsquares is a collection of functions to implement the method of least squares to estimate parameters for a model from numerical data.

72.2 Functions and Variables for lsquares

lsquares_estimates (D, x, e, a)
lsquares_estimates (D, x, e, a, initial = L, tol = t)

Estimate parameters a to best fit the equation e in the variables x and a to the data D, as determined by the method of least squares. lsquares_estimates first seeks an exact solution, and if that fails, then seeks an approximate solution.

The return value is a list of lists of equations of the form [a = ..., b = ..., c = ...]. Each element of the list is a distinct, equivalent minimum of the mean square error.

The data D must be a matrix. Each row is one datum (which may be called a ‘record’ or ‘case’ in some contexts), and each column contains the values of one variable across all data. The list of variables x gives a name for each column of D, even the columns which do not enter the analysis. The list of parameters a gives the names of the parameters for which estimates are sought. The equation e is an expression or equation in the variables x and a; if e is not an equation, it is treated the same as e = 0.

Additional arguments to lsquares_estimates are specified as equations and passed on verbatim to the function lbfgs which is called to find estimates by a numerical method when an exact result is not found.

If some exact solution can be found (via solve), the data D may contain non-numeric values. However, if no exact solution is found, each element of D must have a numeric value. This includes numeric constants such as %pi and %e as well as literal numbers (integers, rationals, ordinary floats, and bigfloats). Numerical calculations are carried out with ordinary floating-point arithmetic, so all other kinds of numbers are converted to ordinary floats for calculations.

If lsquares_estimates needs excessive amounts of time or runs out of memory lsquares_estimates_approximate, which skips the attempt to find an exact solution, might still succeed.

load("lsquares") loads this function.

See also lsquares_estimates_exact, lsquares_estimates_approximate, lsquares_mse, lsquares_residuals, and lsquares_residual_mse.

Examples:

A problem for which an exact solution is found.

(\%i1) load ("lsquares")$
(\%i2) M : matrix (
\([1, 1, 1], [3/2, 1, 2], [9/4, 2, 1], [3, 2, 2], [2, 2, 1]\);
\[
\begin{array}{cccc}
1 & 1 & 1 \\
3 & & \\
-1 & 2 & \\
& 2 & \\
3 & 2 & 2 \\
& & 2 & 2 & 1 \\
\end{array}
\]

\%(o2)\[
\begin{array}{cccc}
9 & \\
-2 & 1 & \\
4 & \\
& \\
3 & 2 & 2 \\
& \\
2 & 2 & 1 \\
\end{array}
\]

\%(i3) \texttt{lsquares\_estimates (}
\texttt{M, [z,x,y], (z+D)^2 = A*x+B*y+C}}, \texttt{[A,B,C,D]);}
\[
59 & 27 & 10921 & 107
\]

\%(o3) \[
\begin{array}{cccc}
[A = --, B = --, C = ------, D = -- -]
16 & 16 & 1024 & 32
\end{array}
\]

A problem for which no exact solution is found, so \texttt{lsquares\_estimates} resorts to numerical approximation.

\%(i1) \texttt{load ("lsquares")}$
\%(i2) \texttt{M : matrix ([1, 1], [2, 7/4], [3, 11/4], [4, 13/4]);}
\[
\begin{array}{cccc}
1 & 1 \\
7 & \\
2 & \\
4 & \\
11 & \\
3 & \\
4 & \\
13 & \\
4 & \\
4 & \\
\end{array}
\]

\%(o2) \[
\begin{array}{cccc}
9 & \\
-2 & 1 & \\
4 & \\
& \\
3 & 2 & 2 \\
& \\
2 & 2 & 1 \\
\end{array}
\]

\%(i3) \texttt{lsquares\_estimates (}
\texttt{M, [x,y], y=a*x^b+c}}, \texttt{[a,b,c], initial=[3,3,3], iprint=[-1,0]);}
\[
\begin{array}{cccc}
A = 1.375751433061394, & B = 0.7148891534417651, \\
c = -0.4020908910062951
\end{array}
\]

Exponential functions aren't well-conditioned for least min square fitting. In case that fitting to them fails it might be possible to get rid of the exponential function using an logarithm.

\%(i1) \texttt{load ("lsquares")}$
\%(i2) \texttt{yvalues: [1,3,5,60,200,203,80]$}
\%(i3) \texttt{time: [1,2,4,5,6,8,10]$}
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(%i4) f: y=a*exp(b*t);

(%o4) y = a \cdot e^{b \cdot t}

(%i5) yvalues_log: log(yvalues)$

(%i6) f_log: log(subst(y=exp(y),f));

(%o6) y = \log(a \cdot e^{b \cdot t})

(%i7) lsquares_estimates (transpose(matrix(yvalues_log,time)), [y,t], f_log, [a,b]);

*************************************************
N= 2 NUMBER OF CORRECTIONS=25
INITIAL VALUES
F= 6.802906290754687D+00 GNORM= 2.851243373781393D+01
*************************************************

I NFN FUNC GNORM STEPLENGTH
1 3 1.14183876593467D+00 1.067358003667488D-01 1.390943719972406D-02
2 5 1.141118195694385D+00 1.23797783303414D-01 5.000000000000000D+00
3 6 1.136945723147959D+00 3.80669691691383D-01 1.000000000000000D+00
4 7 1.13958243220262D+00 3.865103650379243D-02 1.000000000000000D+00
5 8 1.131725773805499D+00 2.922582311540262D-02 1.000000000000000D+00
6 9 1.131625585689168D+00 2.66440547017370D-03 1.000000000000000D+00
7 10 1.13162056485699D+00 2.519366958715444D-04 1.000000000000000D+00

THE MINIMIZATION TERMINATED WITHOUT DETECTING ERRORS.
IFLAG = 0
(%o7) \{[a = 1.155904145765554, b = 0.5772666876959847]\]

lsquares_estimates_exact (MSE, a) [Function]

Estimate parameters a to minimize the mean square error MSE, by constructing a
system of equations and attempting to solve them symbolically via solve. The mean
square error is an expression in the parameters a, such as that returned by lsquares_mse.

The return value is a list of lists of equations of the form [a = ..., b = ..., c =
...]. The return value may contain zero, one, or two or more elements. If two or
more elements are returned, each represents a distinct, equivalent minimum of the
mean square error.

See also lsquares_estimates, lsquares_estimates_approximate, lsquares_mse,
lsquares_residuals, and lsquares_residual_mse.

Example:

(%i1) load ("lsquares")$

(%i2) M : matrix ( [1,1,1], [3/2,1,2], [9/4,2,1], [3,2,2], [2,2,1]);

[ 1 1 1 ]
[     ]
lsquares_estimates_approximate (MSE, a, initial = L, tol = t) [Function]
Estimate parameters \( a \) to minimize the mean square error \( MSE \), via the numerical minimization function lbfgs. The mean square error is an expression in the parameters \( a \), such as that returned by lsquares_mse.

The solution returned by lsquares_estimates_approximate is a local (perhaps global) minimum of the mean square error. For consistency with lsquares_estimates_exact, the return value is a nested list which contains one element, namely a list of equations of the form \([a = \ldots, b = \ldots, c = \ldots]\).

Additional arguments to lsquares_estimates_approximate are specified as equations and passed on verbatim to the function lbfgs.

\( MSE \) must evaluate to a number when the parameters are assigned numeric values. This requires that the data from which \( MSE \) was constructed comprise only numeric constants such as \( \%\pi \) and \( \%e \) and literal numbers (integers, rationals, ordinary floats, and bigfloats). Numerical calculations are carried out with ordinary floating-point arithmetic, so all other kinds of numbers are converted to ordinary floats for calculations.

load("lsquares") loads this function.

See also lsquares_estimates, lsquares_estimates_exact, lsquares_mse, lsquares_residuals, and lsquares_residual_mse.

Example:
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(%i1) load ("lsquares")$
(%i2) M : matrix ([1,1,1], [3/2,1,2], [9/4,2,1], [3,2,2], [2,2,1]);

(%o2)
[ 1 1 1 ]
[ 3/2 1 2 ]
[ 9/4 2 1 ]
[ 3 2 2 ]
[ 2 2 1 ]

(%i3) mse : lsquares_mse (M, [z, x, y], (z + D)^2 = A*x + B*y + C);

5
====
\ 2 2
> ((- B M ) - A M + (M + D) - C)
/ i, 3 i, 2 i, 1
====
 i = 1

(%o3)-------------------------------------------------------------------------- 5

(%i4) lsquares_estimates_approximate (mse, [A, B, C, D], iprint = [-1, 0]);

(%o4) [[A = - 3.67850494701971, B = - 1.683070351177937,  
 C = 10.63469950148714, D = - 3.340357993175297]]

lsquares_mse (D, x, e) [Function]
Returns the mean square error (MSE), a summation expression, for the equation e in
the variables x, with data D.
The MSE is defined as:

\[
\frac{1}{n}\sum_{i=1}^{n}[\text{lhs}(e_i) - \text{rhs}(e_i)]^2,
\]

where n is the number of data and e[i] is the equation e evaluated with the variables
in x assigned values from the i-th datum, D[i].
load("lsquares") loads this function.
Example:
(%i1) load ("lsquares")$
(%i2) M : matrix ([1,1,1], [3/2,1,2], [9/4,2,1], [3,2,2], [2,2,1]);
\begin{align*}
\text{(\%o2)} & \quad \begin{bmatrix}
1 & 1 & 1 \\
\end{bmatrix} \\
\text{(\%i3)} & \quad \text{mse : \texttt{lsquares_mse} (M, [z, x, y], (z + D)^2 = A*x + B*y + C);} \\
& \quad 5 \quad \begin{align*}
& \quad \text{====} \\
& \quad \begin{array}{c}
\text{\textbackslash } \\
\text{2} \\
\text{2} \\
\end{array} \\
& \quad \begin{array}{c}
\text{\textbackslash } \\
\text{\textbackslash } \\
\text{i, 1} \\
\text{i, 3} \\
\text{i, 2} \\
\text{i, 1} \\
\end{array} \\
& \quad \text{====} \\
& \quad i = 1 \\
\end{align*} \\
\text{(\%o3)} & \quad \begin{align*}
& \quad \begin{bmatrix}
9 \\
- 2 & 1 \\
4 \\
\end{bmatrix} \\
\text{(\%i4)} & \quad \text{diff (mse, D);} \\
\text{(\%o4)} & \quad 5 \quad \begin{align*}
& \quad \text{====} \\
& \quad \begin{array}{c}
\text{\textbackslash } \\
\text{4} \\
\text{2} \\
\end{array} \\
& \quad \begin{array}{c}
\text{\textbackslash } \\
\text{\textbackslash } \\
\text{i, 1} \\
\text{i, 3} \\
\text{i, 2} \\
\text{i, 1} \\
\end{array} \\
& \quad \text{====} \\
& \quad i = 1 \\
\end{align*} \\
\text{(\%i5)} & \quad \text{\texttt{\textbackslash\textbackslash}mse, nouns;} \\
\text{(\%o5)} & \quad \begin{align*}
& \quad \frac{\text{2} \quad 2 \quad 9 \quad 2 \quad 2}{4} \\
& \quad + \frac{\text{2} \quad 3 \quad 2 \quad 2}{2} \\
& \quad + \frac{\text{2} \quad 2 \quad 2}{2} \\
& \quad + \frac{\text{2} \quad 2 \quad 2}{2} \\
& \quad + \frac{\text{2} \quad 2 \quad 2}{2} \\
\end{align*}
\end{align*}
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%i3) mse : lsquares_mse (M, [z, x, y], (z + D)^2 = A*x + B*y + C);
5
====
\  2  2
> ((D + M ) - C - M B - M A)
/ i, 1 i, 3 i, 2
====
i = 1
%o3)
----------------------------------------------------------

%i4) diff (mse, D);
5
====
\  4
> (D + M ) ((D + M ) - C - M B - M A)
/ i, 1 i, 1 i, 3 i, 2
====
i = 1
%o4)----------------------------------------------------------

%i5) 'mse, nouns;
2 2 2 2 9 2 2
%o5) (((D + 3) - C - 2 B - 2 A) + ((D + -) - C - B - 2 A)
2 2 2 2 3 2 2
+ ((D + 2) - C - B - 2 A) + ((D + -) - C - 2 B - A)
2
2
+ ((D + 1) - C - B - A))/5
lsquares_residuals (D, x, e, a)
[Function]
Returns the residuals for the equation e with specified parameters a and data D.
D is a matrix, x is a list of variables, e is an equation or general expression; if not an
equation, e is treated as if it were e = 0. a is a list of equations which specify values
for any free parameters in e aside from x.
The residuals are defined as:

\text{llhs}(e_i) - \text{rhs}(e_i),

where \(e[i]\) is the equation e evaluated with the variables in x assigned values from
the i-th datum, \(D[i]\), and assigning any remaining free variables from a.
load("lsquares") loads this function.
Example:

%i1) load("lsquares")$
%i2) M : matrix(
   [1,1,1], [3/2,1,2], [9/4,2,1], [3,2,2], [2,2,1]);
(%i2) % 

(%i3) a : lsquares_estimates (
M, [z,x,y], (z+D)^2 = A*x+B*y+C, [A,B,C,D]);

(%o3) [[A = - --, B = - --, C = ------, D = - ----]]
16 16 1024 32

(%i4) lsquares_residuals (M, [z,x,y], (z+D)^2 = A*x+B*y+C, first(a));

(%o4) [--, --, --, --, --]
64 64 32 64 64

lsquares_residual_mse (D, x, e, a) [Function]
Returns the residual mean square error (MSE) for the equation e with specified parameters a and data D.
The residual MSE is defined as:
\[
\frac{1}{n} \sum_{i=1}^{n} [\text{lhs}(e_i) - \text{rhs}(e_i)]^2,
\]
where e[i] is the equation e evaluated with the variables in x assigned values from the i-th datum, D[i], and assigning any remaining free variables from a.
load("lsquares") loads this function.

Example:
(%i11) load ("lsquares")$

(%i12) M : matrix (1,1,1, 3/2,1,2, [9/4,2,1], [3,2,2], [2,2,1]);

(%o12) [ 1 1 1 ]
[ 3 ]
[ - 1 2 ]
[ 2 ]
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\[
\begin{bmatrix}
-2 & 1 \\
4 & \\
3 & 2 & 2 \\
2 & 2 & 1
\end{bmatrix}
\]

(%i3) a : lsquares_estimates (M, [z,x,y], (z+D)^2 = A*x+B*y+C, [A,B,C,D]);

\[
[[A = - --, B = - --, C = ------, D = - ---]]
\]

(%i4) lsquares_residual_mse (M, [z,x,y], (z + D)^2 = A*x + B*y + C, first (a));

\[
\frac{169}{2560}
\]

plsquares

gives

\[
\frac{11 \ y - 9 \ x - 14}{3}
\]

The same example without degree restrictions:

(%i11) load("plsquares")$

(%i12) plsquares(matrix([[1,2,0],[3,5,4],[4,7,9],[5,8,10]],
[x,y,z],z);

Determination Coefficient for z = .9897039897039897

\[
11 \ y - 9 \ x - 14
\]

\[
\frac{z = \text{-------------}}{3}
\]
(%i3) plsquares(matrix([1,2,0],[3,5,4],[4,7,9],[5,8,10]),
[x,y,z],z,1,0);
Determination Coefficient for z = 1.0
x y + 23 y - 29 x - 19
(%o3) z = ----------------------
6

How many diagonals does a \(N\)-sides polygon have? What polynomial degree should be used?

(%i4) plsquares(matrix([3,0],[4,2],[5,5],[6,9],[7,14],[8,20]),
[N,diagonals],diagonals,5);
Determination Coefficient for diagonals = 1.0
\[ \frac{2}{N - 3 N} \]
(%o4) diagonals = -------
2
(%i5) ev(%[1], N=9); /* Testing for a 9 sides polygon */
(%o5) diagonals = 27

How many ways do we have to put two queens without they are threatened into a \(n \times n\) chessboard?

(%i6) plsquares(matrix([0,0],[1,0],[2,0],[3,8],[4,44]),
[n,positions],[positions],4);
Determination Coefficient for \([\text{positions}]=1.0\]
\[ \frac{4 \ 3 \ 2}{3 n - 10 n + 9 n - 2 n} \]
(%o6) \text{positions} = -------------------------
6
(%i7) ev(%[1], n=8); /* Testing for a \((8 \times 8)\) chessboard */
(%o7) positions = 1288

An example with six dependent variables:

(%i8) mtrx:matrix([0,0,0,0,0,1,1,1],[0,1,0,1,1,1,0,0],
[1,0,0,1,1,1,0,0],[1,1,1,1,0,0,0,1])$
(%i8) plsquares(mtrx,[a,b,_And,_Or,_Xor,_Nand,_Nor,_Nxor],
[\text{\_And, \_Or, \_Xor, \_Nand, \_Nor, \_Nxor}],1,0);
Determination Coefficient for
\text{\_And, \_Or, \_Xor, \_Nand, \_Nor, \_Nxor} =
\text{\text{[1.0, 1.0, 1.0, 1.0, 1.0]} \}
(%o2) \text{\_And = a b, \_Or = a b + b + a,}
\text{\_Xor = - 2 a b + b + a, \_Nand = 1 - a b,}
\text{\_Nor = a b - b - a + 1, \_Nxor = 2 a b - b - a + 1}]

To use this function write first load("lsquares").
73 minpack

73.1 Introduction to minpack

Minpack is a Common Lisp translation (via f2cl) of the Fortran library MINPACK, as obtained from Netlib.

73.2 Functions and Variables for minpack

**minpack_lsquares**

```lisp
minpack_lsquares (flist, varlist, guess)
minpack_lsquares (...,'tolerance = tolerance)
minpack_lsquares (...,'jacobian = jacobian)
```

Compute the point that minimizes the sum of the squares of the functions in the list `flist`. The variables are in the list `varlist`. An initial guess of the optimum point must be provided in `guess`.

The optional keyword arguments, `tolerance` and `jacobian` provide some control over the algorithm. `tolerance` is the estimated relative error desired in the sum of squares. `jacobian` can be used to specify the Jacobian. If `jacobian` is not given or is `true` (the default), the Jacobian is computed from `flist`. If `jacobian` is `false`, a numerical approximation is used.

`minpack_lsquares` returns a list. The first item is the estimated solution; the second is the sum of squares, and the third indicates the success of the algorithm. The possible values are

0 improper input parameters.
1 algorithm estimates that the relative error in the sum of squares is at most `tolerance`.
2 algorithm estimates that the relative error between x and the solution is at most `tolerance`.
3 conditions for info = 1 and info = 2 both hold.
4 fvec is orthogonal to the columns of the jacobian to machine precision.
5 number of calls to fcn with iflag = 1 has reached 100*(n+1).
6 tol is too small. no further reduction in the sum of squares is possible.
7 tol is too small. no further improvement in the approximate solution x is possible.

/* Problem 6: Powell singular function */

```lisp
(%i1) powell(x1,x2,x3,x4) :=
    [x1+10*x2, sqrt(5)*(x3-x4), (x2-2*x3)^2, sqrt(10)*(x1-x4)^2]$

(%i2) minpack_lsquares(powell(x1,x2,x3,x4), [x1,x2,x3,x4], [3,-1,0,1]);
(%o2) [[1.652117596168394e-17, - 1.652117596168393e-18,
```

2.643388153869468e-18, 2.643388153869468e-18],
   6.109327859207777e-34, 4]
/* Same problem but use numerical approximation to Jacobian */
(%i3) minpack_lsquares(powell(x1,x2,x3,x4), [x1,x2,x3,x4],
   [3,-1,0,1], jacobian = false);
(%o3) [[5.060282149485331e-11, - 5.060282149491206e-12,
   2.179447843547218e-11, 2.179447843547218e-11],
   3.534491794847031e-21, 5]

minpack_solve

minpack_solve (flist, varlist, guess)
minpack_solve (...,'tolerance = tolerance)
minpack_solve (...,'jacobian = jacobian)

Solve a system of \( n \) equations in \( n \) unknowns. The \( n \) equations are given in the list \( f\text{list} \), and the unknowns are in \( \text{varlist} \). An initial guess of the solution must be provided in \( \text{guess} \).

The optional keyword arguments, \( \text{tolerance} \) and \( \text{jacobian} \) provide some control over the algorithm. \( \text{tolerance} \) is the estimated relative error desired in the sum of squares. \( \text{jacobian} \) can be used to specify the Jacobian. If \( \text{jacobian} \) is not given or is \text{true} \) (the default), the Jacobian is computed from \( f\text{list} \). If \( \text{jacobian} \) is \text{false}, a numerical approximation is used.

\( \text{minpack_solve} \) returns a list. The first item is the estimated solution; the second is the sum of squares, and the third indicates the success of the algorithm. The possible values are

0 improper input parameters.
1 algorithm estimates that the relative error in the solution is at most \( \text{tolerance} \).
2 number of calls to \( f\text{cn} \) with iflag = 1 has reached \( 100\times(n+1) \).
3 tol is too small. no further reduction in the sum of squares is possible.
4 Iteration is not making good progress.

/* Problem 6: Powell singular function */
(%i1) powell(x1,x2,x3,x4) :=
   [x1+10*x2, sqrt(5)*(x3-x4), (x2-2*x3)^2,
   sqrt(10)*(x1-x4)^2]$
(%i2) minpack_lsquares(powell(x1,x2,x3,x4), [x1,x2,x3,x4],
   [3,-1,0,1]);
(%o2) [[8.586306796471285e-19, - 8.586306796471285e-20,
   2.19447843547218e-11, 2.19447843547218e-11],
   3.53491794847031e-21, 5]

In this particular case, we can solve this analytically:

(%i3) solve(powell(x1,x2,x3,x4),[x1,x2,x3,x4]);
(%o3) [[x1 = 0, x2 = 0, x3 = 0, x4 = 0]]

and we see that the numerical solution is quite close the analytical one.
74 makeOrders

74.1 Functions and Variables for makeOrders

\textbf{makeOrders (indvarlist,orderlist)} \hspace{1cm} \textbf{[Function]}

Returns a list of all powers for a polynomial up to and including the arguments.

\begin{verbatim}
(%i1) load("makeOrders")$

(%i2) makeOrders([a,b],[2,3]);
(%o2) [[0, 0], [0, 1], [0, 2], [0, 3], [1, 0], [1, 1],
     [1, 2], [1, 3], [2, 0], [2, 1], [2, 2], [2, 3]]

(%i3) expand((1+a+a^2)*(1+b+b^2+b^3));
(\%o3) a b + a b + b + a b + a b + b + a b + a b
       + b + a + a + 1
\end{verbatim}

where [0, 1] is associated with the term \(b\) and [2, 3] with \(a^2b^3\).

To use this function write first \texttt{load("makeOrders")}.
75 mnewton

75.1 Introduction to mnewton

mnewton is an implementation of Newton’s method for solving nonlinear equations in one or more variables.

75.2 Functions and Variables for mnewton

newtonepsilon [Option variable]

Default value: $10.0^{-\text{fpprec}/2}$

Precision to determine when the mnewton function has converged towards the solution. If newtonepsilon is a bigfloat, then mnewton computations are done with bigfloats. See also mnewton.

newtonmaxiter [Option variable]

Default value: 50

Maximum number of iterations to stop the mnewton function if it does not converge or if it converges too slowly. See also mnewton.

mnewton [Function]

mnewton(FuncList, VarList, GuessList)
mnewton(FuncList, VarList, GuessList, DF)

Multiple nonlinear functions solution using the Newton method. FuncList is the list of functions to solve, VarList is the list of variable names, and GuessList is the list of initial approximations. The optional argument DF is the Jacobian matrix of the list of functions; if not supplied, it is calculated automatically from FuncList.

The solution is returned in the same format that solve() returns. If the solution is not found, [] is returned.

This function is controlled by global variables newtonepsilon and newtonmaxiter. See also realroots, allroots, find_root and newton.

(%i1) load("mnewton")$

(%i2) mnewton([x1+3*log(x1)-x2^2, 2*x1^2-x1*x2-5*x1+1], [x1, x2], [5, 5]);
(%o2) [[x1 = 3.756834008012769, x2 = 2.779849592817897]]
(%i3) mnewton([2*a^a-5], [a], [1]);
(%o3) [[a = 1.70927556786144]]
(%i4) mnewton([2*3^u-v/u-5, u+2^v-4], [u, v], [2, 2]);
(%o4) [[u = 1.066618389595407, v = 1.552564766841786]]

The variable newtonepsilon controls the precision of the approximations. It also controls if computations are performed with floats or bigfloats.

(%i11) load("mnewton")$
(%i2) (fpprec : 25, newtonepsilon : bfloat(10^(-fpprec+5)))$

(%i3) mnewton([2*3^u-v/u-5, u+2^v-4], [u, v], [2, 2]);
(%o3) [[u = 1.066618389595406772591173b0,
       v = 1.552564766841786450100418b0]]

To use this function write first load("mnewton"). See also newtonepsilon and newtonmaxiter.
76 numericalio

76.1 Introduction to numericalio

numericalio is a collection of functions to read and write files and streams. Functions for plain-text input and output can read and write numbers (integer, float, or bigfloat), symbols, and strings. Functions for binary input and output can read and write only floating-point numbers.

If there already exists a list, matrix, or array object to store input data, numericalio input functions can write data into that object. Otherwise, numericalio can guess, to some degree, the structure of an object to store the data, and return that object.

76.1.1 Plain-text input and output

In plain-text input and output, it is assumed that each item to read or write is an atom: an integer, float, bigfloat, string, or symbol, and not a rational or complex number or any other kind of nonatomic expression. The numericalio functions may attempt to do something sensible faced with nonatomic expressions, but the results are not specified here and subject to change.

Atoms in both input and output files have the same format as in Maxima batch files or the interactive console. In particular, strings are enclosed in double quotes, backslash \ prevents any special interpretation of the next character, and the question mark ? is recognized at the beginning of a symbol to mean a Lisp symbol (as opposed to a Maxima symbol). No continuation character (to join broken lines) is recognized.

76.1.2 Separator flag values for input

The functions for plain-text input and output take an optional argument, separator_flag, that tells what character separates data.

For plain-text input, these values of separator_flag are recognized: comma for comma separated values, pipe for values separated by the vertical bar character |, semicolon for values separated by semicolon ;, and space for values separated by space or tab characters. Equivalently, the separator may be specified as a string of one character: "," (comma), "|" (pipe), ";" (semicolon), " " (space), or " " (tab).

If the file name ends in .csv and separator_flag is not specified, comma is assumed. If the file name ends in something other than .csv and separator_flag is not specified, space is assumed.

In plain-text input, multiple successive space and tab characters count as a single separator. However, multiple comma, pipe, or semicolon characters are significant. Successive comma, pipe, or semicolon characters (with or without intervening spaces or tabs) are considered to have false between the separators. For example, 1234,,Foo is treated the same as 1234,false,Foo.

76.1.3 Separator flag values for output

For plain-text output, tab, for values separated by the tab character, is recognized as a value of separator_flag, as well as comma, pipe, semicolon, and space.
In plain-text output, false atoms are written as such; a list [1234, false, Foo] is written 1234,false,Foo, and there is no attempt to collapse the output to 1234,,Foo.

76.1.4 Binary floating-point input and output

Numericalio functions can read and write 8-byte IEEE 754 floating-point numbers. These numbers can be stored either least significant byte first or most significant byte first, according to the global flag set by assume_external_byte_order. If not specified, numericalio assumes the external byte order is most-significant byte first.

Other kinds of numbers are coerced to 8-byte floats; numericalio cannot read or write binary non-numeric data.

Some Lisp implementations do not recognize IEEE 754 special values (positive and negative infinity, not-a-number values, denormalized values). The effect of reading such values with numericalio is undefined.

Numericalio includes functions to open a stream for reading or writing a stream of bytes.

76.2 Functions and Variables for plain-text input and output

read_matrix

read_matrix (S)
read_matrix (S, M)
read_matrix (S, separator_flag)
read_matrix (S, M, separator_flag)

Reads the source S and returns its entire content as a matrix. The size of the matrix is inferred from the input data; each line of the file becomes one row of the matrix. If some lines have different lengths, read_matrix complains.

read_matrix (S, M) read the source S into the matrix M, until M is full or the source is exhausted. Input data are read into the matrix in row-major order; the input need not have the same number of rows and columns as M.

The source S may be a file name or a stream which for example allows skipping the very first line of a file (that may be useful, if you read CSV data, where the first line often contains the description of the columns):

s : openr("data.txt");
readline(s); /* skip the first line */
M : read_matrix(s, 'comma); /* read the following (comma-separated) lines into matrix M */
close(s);

The recognized values of separator_flag are comma, pipe, semicolon, and space. Equivalently, the separator may be specified as a string of one character: "," (comma), "|" (pipe), ";" (semicolon), " " (space), or " " (tab). If separator_flag is not specified, the file is assumed space-delimited.

See also openr, read_array, read_hashed_array, read_list, read_binary_matrix, write_data and read_nested_list.
read_array

    read_array (S, A)
    read_array (S, A, separator_flag)

Reads the source $S$ into the array $A$, until $A$ is full or the source is exhausted. Input data are read into the array in row-major order; the input need not conform to the dimensions of $A$.

The source $S$ may be a file name or a stream.

The recognized values of separator_flag are comma, pipe, semicolon, and space. Equivalently, the separator may be specified as a string of one character: "," (comma), "|" (pipe), ";" (semicolon), " " (space), or " " (tab). If separator_flag is not specified, the file is assumed space-delimited.

See also openr, read_matrix, read_hashed_array, read_list, read_binary_array and read_nested_list.

read_hashed_array

    read_hashed_array (S, A)
    read_hashed_array (S, A, separator_flag)

Reads the source $S$ and returns its entire content as a hashed array. The source $S$ may be a file name or a stream.

read_hashed_array treats the first item on each line as a hash key, and associates the remainder of the line (as a list) with the key. For example, the line 567 12 17 32 55 is equivalent to $A[567]: [12, 17, 32, 55]$. Lines need not have the same numbers of elements.

The recognized values of separator_flag are comma, pipe, semicolon, and space. Equivalently, the separator may be specified as a string of one character: "," (comma), "|" (pipe), ";" (semicolon), " " (space), or " " (tab). If separator_flag is not specified, the file is assumed space-delimited.

See also openr, read_matrix, read_array, read_list and read_nested_list.

read_nested_list

    read_nested_list (S)
    read_nested_list (S, separator_flag)

Reads the source $S$ and returns its entire content as a nested list. The source $S$ may be a file name or a stream.

read_nested_list returns a list which has a sublist for each line of input. Lines need not have the same numbers of elements. Empty lines are not ignored: an empty line yields an empty sublist.

The recognized values of separator_flag are comma, pipe, semicolon, and space. Equivalently, the separator may be specified as a string of one character: "," (comma), "|" (pipe), ";" (semicolon), " " (space), or " " (tab). If separator_flag is not specified, the file is assumed space-delimited.

See also openr, read_matrix, read_array, read_list and read_hashed_array.
read_list

- read_list(S)
- read_list(S, L)
- read_list(S, separator_flag)
- read_list(S, L, separator_flag)

read_list(S) reads the source S and returns its entire content as a flat list.
read_list(S, L) reads the source S into the list L, until L is full or the source is exhausted.

The source S may be a file name or a stream.
The recognized values of separator_flag are comma, pipe, semicolon, and space. Equivalently, the separator may be specified as a string of one character: "," (comma), "|" (pipe), ";" (semicolon), " " (space), or " " (tab). If separator_flag is not specified, the file is assumed space-delimited.

See also openr, read_matrix, read_array, read_nested_list, read_binary_list and read_hashed_array.

write_data

- write_data(X, D)
- write_data(X, D, separator_flag)

Writes the object X to the destination D.
write_data writes a matrix in row-major order, with one line per row.
write_data writes an array created by array or make_array in row-major order, with a new line at the end of every slab. Higher-dimensional slabs are separated by additional new lines.
write_data writes a hashed array with each key followed by its associated list on one line.
write_data writes a nested list with each sublist on one line.
write_data writes a flat list all on one line.

The destination D may be a file name or a stream. When the destination is a file name, the global variable file_output_append governs whether the output file is appended or truncated. When the destination is a stream, no special action is taken by write_data after all the data are written; in particular, the stream remains open.
The recognized values of separator_flag are comma, pipe, semicolon, space, and tab. Equivalently, the separator may be specified as a string of one character: "," (comma), "|" (pipe), ";" (semicolon), " " (space), or " " (tab). If separator_flag is not specified, the file is assumed space-delimited.

See also openw and read_matrix.

76.3 Functions and Variables for binary input and output

assume_external_byte_order(byte_order_flag)  [Function]
Tells numericalio the byte order for reading and writing binary data. Two values of byte_order_flag are recognized: lsb which indicates least-significant byte first, also called little-endian byte order; and msb which indicates most-significant byte first, also called big-endian byte order.
If not specified, `numericalio` assumes the external byte order is most-significant byte first.

**Function**

openr_binary (`file_name`)  
Returns an input stream of 8-bit unsigned bytes to read the file named by `file_name`.  
See also `openw_binary` and `openr`.

**Function**

openw_binary (`file_name`)  
Returns an output stream of 8-bit unsigned bytes to write the file named by `file_name`.  
See also `openr_binary`, `opena_binary` and `openw`.

**Function**

opena_binary (`file_name`)  
Returns an output stream of 8-bit unsigned bytes to append the file named by `file_name`.

**Function**

read_binary_matrix (`S`, `M`)  
Reads binary 8-byte floating point numbers from the source `S` into the matrix `M` until `M` is full, or the source is exhausted. Elements of `M` are read in row-major order.  
The source `S` may be a file name or a stream.  
The byte order in elements of the source is specified by `assume_external_byte_order`.  
See also `read_matrix`.

**Function**

read_binary_array (`S`, `A`)  
Reads binary 8-byte floating point numbers from the source `S` into the array `A` until `A` is full, or the source is exhausted. `A` must be an array created by `array` or `make_array`. Elements of `A` are read in row-major order.  
The source `S` may be a file name or a stream.  
The byte order in elements of the source is specified by `assume_external_byte_order`.  
See also `read_array`.

**Function**

read_binary_list  
read_binary_list (`S`)  
read_binary_list (`S`, `L`)  
read_binary_list (`S`) reads the entire content of the source `S` as a sequence of binary 8-byte floating point numbers, and returns it as a list. The source `S` may be a file name or a stream.  
read_binary_list (`S`, `L`) reads 8-byte binary floating point numbers from the source `S` until the list `L` is full, or the source is exhausted.  
The byte order in elements of the source is specified by `assume_external_byte_order`.  
See also `read_list`.  

write_binary_data (X, D)  [Function]

Writes the object X, comprising binary 8-byte IEEE 754 floating-point numbers, to the destination D. Other kinds of numbers are coerced to 8-byte floats. write_binary_data cannot write non-numeric data.

The object X may be a list, a nested list, a matrix, or an array created by array or make_array; X cannot be a hashed array or any other type of object. write_binary_data writes nested lists, matrices, and arrays in row-major order.

The destination D may be a file name or a stream. When the destination is a file name, the global variable file_output_append governs whether the output file is appended or truncated. When the destination is a stream, no special action is taken by write_binary_data after all the data are written; in particular, the stream remains open.

The byte order in elements of the destination is specified by assume_external_byte_order.

See also write_data.
1099

77 odepack
77.1 Introduction to ODEPACK
ODEPACK is a collection of Fortran solvers for the initial value problem for ordinary differential equation systems. It consists of nine solvers, namely a basic
solver called LSODE and eight variants of it – LSODES, LSODA, LSODAR,
LSODPK, LSODKR, LSODI, LSOIBT, and LSODIS. The collection is suitable for both stiff and nonstiff systems. It includes solvers for systems given in
explicit form, dy/dt = f(t,y), and also solvers for systems given in linearly implicit form, A(t,y) dy/dt = g(t,y). Two of the solvers use general sparse matrix
solvers for the linear systems that arise. Two others use iterative (preconditioned Krylov) methods instead of direct methods for these linear systems. The
most recent addition is LSODIS, which solves implicit problems with general
sparse treatment of all matrices involved.
1

References: [1] Fortran Code is from https://www.netlib.org/odepack/

77.1.1 Getting Started with ODEPACK
Of the eight variants of the solver, Maxima currently only has an interface to dlsode.
Let’s say we have this system of equations to solve:
f1 = -.04d0*y1 + 1d4*y2*y3
f3 = 3d7*y2*y2
dy1/dt = f1
dy2/dt = -f1 - f3
dy3/dt = f3
The independent variable is t; the dependent variables are y1, y2, and y3,
To start the solution, set up the differential equations to solved:
load("dlsode");
f1: -.04d0*y1 + 1d4*y2*y3$
f3: 3d7*y2*y2$
f2: -f1 - f3$
fex: [f1, f2, f3];
Initialize the solver, where we have selected method 21:
(%i6) state : dlsode_init(fex, ['t,y1,y2,y3], 21);
(%o6) [[f, #<Function "LAMBDA ($T $Y1 $Y2 $Y3)" {49DAC061}>],
[vars, [t, y1, y2, y3]], [mf, 21], [neq, 3], [lrw, 58], [liw, 23], [rwork, {Li\
sp Array: #(0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0)}],
[iwork, {Lisp Array: #(0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0)}],
1

From https://www.netlib.org/odepack/opkd-sum


The arrays rwork and iwork carry state between calls to `dlsode_step`, so they should not be modified by the user. In fact, this state should not be modified by the user at all.

Now that the algorithm has been initialized we can compute solutions to the differential equation, using the state returned above.

For this example, we want to compute the solution at times $0.4 \times 10^k$ for $k$ from 0 to 11, with the initial values of 1, 0, 0 for the dependent variables and with a relative tolerance of 1d-4 and absolute tolerances of 1e-6, 1e-10, and 1d-6 for the dependent variables.

Then

```plaintext
y: [1d0, 0d0, 0d0];
t: 0d0;
rtol : 1d-4;
atol: [1d-6, 1d-10, 1d-6];
istate: 1;
t:0d0;
tout:.4d0;

for k : 1 thru 12 do
  block([],
    result: dlsode_step(y, t, tout, rtol, atol, istate, state),
    printf(true, "At t = ~12,4,2e y = ~{~14,6,2e~}~%", result[1], result[2]),
    istate : result[3],
    tout : tout * 10);
```

This produces the output:

```
At t = 4.0000e-01 y = 9.851726e-01 3.386406e-05 1.479357e-02
At t = 4.0000e+00 y = 9.055142e-01 2.240418e-05 9.446344e-02
At t = 4.0000e+01 y = 7.158050e-01 9.184616e-06 2.841585e-01
At t = 4.0000e+02 y = 4.504846e-01 3.222434e-06 5.495122e-01
At t = 4.0000e+03 y = 1.831701e-01 8.940379e-07 8.168290e-01
At t = 4.0000e+04 y = 4.935213e-03 1.983756e-08 9.950648e-01
At t = 4.0000e+05 y = 5.159269e-04 2.064759e-09 9.994814e-01
At t = 4.0000e+06 y = 5.306413e-05 2.122677e-10 9.999469e-01
At t = 4.0000e+07 y = 5.494530e-06 2.197824e-11 9.999945e-01
At t = 4.0000e+08 y = 5.129458e-07 2.051784e-12 9.999995e-01
At t = 4.0000e+09 y = -7.170563e-08 -2.868225e-13 1.000000e+00
```

### 77.2 Functions and Variables for odepack

`dlsode_init (fex, vars, method)`

This must be called before running the solver. This function returns a state object for use in the solver. The user must not modify the state.

The ODE to be solved is given in `fex`, which is a list of the equations. `vars` is a list of independent variable and the dependent variables. The list of dependent variables
must be in the same order as the equations if \( fex \). Finally, \textit{method} indicates the method to be used by the solver:

10  Nonstiff (Adams) method, no Jacobian used.
21  Stiff (BDF) method, user-supplied full Jacobian.
22  Stiff method, internally generated full Jacobian.

The returned state object is a list of lists. The sublist is a list of two elements:

- \texttt{f}  The compiled function for the ODE.
- \texttt{vars}  The list independent and dependent variables (\texttt{vars}).
- \texttt{mf}  The method to be used (\texttt{method}).
- \texttt{neq}  The number of equations.
- \texttt{lrw}  Length of the work vector for real values.
- \texttt{liw}  Length of the work vector for integer values.
- \texttt{rwork}  Lisp array holding the real-valued work vector.
- \texttt{iwork}  Lisp array holding the integer-valued work vector.
- \texttt{fjac}  Compiled analytical Jacobian of the equations

See also \texttt{dlsode_step}. See Section 77.1.1 [Getting Started with ODEPACK], page 1099, for an example of usage.

\texttt{dlsode_step (inity, t, tout, rtol, atol, istate, state)}  \[\text{[Function]}\]

Performs one step of the solver, returning the values of the independent and dependent variables, a success or error code.

- \texttt{inity}  For the first call (when \texttt{istate} = 1), the initial values
- \texttt{t}  Current value of the independent value
- \texttt{tout}  Next point where output is desired which must not be equal to \texttt{t}.
- \texttt{rtol}  relative tolerance parameter
- \texttt{atol}  Absolute tolerance parameter, scalar of vector. If scalar, it applies to all dependent variables. Otherwise it must be the tolerance for each dependent variable.
  Use \texttt{rtol} = 0 for pure absolute error and use \texttt{atol} = 0 for pure relative error.
- \texttt{istate}  1 for the first call to dlsode, 2 for subsequent calls.
- \texttt{state}  state returned by dlsode\_init.

The output is a list of the following items:

- \texttt{t}  independent variable value
- \texttt{y}  list of values of the dependent variables at time \texttt{t}. 

**istate**  Integration status:
1  no work because tout = tt
2  successful result
-1  Excess work done on this call
-2  Excess accuracy requested
-3  Illegal input detected
-4  Repeated error test failures
-5  Repeated convergence failures (perhaps bad Jacobian or wrong choice of mf or tolerances)
-6  Error weight because zero during problem (solution component is vanished and atol(i) = 0.

**info**  association list of various bits of information:

- **n_steps**  total steps taken thus far
- **n_f_eval**  total number of function evals
- **n_j_eval**  total number of Jacobian evals
- **method_order**  method order
- **len_rwork**  Actual length used for real work array
- **len_iwork**  Actual length used for integer work array

See also **dlsode_init**. See Section 77.1.1 [Getting Started with ODEPACK], page 1099, for an example of usage.
78  operating system

78.1 Introduction to operating system
Package operating system contains functions for operating system-tasks, like file system operations.

78.2 Directory operations

chdir (dir)  [Function]  
Change to directory dir

mkdir (dir)  [Function]  
Create directory dir

rmdir (dir)  [Function]  
remove directory dir

gGetCurrentDirectory ()  [Function]  
returns the current working directory.
See also directory.

Examples:
(%%i1) load("operatingsystem")$
(%%i2) mkdir("testdirectory")$
(%%i3) chdir("testdirectory")$
(%%i4) chdir("..")$
(%%i5) rmdir("testdirectory")$

78.3 File operations

copy_file (file1, file2)  [Function]  
copies file file1 to file2

rename_file (file1, file2)  [Function]  
renames file file1 to file2

delete_file (file1)  [Function]  
deletes file file1

78.4 Environment operations

getenv (env)  [Function]  
Get the value of the environment variable env

Example:
(%%i1) load("operatingsystem")$
(%%i2) getenv("PATH");
(%%o2) /usr/local/sbin:/usr/local/bin:/usr/sbin:/usr/bin:/sbin:/bin
79 opsubst

79.1 Functions and Variables for opsubst

\textbf{opsubst} \hfill [Function]

\begin{align*}
\text{opsubst} & (f, g, e) \\
\text{opsubst} & (g = f, e) \\
\text{opsubst} & ([g1 = f1, g2 = f2, \ldots, gn = fn], e)
\end{align*}

The function \text{opsubst} is similar to the function \text{subst}, except that \text{opsubst} only makes substitutions for the operators in an expression. In general, when \( f \) is an operator in the expression \( e \), substitute \( g \) for \( f \) in the expression \( e \).

To determine the operator, \text{opsubst} sets \text{inflag} to true. This means \text{opsubst} substitutes for the internal, not the displayed, operator in the expression.

Examples:

\begin{verbatim}
(%i1) load ("opsubst")$

(%i2) opsubst(f, g, g(g(x)));
(%o2) f(f(x))

(%i3) opsubst(f, g, g(g));
(%o3) f(g)

(%i4) opsubst(f, g[x], g[x](z));
(%o4) f(z)

(%i5) opsubst([f=g, g=fn], f(x));
(%o5) g(z)

(%i6) opsubst(tan, sin, sin(sin));
(%o6) tan(sin)

(%i7) opsubst([f=g, g=h], f(x));
(%o7) h(x)

(%i8) opsubst("+","-",a-b);
(%o8) a - b

(%i9) opsubst("f","-",-a);
(%o9) - a

(%i10) opsubst("\^\-\^","/",a/b);
(%o10) -

(%i11) opsubst(a+b,f, f(x));
\end{verbatim}

Internally, Maxima does not use the unary negation, division, or the subtraction operators; thus:

\begin{verbatim}
(%i8) opsubst("+","-",a-b);
(%o8) a - b

(%i9) opsubst("f","-",-a);
(%o9) - a

(%i10) opsubst("\^\-\^","/",a/b);
(%o10)

(%i11) opsubst("[","*", -a*b);
(%o11) [- 1, a, b]
\end{verbatim}

When either operator isn't a Maxima symbol, generally some other function will signal an error:

\begin{verbatim}
(%i12) opsubst(a+b,f, f(x));
\end{verbatim}
Improper name or value in functional position:
\[ b + a \]
-- an error. Quitting. To debug this try debugmode(true);

However, subscripted operators are allowed:
\[
(\%i13) \text{opsubst}(g[5], f, f(x));
(\%o13) \quad g(x)^5
\]

To use this function write first \texttt{load("opsubst")}. 
80 orthopoly

80.1 Introduction to orthogonal polynomials

orthopoly is a package for symbolic and numerical evaluation of several kinds of orthogonal polynomials, including Chebyshev, Laguerre, Hermite, Jacobi, Legendre, and ultraspherical (Gegenbauer) polynomials. Additionally, orthopoly includes support for the spherical Bessel, spherical Hankel, and spherical harmonic functions.

For the most part, orthopoly follows the conventions of Abramowitz and Stegun Handbook of Mathematical Functions, Chapter 22 (10th printing, December 1972); additionally, we use Gradshteyn and Ryzhik, Table of Integrals, Series, and Products (1980 corrected and enlarged edition), and Eugen Merzbacher Quantum Mechanics (2nd edition, 1970).

Barton Willis of the University of Nebraska at Kearney (UNK) wrote the orthopoly package and its documentation. The package is released under the GNU General Public License (GPL).

80.1.1 Getting Started with orthopoly

load ("orthopoly") loads the orthopoly package.

To find the third-order Legendre polynomial,

\[
\text{legendre}_p(3, x)
\]

\[
\frac{5 (1 - x)^2}{2} - \frac{15 (1 - x)}{2} - 6 (1 - x) + 1
\]

To express this as a sum of powers of \(x\), apply ratsimp or rat to the result.

\[
\text{ratsimp (%, rat ())}
\]

Alternatively, make the second argument to legendre_p (its “main” variable) a canonical rational expression (CRE).

\[
\text{legendre}_p(3, \text{rat}(x))
\]

For floating point evaluation, orthopoly uses a running error analysis to estimate an upper bound for the error. For example,

\[
\text{jacobi}_p(150, 2, 3, 0.2)
\]

Intervals have the form interval (c, r), where c is the center and r is the radius of the interval. Since Maxima does not support arithmetic on intervals, in some situations, such
as graphics, you want to suppress the error and output only the center of the interval. To
do this, set the option variable \texttt{orthopoly\_returns\_intervals} to \texttt{false}.

\begin{verbatim}
(%i1) orthopoly\_returns\_intervals : false;
(%o1) false
(%i2) jacobi\_p (150, 2, 3, 0.2);
(%o2) - 0.062017037936715
\end{verbatim}

Refer to the section see \cite{Floating point Evaluation}, page 1111, for more information.

Most functions in \texttt{orthopoly} have a \texttt{gradef} property; thus

\begin{verbatim}
(%i1) diff (hermite (n, x), x);
(%o1) 2 n H \_ \_ (x) \\
\_ \_ (x) \\
n - 1

(%i2) diff (gen\_laguerre (n, a, x), x);
(a) (a) \\
\_ (x) - (n + a) \_ (x) unit\_step(n) \\
n \_ \_ \_ \_ (x) \\
n - 1 x
\end{verbatim}

The unit step function in the second example prevents an error that would otherwise arise by evaluating with \( n \) equal to 0.

\begin{verbatim}
(%i3) ev (%, n = 0);
(%o3) 0
\end{verbatim}

The \texttt{gradef} property only applies to the “main” variable; derivatives with respect other arguments usually result in an error message; for example

\begin{verbatim}
(%i1) diff (hermite (n, x), x);
(%o1) 2 n H \_ \_ (x) \\
\_ \_ (x) \\
n - 1

(%i2) diff (hermite (n, x), n);
\end{verbatim}

Maxima doesn’t know the derivative of hermite with respect the first argument
-- an error. Quitting. To debug this try debugmode(true);

Generally, functions in \texttt{orthopoly} map over lists and matrices. For the mapping to fully evaluate, the option variables \texttt{doallmxops} and \texttt{listarith} must both be \texttt{true} (the defaults). To illustrate the mapping over matrices, consider

\begin{verbatim}
(%i1) hermite (2, x);
2
(%o1) - 2 (1 - 2 x )

(%i2) m : matrix ([0, x], [y, 0]);
[ 0 x ]
[ y 0 ]

(%o2) 

(%i3) hermite (2, m);
[ 2 ]
[ - 2 - 2 (1 - 2 x ) ]

(%o3) 
\end{verbatim}
In the second example, the $i, j$ element of the value is $\text{hermite}(2, m[i,j])$; this is not the same as computing $-2 + 4 m \cdot m$, as seen in the next example.

\begin{verbatim}
(%i4) 2 * matrix ([1, 0], [0, 1]) + 4 * m . m;

(%o4)
[ 4 x y - 2 0 ]
[ 0 4 x y - 2 ]
\end{verbatim}

If you evaluate a function at a point outside its domain, generally $\text{orthopoly}$ returns the function unevaluated. For example,

\begin{verbatim}
(%i1) legendre_p (2/3, x);
(%o1) $P^{2/3}(x)$
\end{verbatim}

$\text{orthopoly}$ supports translation into TeX; it also does two-dimensional output on a terminal.

\begin{verbatim}
(%i1) spherical_harmonic (1, m, theta, phi);
(%o1) $Y^m_{\theta, \varphi}$
\end{verbatim}

$\text{orthopoly}$ supports translation into TeX; it also does two-dimensional output on a terminal.

\begin{verbatim}
(%i1) jacobi_p (1, 2, x/2);
(%o1) $P_n^2\left(\frac{x}{2}\right)$
\end{verbatim}

80.1.2 Limitations

When an expression involves several orthogonal polynomials with symbolic orders, it’s possible that the expression actually vanishes, yet Maxima is unable to simplify it to zero. If you divide by such a quantity, you’ll be in trouble. For example, the following expression vanishes for integers $n$ greater than 1, yet Maxima is unable to simplify it to zero.

\begin{verbatim}
(%i1) (2*n - 1) * legendre_p (n - 1, x) * x - n * legendre_p (n, x) + (1 - n) * legendre_p (n - 2, x);
(%o1) (2 n - 1) $P^{n-1}(x) x - n P^n(x) + (1 - n) P^{n-2}(x)$
\end{verbatim}

For a specific $n$, we can reduce the expression to zero.

\begin{verbatim}
(%i2) ev (% , n = 10, ratsimp);
(%o2) 0
\end{verbatim}

Generally, the polynomial form of an orthogonal polynomial is ill-suited for floating point evaluation. Here’s an example.

\begin{verbatim}
(%i1) p : jacobi_p (100, 2, 3, x);
\end{verbatim}
(%i2) subst (0.2, x, p);  
(%o2) 3.4442767023833592E+35  
(%i3) jacobi_p (100, 2, 3, 0.2);  
(%o3) interval(0.18413609135169, 6.8990300925815987E-12)  
(%i4) float(jacobi_p (100, 2, 3, 2/10));  
(%o4) 0.18413609135169  

The true value is about 0.184; this calculation suffers from extreme subtractive cancellation error. Expanding the polynomial and then evaluating, gives a better result.

(%i5) p : expand(p)$  
(%i6) subst (0.2, x, p);  
(%o6) 0.18413609766122982  

This isn't a general rule; expanding the polynomial does not always result in an expression that is better suited for numerical evaluation. By far, the best way to do numerical evaluation is to make one or more of the function arguments floating point numbers. By doing that, specialized floating point algorithms are used for evaluation.

Maxima's float function is somewhat indiscriminate; if you apply float to an expression involving an orthogonal polynomial with a symbolic degree or order parameter, these parameters may be converted into floats; after that, the expression will not evaluate fully. Consider

(%i1) assoc_legendre_p (n, 1, x);  
(1)  
(%o1) P (x)  
(2)  
(%i2) float (%);  
(3)  
(%o2) P (x)  
(4)  
(%i3) ev (%; n=2, x=0.9);  
(5)  
(%o3) P (0.9)  
(6)  

The expression in (%o3) will not evaluate to a float; orthopoly doesn't recognize floating point values where it requires an integer. Similarly, numerical evaluation of the pochhammer function for orders that exceed pochhammer_max_index can be troublesome; consider

(%i1) x : pochhammer (1, 10), pochhammer_max_index : 5;  
(1)  
(%o1) (1)  
(2)  
Applying float doesn't evaluate x to a float

(%i2) float (x);  
(3)  
(%o2) (1.0)  
(4)  

To evaluate x to a float, you'll need to bind pochhammer_max_index to 11 or greater and apply float to x.

(%i3) float (x), pochhammer_max_index : 11;
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(3628800.0)

The default value of **pochhammer_max_index** is 100; change its value after loading **orthopoly**.

Finally, be aware that reference books vary on the definitions of the orthogonal polynomials; we’ve generally used the conventions of Abramowitz and Stegun.

Before you suspect a bug in **orthopoly**, check some special cases to determine if your definitions match those used by **orthopoly**. Definitions often differ by a normalization; occasionally, authors use “shifted” versions of the functions that makes the family orthogonal on an interval other than \((-1,1)\). To define, for example, a Legendre polynomial that is orthogonal on \((0,1)\), define

\[
\text{(i1)} \quad \text{shifted_legendre_p (n, x) := legendre_p (n, 2*x - 1)}$
\]

\[
\text{(i2)} \quad \text{shifted_legendre_p (2, rat (x));}
\]

\[\begin{align*}
\text{(o2)} & \quad 6 x - 6 x + 1 \\
\end{align*}\]

\[
\text{(i3)} \quad \text{legendre_p (2, rat (x));}
\]

\[\frac{3 x - 1}{2}
\]

\[
\text{(o3)}\]

**80.1.3 Floating point Evaluation**

Most functions in **orthopoly** use a running error analysis to estimate the error in floating point evaluation; the exceptions are the spherical Bessel functions and the associated Legendre polynomials of the second kind. For numerical evaluation, the spherical Bessel functions call SLATEC functions. No specialized method is used for numerical evaluation of the associated Legendre polynomials of the second kind.

The running error analysis ignores errors that are second or higher order in the machine epsilon (also known as unit roundoff). It also ignores a few other errors. It’s possible (although unlikely) that the actual error exceeds the estimate.

Intervals have the form **interval** \((c, r)\), where \(c\) is the center of the interval and \(r\) is its radius. The center of an interval can be a complex number, and the radius is always a positive real number.

Here is an example.

\[
\text{(i1)} \quad \text{fpprec : 50}$
\]

\[
\text{(i2)} \quad \text{y0 : jacobi_p (100, 2, 3, 0.2);}
\]

\[
\text{(o2)} \quad \text{interval(0.1841360913516871, 6.8990300925815987E-12)}
\]

\[
\text{(i3)} \quad \text{y1 : bfloat (jacobi_p (100, 2, 3, 1/5));}
\]

\[
\text{(o3)} \quad 1.841360913516856309137024958913493690868904463668b-1
\]

Let’s test that the actual error is smaller than the error estimate

\[
\text{(i4)} \quad \text{is (abs (part (y0, 1) - y1) < part (y0, 2));}
\]

\[
\text{(o4)} \quad \text{true}
\]

Indeed, for this example the error estimate is an upper bound for the true error.
Maxima does not support arithmetic on intervals.

(%i1) legendre_p (7, 0.1) + legendre_p (8, 0.1);
(%o1) interval(0.18032072148437508, 3.1477135311021797E-15)
    + interval(- 0.19949294375000004, 3.3769353084291579E-15)

A user could define arithmetic operators that do interval math. To define interval addition, we can define

(%i1) infix ("@+")
(%i2) "@+"(x,y) := interval (part (x, 1) + part (y, 1), part (x, 2)
    + part (y, 2))$
(%i3) legendre_p (7, 0.1) @+ legendre_p (8, 0.1);
(%o3) interval(- 0.019172222265624955, 6.5246488395313372E-15)

The special floating point routines get called when the arguments are complex. For example,

(%i1) legendre_p (10, 2 + 3.0*%i);
(%o1) interval(- 3.876378825E+7 %i - 6.0787748E+7,
    1.2089173052721777E-6)

Let’s compare this to the true value.

(%i1) float (expand (legendre_p (10, 2 + 3*%i)));
(%o1) - 3.876378825E+7 %i - 6.0787748E+7

Additionally, when the arguments are big floats, the special floating point routines get called; however, the big floats are converted into double floats and the final result is a double.

(%i1) ultraspherical (150, 0.5b0, 0.9b0);
(%o1) interval(- 0.043009481257265, 3.3750051301228864E-14)

80.1.4 Graphics and orthopoly

To plot expressions that involve the orthogonal polynomials, you must do two things:

1. Set the option variable orthopoly_returns_intervals to false,
2. Quote any calls to orthopoly functions.

If function calls aren’t quoted, Maxima evaluates them to polynomials before plotting; consequently, the specialized floating point code doesn’t get called. Here is an example of how to plot an expression that involves a Legendre polynomial.

(%i1) plot2d ('(legendre_p (5, x)), [x, 0, 1]),
    orthopoly_returns_intervals : false;
(%o1)
The entire expression `legendre_p (5, x)` is quoted; this is different than just quoting the function name using `'legendre_p (5, x)`.

### 80.1.5 Miscellaneous Functions

The `orthopoly` package defines the Pochhammer symbol and a unit step function. `orthopoly` uses the Kronecker delta function and the unit step function in `gradef` statements.

To convert Pochhammer symbols into quotients of gamma functions, use `makegamma`.

```lisp
(%i1) makegamma (pochhammer (x, n));
   gamma(x + n)
---
    gamma(x)
(%o1) ___________________________

(%i2) makegamma (pochhammer (1/2, 1/2));
                   1
(%o2) -----------
                sqrt(%pi)
```

Derivatives of the Pochhammer symbol are given in terms of the `psi` function.

```lisp
(%i1) diff (pochhammer (x, n), x);
   n 0
   (x) (psi (x + n) - psi (x))
   0
(%o1) _______________________________

(%i2) diff (pochhammer (x, n), n);
   n 0
   (x) psi (x + n)
   0
(%o2) _______________________________
```

You need to be careful with the expression in (%o1); the difference of the `psi` functions has polynomials when \( x = -1, -2, \ldots, -n \). These polynomials cancel with factors in `pochhammer (x, n)` making the derivative a degree \( n - 1 \) polynomial when \( n \) is a positive integer.

The Pochhammer symbol is defined for negative orders through its representation as a quotient of gamma functions. Consider

```lisp
(%i1) q : makegamma (pochhammer (x, n));
   gamma(x + n)
---
    gamma(x)
(%o1) ___________________________

(%i2) sublis ([x=11/3, n= -6], q);
```
Alternatively, we can get this result directly.

(%i1) pochhammer (11/3, -6);
(%o1) \frac{729}{2240}

The unit step function is left-continuous; thus

(%i1) [unit_step (-1/10), unit_step (0), unit_step (1/10)];
(%o1) [0, 0, 1]

If you need a unit step function that is neither left or right continuous at zero, define your own using \texttt{signum}; for example,

(%i1) xunit_step (x) := (1 + \texttt{signum} (x))/2$

(%i2) [xunit_step (-1/10), xunit_step (0), xunit_step (1/10)];
(%o2) [0, -, 1]

Do not redefine \texttt{unit_step} itself; some code in \texttt{orthopoly} requires that the unit step function be left-continuous.

### 80.1.6 Algorithms

Generally, \texttt{orthopoly} does symbolic evaluation by using a hypergeometric representation of the orthogonal polynomials. The hypergeometric functions are evaluated using the (un- documented) functions \texttt{hypergeo11} and \texttt{hypergeo21}. The exceptions are the half-integer Bessel functions and the associated Legendre function of the second kind. The half-integer Bessel functions are evaluated using an explicit representation, and the associated Legendre function of the second kind is evaluated using recursion.

For floating point evaluation, we again convert most functions into a hypergeometric form; we evaluate the hypergeometric functions using forward recursion. Again, the exceptions are the half-integer Bessel functions and the associated Legendre function of the second kind. Numerically, the half-integer Bessel functions are evaluated using the SLATEC code.

### 80.2 Functions and Variables for orthogonal polynomials

\texttt{assoc_legendre_p} \([n, m, x] \)

The associated Legendre function of the first kind of degree \(n\) and order \(m\), \(P_n^m(z)\), is a solution of the differential equation:

\[
(1 - z^2) \frac{d^2w}{dz^2} - 2z \frac{dw}{dz} + \left[ n(n + 1) - \frac{m^2}{1 - z^2} \right] w = 0
\]

This is related to the Legendre polynomial, \(P_n(x)\) via
$P_n^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$

Reference: A&S eqn 22.5.37, A&S eqn 8.6.6, and A&S eqn 8.2.5.

Some examples:

(%i1) assoc_legendre_p(2,0,x);
(3 (1 - x)) + ------------ + 1
2
(%o1)

(%i2) factor(%);
3 x - 1
2
(%o2)

(%i3) factor(assoc_legendre_p(2,1,x));
- 3 x sqrt(1 - x )
(%o3)

(%i4) (-1)^1*(1-x^2)^(1/2)*diff(legendre_p(2,x),x);
- (3 - 3 (1 - x)) sqrt(1 - x )
(%o4)

(%i5) factor(%);
- 3 x sqrt(1 - x )
(%o5)

assoc_legendre_q(n, m, x) [Function]

The associated Legendre function of the second kind of degree $n$ and order $m$, $Q_n^m(z)$, is a solution of the differential equation:

$$(1 - z^2) \frac{d^2 w}{dz^2} - 2z \frac{dw}{dz} + \left[n(n + 1) - \frac{m^2}{1 - z^2}\right] w = 0$$

Reference: Abramowitz and Stegun, equation 8.5.3 and 8.1.8.

Some examples:

(%i1) assoc_legendre_q(0,0,x);
\frac{x + 1}{x - 1} \log(- \frac{x}{x - 1})
(%o1)

(%i2) assoc_legendre_q(1,0,x);
\frac{x + 1}{x - 1} \log(- \frac{x}{x - 1}) x - 2
(%o2)
The Chebyshev polynomial of the first kind of degree \( n \), \( T_n(x) \).

Reference: A&S eqn 22.5.47.

The polynomials \( T_n(x) \) can be written in terms of a hypergeometric function:

\[
T_n(x) = \, _2F_1\left(-n, n; \frac{1}{2}, \frac{1 - x}{2}\right)
\]

The polynomials can also be defined in terms of the sum

\[
T_n(x) = \frac{n}{2} \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \frac{(n - r)}{n - r} \binom{n}{k} (2x)^{n-2r}
\]

or the Rodrigues formula

\[
T_n(x) = \frac{1}{\kappa_n w(x)} \frac{d^n}{dx^n} (w(x)(1 - x^2)^n)
\]

where

\[
w(x) = 1/\sqrt{1 - x^2}
\]

\[
\kappa_n = (-2)^n \left(\frac{1}{2}\right)^n
\]

Some examples:

\[
(\%i1) \quad \text{chebyshev_t}(2,x);
\]

\[
(\%o1) \quad (-4(1-x)) + 2(1-x) + 1
\]

\[
(\%i2) \quad \text{factor}(\%); \quad 2
\]

\[
(\%o2) \quad 2x - 1
\]

\[
(\%i3) \quad \text{factor}(\text{chebyshev_t}(3,x)); \quad 2
\]

\[
(\%o3) \quad x(4x - 3)
\]

\[
(\%i4) \quad \text{factor}(\text{hgfred}([-3,3],[1/2],(1-x)/2)); \quad 2
\]

\[
(\%o4) \quad x(4x - 3)
\]
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**chebyshev_u (n, x)**

The Chebyshev polynomial of the second kind of degree \( n \), \( U_n(x) \).

Reference: A&S eqn 22.5.48.

The polynomials \( U_n(x) \) can be written in terms of a hypergeometric function:

\[
U_n(x) = (n + 1) \, \frac{\Gamma(n + 2)}{\Gamma(n + 1)} \, \sin\left(\frac{n\pi}{2}\right) \, \left(-\frac{1}{2}\right)^n \, \frac{\Gamma(-n + 1)}{\Gamma(-n + 2)} \, \frac{\Gamma(n + 1)}{\Gamma(n + 2)} \, \left(\frac{1}{2}\right)^n \, \left(\frac{1 - x}{2}\right)^n \, _2F_1\left(-n, n + 2; \frac{3}{2}; \frac{1 - x}{2}\right)
\]

The polynomials can also be defined in terms of the sum

\[
U_n(x) = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \binom{n-r}{r} (2x)^{n-2r}
\]

or the Rodrigues formula

\[
U_n(x) = \frac{1}{\kappa_n w(x)} \frac{d^n}{dx^n} (w(x)(1 - x^2)^n)
\]

where

\[
w(x) = \sqrt{1 - x^2}
\]

\[
\kappa_n = \frac{(-2)^n \binom{n}{\frac{1}{2}}}{n + 1}
\]

(%i1) chebyshev_u(2,x);  
\[
\frac{8 (1 - x)}{3} \quad \frac{4 (1 - x)}{3} + 1
\]

(%o1) \[
3 \left(\frac{- \quad \frac{1 - x}{3}}{3} + \frac{- \quad \frac{1 - x}{3}}{3} + 1\right)
\]

(%i2) expand(%);  
\[
2
\]

(%i3) expand(chebyshev_u(3,x));  
\[
4 x - 1
\]

(%i4) expand(chebyshev_u(3,x));  
\[
8 x - 4 x
\]

(%i5) expand(chebyshev_u(3,x));  
\[
8 x - 4 x
\]

**gen_laguerre (n, a, x)**

The generalized Laguerre polynomial of degree \( n \), \( L_n^{(\alpha)}(x) \).

These can be defined by

\[
L_n^{(\alpha)} = \binom{n + \alpha}{n} \, _1F_1\left(-n; \alpha + 1; x\right)
\]

The polynomials can also be defined by the sum

\[
L_n^{(\alpha)} = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \binom{n-r}{r} (2x)^{n-2r}
\]

or the Rodrigues formula

\[
L_n^{(\alpha)} = \frac{1}{\kappa_n w(x)} \frac{d^n}{dx^n} (w(x)(1 - x^2)^n)
\]

where

\[
w(x) = \sqrt{1 - x^2}
\]

\[
\kappa_n = \frac{(-2)^n \binom{n}{\frac{1}{2}}}{n + 1}
\]
\[ L_n^{(\alpha)} = \sum_{k=0}^{n} \frac{(\alpha + k + 1)_{n-k}(-x)^k}{(n-k)!k!} \]

or the Rodrigues formula

\[ L_n^{(\alpha)}(x) = \frac{1}{\kappa_n w(x) \frac{d^n}{dx^n}(w(x)x^n)} \]

where

\[ w(x) = e^{-x}x^{\alpha} \]
\[ \kappa_n = n! \]

Reference: A&S eqn 22.5.54.

Some examples:

\begin{verbatim}
(%i1) gen_laguerre(1,k,x);
(x + 1) x
(%o1) (k + 1) (1 - -----)
 k + 1

(%i2) gen_laguerre(2,k,x);
2
x
(k + 1) (k + 2) (-------------- - ----- + 1)
 (k + 1) (k + 2) k + 1

(%o2) ---------------------------------------------
2

(%i3) binomial(2+k,2)*hgfred([-2],[1+k],x);
2
x
(k + 1) (k + 2) (-------------- - ----- + 1)
 (k + 1) (k + 2) k + 1

(%o3) ---------------------------------------------
2
\end{verbatim}

\textbf{hermite}(n, x) \quad \textbf{[Function]}

The Hermite polynomial of degree \( n \), \( H_n(x) \).

These polynomials may be defined by a hypergeometric function

\[ H_n(x) = (2x)^n \, \, _2F_0 \left( -\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2}; -\frac{1}{x^2} \right) \]

or by the series

\[ H_n(x) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k(2x)^{n-2k}}{k!(n-2k)!} \]
or the Rodrigues formula

\[ H_n(x) = \frac{1}{\kappa_n w(x)} \frac{d^n}{dx^n} (w(x)) \]

where

\[ w(x) = e^{-x^2/2} \]

\[ \kappa_n = (-1)^n \]

Reference: A&S eqn 22.5.55.

Some examples:

```plaintext
(%i1) hermite(3,x);

2
2 x
(\%o1) - 12 x (1 - ----)
3

(%i2) expand(\%);

3
8 x - 12 x
(\%o2)

(%i3) expand(hermite(4,x));

4
2
16 x - 48 x + 12
(\%o3)

(%i4) expand((2*x)^4*hgfred([-2,-2+1/2],[\,],[\,]-1/x^2));

4
2
16 x - 48 x + 12
(\%o4)

(%i5) expand(4!*sum((-1)^k*(2*x)^(4-2*k)/(k!*(4-2*k)!),k,0,floor(4/2)));

4
2
16 x - 48 x + 12
(\%o5)
```

Function `intervalp (e)`

Return true if the input is an interval and return false if it isn’t.

Function `jacobi_p (n, a, b, x)`

The Jacobi polynomial, \( P_n^{(a,b)}(x) \).

The Jacobi polynomials are actually defined for all \( a \) and \( b \); however, the Jacobi polynomial weight \( (1 - x)^a(1 + x)^b \) isn’t integrable for \( a \leq -1 \) or \( b \leq -1 \).

Reference: A&S eqn 22.5.42.

The polynomial may be defined in terms of hypergeometric functions:

\[ P_n^{(a,b)}(x) = \binom{n + a}{n} _2F_1 \left( \begin{array}{c} \frac{n + a + b + 1}{2} \\ \frac{a + 1}{2} \end{array} ; -n, n + a + b + 1; \frac{1 - x}{2} \right) \]

or the Rodrigues formula

\[ P_n^{(a,b)}(x) = \frac{1}{\kappa_n w(x)} \frac{d^n}{dx^n} \left( w(x) (1 - x^2)^n \right) \]
where

\[ w(x) = (1 - x)^a (1 - x)^b \]

\[ \kappa_n = (-2)^n n! \]

Some examples:

(%i1) jacobi_p(0,a,b,x);
(%o1) 1

(%i2) jacobi_p(1,a,b,x);
(%o2) (b + a + 2) (1 - x)

\[ \frac{(a + 1) (1 - \frac{(b + a + 2) (1 - x)}{2 (a + 1)})}{2 (a + 1)} \]

laguerre (n, x) [Function]

The Laguerre polynomial, \( L_n(x) \) of degree \( n \).

Reference: A&S eqn 22.5.16 and A&S eqn 22.5.54.

These are related to the generalized Laguerre polynomial by

\[ L_n(x) = L_n^{(0)}(x) \]

The polynomials are given by the sum

\[ L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} x^k \]

Some examples:

(%i1) laguerre(1,x);
(%o1) 1 - x

(%i2) laguerre(2,x);
(%o2) \frac{2}{2} x - 2 x + 1

(%i3) gen_laguerre(2,0,x);
(%o3) \frac{2}{2} x - 2 x + 1

(%i4) sum((-1)^k/k!*binomial(2,k)*x^k,k,0,2);
(%o4) \frac{2}{2} x - 2 x + 1

legendre_p (n, x) [Function]

The Legendre polynomial of the first kind, \( P_n(x) \), of degree \( n \).

Reference: A&S eqn 22.5.50 and A&S eqn 22.5.51.
The Legendre polynomial is related to the Jacobi polynomials by

\[ P_n(x) = P_n^{(0,0)}(x) \]

or the Rodrigues formula

\[ P_n(x) = \frac{1}{\kappa_n w(x)} \frac{d^n}{dx^n} \left[ w(x) (1-x^2)^n \right] \]

where

\[ w(x) = 1 \]
\[ \kappa_n = (-2)^n n! \]

Some examples:

```
(%i1) legendre_p(1,x);
(%o1) x
(%i2) legendre_p(2,x);
(%o2) (-3 (1 - x)) + ---------------- + 1
     2
(%i3) expand(%);
(%o3) ------ - -
     2 2
(%i4) expand(legendre_p(3,x));
(%o4) ------ - ----
     2 2
(%i5) expand(jacobi_p(3,0,0,x));
```

**legendre_q(n, x)**

The Legendre function of the second kind, \( Q_n(x) \) of degree \( n \).

Reference: Abramowitz and Stegun, equations 8.5.3 and 8.1.8.

These are related to \( Q_n^m(x) \) by

\[ Q_n(x) = Q_n^0(x) \]

Some examples:

```
(%i11) legendre_q(0,x);
```
\[
\frac{x + 1}{\log(-\frac{x - 1}{x - 2})}
\]

\[(\text{o1})\]

\[(\text{i2})\text{ legendre_q}(1,x) ;\]
\[
\frac{x + 1}{\log(-\frac{x - 1}{x - 2})} \quad \frac{2}{\text{o2)/R/}}
\]

\[(\text{i3})\text{ assoc_legendre_q}(1,0,x) ;\]
\[
\frac{x + 1}{\log(-\frac{x - 1}{x - 2})} \quad \frac{2}{\text{o3)/R/}}
\]

\textbf{orthopoly_recur (f, args)} \quad [\text{Function}]

Returns a recursion relation for the orthogonal function family \texttt{f} with arguments \texttt{args}. The recursion is with respect to the polynomial degree.

\[(\text{i1})\text{ orthopoly_recur (legendre_p, [n, x]) ;}\]
\[
(2 \ n + 1) \ P(x) \ x - n \ P(x) \quad (x) \quad n \quad n - 1
\]
\[(\text{o1}) \quad \text{P} (x) = \frac{\text{-----------------------------}}{n + 1 \quad n + 1}
\]

The second argument to \texttt{orthopoly_recur} must be a list with the correct number of arguments for the function \texttt{f}; if it isn't, Maxima signals an error.

\[(\text{i1})\text{ orthopoly_recur (jacobi_p, [n, x]) ;}\]

\text{Function jacobi_p needs 4 arguments, instead it received 2 -- an error. Quitting. To debug this try debugmode(true);} ;

Additionally, when \texttt{f} isn't the name of one of the families of orthogonal polynomials, an error is signalled.

\[(\text{i1})\text{ orthopoly_recur (foo, [n, x]) ;}\]

\text{A recursion relation for foo isn't known to Maxima -- an error. Quitting. To debug this try debugmode(true);} ;

\textbf{orthopoly_returns_intervals} \quad [\text{Variable}]

Default value: \texttt{true}

When \texttt{orthopoly_returns_intervals} is \texttt{true}, floating point results are returned in the form \texttt{interval (c, r)}, where \texttt{c} is the center of an interval and \texttt{r} is its radius. The center can be a complex number; in that case, the interval is a disk in the complex plane.
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orthopoly_weight (f, args)  [Function]
Returns a three element list; the first element is the formula of the weight for the orthogonal polynomial family f with arguments given by the list args; the second and third elements give the lower and upper endpoints of the interval of orthogonality. For example,

(%i1) w : orthopoly_weight (hermite, [n, x]);
    2
- x
(%o1) [%e , - inf, inf]
(%i2) integrate(w[1]*hermite(3, x)*hermite(2, x), x, w[2], w[3]);
(%o2) 0
The main variable of f must be a symbol; if it isn’t, Maxima signals an error.

pochhammer (x, n)  [Function]
The Pochhammer symbol, \((x)_n\). (See A&S eqn 6.1.22 and DLMF 5.2.iii).
For nonnegative integers \(n\) with \(n \leq \text{pochhammer\_max\_index}\), the expression \((x)_n\) evaluates to the product \(x(x+1)(x+2)\cdots(x+n-1)\) when \(n > 0\) and to 1 when \(n = 0\). For negative \(n\), \((x)_n\) is defined as \((-1)^n/(1-x)^{-n}\).
Thus

(%i1) pochhammer (x, 3);
(%o1) x (x + 1) (x + 2)
(%i2) pochhammer (x, -3);
1
- ------------------------
(1 - x) (2 - x) (3 - x)

To convert a Pochhammer symbol into a quotient of gamma functions, (see A&S eqn 6.1.22) use makegamma; for example

(%i1) makegamma (pochhammer (x, n));
  gamma(x + n)
(%o1) ------------------------
        gamma(x)

When \(n\) exceeds \text{pochhammer\_max\_index} or when \(n\) is symbolic, pochhammer returns a noun form.

(%i1) pochhammer (x, n);
(%o1) (x)

pochhammer\_max\_index  [Variable]
Default value: 100
pochhammer \((n, x)\) expands to a product if and only if \(n \leq \text{pochhammer\_max\_index}\).
Examples:

(%i1) pochhammer (x, 3), pochhammer\_max\_index : 3;
  2
- x
(%o1) x (x + 1) (x + 2)
(%i2) pochhammer (x, 4), pochhammer\_max\_index : 3;
(%o2) (x)
Reference: A&S eqn 6.1.16.

spherical_bessel_j(n, x)  [Function]
The spherical Bessel function of the first kind, \( j_n(x) \).
Reference: A&S eqn 10.1.8 and A&S eqn 10.1.15.
It is related to the Bessel function by

\[
j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x)
\]

Some examples:

(%i1) spherical_bessel_j(1,x);
\[
\frac{\sin(x)}{x} - \frac{\cos(x)}{x}
\]
(%o1) \[
\frac{\sin(x)}{x} - \frac{\cos(x)}{x}
\]
(%i2) spherical_bessel_j(2,x);
\[
\frac{3}{2} \left( - \frac{\sin(x)}{x} - \frac{\cos(x)}{x} \right)
\]
(%o2) \[
\frac{3}{2} \left( - \frac{\sin(x)}{x} - \frac{\cos(x)}{x} \right)
\]
(%i3) expand(%);
\[
\frac{\sin(x)}{x} + \frac{3 \sin(x)}{x} + \frac{3 \cos(x)}{x}
\]
(%o3) \[
\frac{\sin(x)}{x} + \frac{3 \sin(x)}{x} + \frac{3 \cos(x)}{x}
\]
(%i4) expand(sqrt(\(\pi/(2*\pi))\)*bessel_j(2+1/2,x),besselexpand:true);
\[
\frac{\cos(x)}{x} - \frac{\sin(x)}{x}
\]
(%o4) \[
\frac{\cos(x)}{x} - \frac{\sin(x)}{x}
\]

spherical_bessel_y(n, x)  [Function]
The spherical Bessel function of the second kind, \( y_n(x) \).
Reference: A&S eqn 10.1.9 and A&S eqn 10.1.15.
It is related to the Bessel function by

\[
y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x)
\]

(%i1) spherical_bessel_y(1,x);
\[
\cos(x) - \sin(x)
\]
(%o1) \[
\cos(x) - \sin(x)
\]
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(%o1) \frac{x}{-------------------}
     x
x

(%i2) spherical_bessel_y(2,x);
3 \sin(x) 3
------- - (1 - --) \cos(x)
x 2

(%o2) \frac{- --------------------------}{x}
     x

(%i3) expand(%);
3 \sin(x) \cos(x) 3 \cos(x)
(- --------) + ------ - --------
2 x 3
x x

(%i4) expand(sqrt(%pi/(2*x))*bessel_y(2+1/2,x)),besselexpand:true;
3 \sin(x) \cos(x) 3 \cos(x)
(- --------) + ------ - --------
2 x 3
x x

spherical_hankel1 (n, x)
  The spherical Hankel function of the first kind, \( h_n^{(1)}(x) \).
  
  Reference: A&S eqn 10.1.36.

  This is defined by
  
  \[
  h_n^{(1)}(x) = j_n(x) + iy_n(x)
  \]

spherical_hankel2 (n, x)
  The spherical Hankel function of the second kind, \( h_n^{(2)}(x) \).
  

  This is defined by
  
  \[
  h_n^{(2)}(x) = j_n(x) + iy_n(x)
  \]

spherical_harmonic (n, m, theta, phi)
  The spherical harmonic function, \( Y_n^m(\theta, \phi) \).

  Spherical harmonics satisfy the angular part of Laplace’s equation in spherical coordinates.

  For integers \( n \) and \( m \) such that \( n \geq |m| \) and for \( \theta \in [0, \pi] \), Maxima’s spherical harmonic function can be defined by

  \[
  Y_n^m(\theta, \phi) = (-1)^m \sqrt{\frac{2n+1 \cdot (n-m)!}{4\pi \cdot (n+m)!}} P_n^m(\cos \theta)e^{im\phi}
  \]

  Further, when \( n < |m| \), the spherical harmonic function vanishes.
The factor \((-1)^m\), frequently used in Quantum mechanics, is called the Condon-Shortely phase. Some references, including NIST Digital Library of Mathematical Functions omit this factor; see \url{http://dlmf.nist.gov/14.30.E1}. Reference: Merzbacher 9.64.

Some examples:

```maxima
(%i1) spherical_harmonic(1,0,theta,phi);
   sqrt(3) cos(theta)
(%)1 ------------------
   2 sqrt(%pi)
(%i2) spherical_harmonic(1,1,theta,phi);
   %i phi
(%)2 sqrt(3) %e sin(theta)
   ---------------------------------
   3/2
   2 sqrt(%pi)
(%i3) spherical_harmonic(1,-1,theta,phi);
   - %i phi
(%)3 sqrt(3) %e sin(theta)
   ---------------------------------
   3/2
   2 sqrt(%pi)
(%i4) spherical_harmonic(2,0,theta,phi);
   2
(%)4 sqrt(5) ((- 3 (1 - cos(theta))) + ------------------- + 1)
   2
   ------------------------------------------
   2 sqrt(%pi)
(%i5) factor(%);
   2
(%)5 sqrt(5) (3 cos (theta) - 1)
   ------------------------------------------
   4 sqrt(%pi)
```

`unit_step (x)`

The left-continuous unit step function; thus `unit_step (x)` vanishes for \(x \leq 0\) and equals 1 for \(x > 0\).

If you want a unit step function that takes on the value 1/2 at zero, use `hstep`.

`ultraspherical (n, a, x)`

The ultraspherical polynomial, \(C_n^{(a)}(x)\) (also known as the Gegenbauer polynomial).

Reference: A&S eqn 22.5.46.

These polynomials can be given in terms of Jacobi polynomials:

\[
C_n^{(a)}(x) = \frac{\Gamma(a + \frac{1}{2})}{\Gamma(2a)} \frac{\Gamma(n + 2a)}{\Gamma(n + a + \frac{3}{2})} P_n^{(a-1/2,a-1/2)}(x)
\]
or the series
\[
C_n^{(\alpha)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (\alpha)_{n-k}}{k! (n-2k)!} (2x)^{n-2k}
\]
or the Rodrigues formula
\[
C_n^{(\alpha)}(x) = \frac{1}{\kappa_n w(x)} \frac{d^n}{dx^n} \left( w(x) (1 - x^2)^n \right)
\]
where
\[
w(x) = (1 - x^2)^{\alpha - \frac{1}{2}} \\
\kappa_n = \frac{(-2)^n (\alpha + \frac{1}{2})_n n!}{(2\alpha)_n}
\]

Some examples:

(%i1) ultraspherical(1,a,x);
2 a + 1
(%o1) \frac{2 a (1 - \frac{1}{2})}{2 (a + -)}

(%i2) factor(%);
2 a x
(%o2) \frac{2 a x}{2}

(%i3) factor(ultraspherical(2,a,x));
a (2 a x + 2 x - 1)
81 pslq

81.1 Introduction to pslq

Package pslq contains two functions.

(1) \texttt{guess\_exact\_value} tries to find an exact equivalent for an inexact number (float or bigfloat).

(2) \texttt{pslq\_integer\_relation} tries to find integer coefficients such that a linear combination of inexact numbers is approximately zero.

81.2 Functions and Variables for pslq

guess\_exact\_value (x) \hspace{1cm} [Function]

When \( x \) is a floating point number or bigfloat, \texttt{guess\_exact\_value} tries to find an exact expression (in terms of radicals, logarithms, exponentials, and the constant \%pi) which is nearly equal to the given number. If \texttt{guess\_exact\_value} cannot find such an expression, \( x \) is returned unchanged.

When \( x \) is rational number or other mapatom (other than a float or bigfloat), \( x \) is returned unchanged.

Otherwise, \( x \) is a nonatomic expression, and \texttt{guess\_exact\_value} is applied to each of the arguments of \( x \).

Example:

\begin{verbatim}
(%i1) load ("pslq.mac");          \hspace{1cm} %o1 pslq.mac
(%i2) root: float (sin (%pi/12));  \hspace{1cm} %o2 0.2588190451025207
(%i3) guess\_exact\_value (root);  \hspace{1cm} %o3 sqrt(2 - sqrt(3))
                                 \hspace{1cm} --
                                 2
(%i4) L: makelist (root\^i, i, 0, 4); \hspace{1cm} %o4 [1.0, 0.2588190451025207, 0.06698729810778066, 0.01733758853025369, 0.004487298107780675]
(%i5) m: pslq\_integer\_relation(%); \hspace{1cm} %o5 [-1, 0, 16, 0, -16]
(%i6) makelist (x\^i, i, 0, 4) . m; \hspace{1cm} %o6 4 2
                                 \hspace{1cm} x^2 + 16 x - 1
(%i7) solve(%); \hspace{1cm} %o7 [x = - -----------------, x = - -----------------,
                                 \hspace{1cm} 2 2
                                 \hspace{1cm} sqrt(2 - sqrt(3)) sqrt(2 - sqrt(3))
                                 \hspace{1cm} x = - -----------------, x = - -----------------]
                                 \hspace{1cm} 2 2
\end{verbatim}
pslq_integer_relation (L)  
[Function]  
Implements the PSLQ algorithm [1] to find integer relations between bigfloat numbers.  
For a given list $L$ of floating point numbers, $\text{pslq\_integer\_relation}$ returns a list of integers $m$ such that $m \cdot L = 0$ (with absolute residual error less than $\text{pslq\_threshold}$).  

Example:  
\begin{verbatim}
(%i1) load ("pslq.mac");
(%o1) pslq.mac
(%i2) root: float (sin (%pi/12));
(%o2) 0.2588190451025207
(%i3) L: makelist (root^i, i, 0, 4);
(%o3) [1.0, 0.2588190451025207, 0.06698729810778066, 0.01733758853025369, 0.004487298107780675]
(%i4) m: pslq_integer_relation(%);
(%o4) [- 1, 0, 16, 0, - 16]
(%i5) m . L;
(%o5) - 2.359223927328458E-16
(%i6) float (10^(2 - fpprec));
(%o6) 1.0E-14
(%i7) is (abs (m . L) < 10^(2 - fpprec));
(%o7) true
\end{verbatim}

\text{pslq\_precision}  
[Variable]  
Default value: $10^{-(fpprec - 2)}$  
Maximum magnitude of some intermediate results in $\text{pslq\_integer\_relation}$.  
The search fails if one of the intermediate results has elements larger than $\text{pslq\_precision}$.  

\text{pslq\_threshold}  
[Variable]  
Default value: $10^{-(2 - fpprec)}$  
Threshold for absolute residual error of integer relation found by $\text{pslq\_integer\_relation}$.  

\text{pslq\_depth}  
[Variable]  
Default value: $20 \times n$  
Number of iterations of the PSLQ algorithm.  
The default value is 20 times $n$, where $n$ is the length of the list of numbers supplied to $\text{pslq\_integer\_relation}$.  

\text{pslq\_status}  
[Variable]  
Indicates success or failure for an integer relation search by $\text{pslq\_integer\_relation}$.  
When $\text{pslq\_status}$ is 1, it indicates an integer relation was found, and the absolute residual error is less than $\text{pslq\_threshold}$.  
When $\text{pslq\_status}$ is 2, it indicates an integer relation was not found because some intermediate results are larger than $\text{pslq\_precision}$.  

When `pslq_status` is 3, it indicates an integer relation was not found because the number of iterations `pslq_depth` was reached.
82 pytranslate

82.1 Introduction to pytranslate

pytranslate package provides Maxima to Python translation functionality. The package is experimental, and the specifications of the functions in this package might change. It was written as a Google Summer of Code project by Lakshya A Agrawal (Undergraduate Student, IIIT-Delhi) in 2019. A detailed project report is available as a GitHub Gist.

The package needs to be loaded in a Maxima instance for use, by executing:

```
load("pytranslate");
```

The statements are converted to python3 syntax. The file pytranslate.py must be imported for all translations to run, as shown in example.

Example:

```
(%i1) load ("pytranslate")$
/* Define an example function to calculate factorial */
(%i2) pytranslate(my_factorial(x) := if (x = 1 or x = 0) then 1
   else x * my_factorial(x - 1));
(%o2)
def my_factorial(x, v = v):
    v = Stack({}, v)
    v.ins({"x" : x})
    return((1 if ((v["x"] == 1) or (v["x"] == 0)) \n        else (v["x"] * my_factorial((v["x"] + (-1)))))

m["my_factorial"] = my_factorial
(%i3) my_factorial(5);
(%o3) 120
```

The Maxima to Python Translator works in two stages:
1. Conversion of the internal Maxima representation to a defined Intermediate Representation, henceforth referred as IR(mapping is present in share/pytranslate/maxima-to-ir.html)
2. The conversion of IR to Python.

Supported Maxima forms:
1. Section 5.1 Numbers(including complex numbers)
2. Section 7.6 Assignment operators
3. Section 7.2 Arithmetic operators(+, -, *, ^, /, !)
4. Section 7.4 Logical operators (and, or, not)
5. Section 7.3 Relational operators (>, <, >=, <=, !=, ==)
6. Section 5.4 Lists
7. Section 5.5 Arrays
8. block
9. Section 36.2 Function and function calls
10. if-else converted to Python conditionals
11. for loops
12. lambda form

82.1.1 Tests for pytranslate

The tests for pytranslate are present at share/pytranslate/rtest_pytranslate.mac and can be run by executing batch(rtest_pytranslate, test);

82.2 Functions in pytranslate

pytranslate (expr, [print-ir])  [Function]
Translates the expression expr to equivalent python3 statements. Output is printed in the stdout.

Example:

(%i1) load ("pytranslate");
(%i2) pytranslate('for i:8 step -1 unless i<3 do (print(i)))
(%o2)
v['i'] = 8
while not((v['i'] < 3)):
    m['print'](v['i'])
    v['i'] = (v['i'] + -1)
del v['i']

expr is evaluated, and the return value is used for translation. Hence, for statements like assignment, it might be useful to quote the statement:

(%i1) load ("pytranslate");
(%i2) pytranslate(x:20);
(%o2) 20
(%i3) pytranslate('x:20);
(%o3) v['x'] = 20

Passing the optional parameter (print-ir) to pytranslate as t, will print the internal IR representation of expr and return the translated python3 code.

(%i1) load("pytranslate");
(%o1) pytranslate
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(\%i2) pytranslate('plot3d(lambda([x, y], x^2+y^-1), [x, 1, 10], [y, 1, 10]), t);

(body
(funcall (element-array "m" (string "plot3d"))
(lambda
  ((symbol "x") (symbol "y")
    (op-no-bracket
      =
      (symbol "v")
      (funcall (symbol "stack") (dictionary) (symbol "v"))))
    (op +
      (funcall (element-array (symbol "m") (string "pow"))
        (symbol "x") (num 2 0))
      (funcall (element-array (symbol "m") (string "pow"))
        (symbol "y") (unary-op - (num 1 0))))))
  (struct-list (string "x") (num 1 0) (num 10 0))
  (struct-list (string "y") (num 1 0) (num 10 0))))

m["plot3d"](lambda x, y, v = Stack({}, v): (m["pow"](x, 2) + m["\pow"](y, (-1))), ["x", 1, 10], ["y", 1, 10])

show_form (expr)
  Displays the internal maxima form of expr

(\%i4) show_form(a^b);
((mexpt) $a$ $b$)
(\%o4) a^b

82.3 Extending pytranslate

Working of pytranslate:

- The entry point for pytranslate is the function $pytranslate$ defined in share/pytranslate/pytranslate.lisp.
- $pytranslate$ calls the function maxima-to-ir with the Maxima expression as an argument(henceforth referred as expr).
- maxima-to-ir determines if expr is atomic or non-atomic(lisp cons form). If atomic, atom-to-ir is called with expr which returns the IR for the atomic expression.
  To define/modify translation for atomic expressions, make changes to the definition of atom-to-ir in accordance with the IR.
- If expr is non-atomic, the function cons-to-ir is called with expr as an argument.

- cons-to-ir looks for (caar expr) which specifies the type of expr, in hash-table *maxima-direct-ir-map* and if the type is found, then appends the retrieved IR with the result of lisp call (mapcar #'maxima-to-ir (cdr expr)), which applies maxima-to-ir function to all the elements present in the list. Effectively, recursively generate IR for all the elements present in expr and append them to the IR map
for the type.

Example:

\[
\begin{align*}
(%i9) & \text{show_form(a+b);} \\
& ((\text{MPLUS}) B \ $A) \\
(%i10) & \text{pytranslate(a+b, t);} \\
& \text{(body (op + (element-array (symbol "v") (string "b")) \} \\
& \text{ (element-array (symbol "v") (string "a")))))} \\
& (\%o10) \\
& (v["b"] + v["a"])
\end{align*}
\]

Here, operator + with internal maxima representation, (mplus) is present in *maxima-direct-ir-map* and mapped to (op +) to which the result of generating IR for all other elements of the list (a b), i.e. (ELEMENT-ARRAY (SYMBOL "v") (STRING "b")). (ELEMENT-ARRAY (SYMBOL "v") (STRING "a")) is appended.

- If (caar expr) is not found in *maxima-direct-ir-map*, then cons-to-ir looks for the type in *maxima-special-ir-map* which returns the function to handle the translation of the type of expr. cons-to-ir then calls the returned function with argument expr as an argument.

Example:

\[
\begin{align*}
(%i11) & \text{show_form(g(x) := x^2);} \\
& ((\text{mdefine simp}) ((\$g) $x) ((\text{mexpt}) $x 2)) \\
(%i12) & \text{pytranslate(g(x):=x^2, t);} \\
& \text{(body} \\
& \text{(body} \\
& \text{(func-def (symbol "g")} \\
& \text{((symbol "x") (op-no-bracket = (symbol "v") (symbol "v")))} \\
& \text{(body-indentned} \\
& \text{ (op-no-bracket = (symbol "v") (funcall (symbol "stack") \} \\
& \text{ (dictionary) (symbol "v")))} \\
& \text{ (obj-funcall (symbol "v") (symbol "ins") (dictionary \} \\
& \text{ ((string "x") (symbol "x")))} \\
& \text{ (funcall (symbol "return")} \\
& \text{ (funcall (element-array (symbol "f") (string "pow")))} \\
& \text{ (element-array (symbol "v") (string "x")))} \\
& \text{ (num 2 0)))} \\
& \text{(op-no-bracket = (element-array (symbol "f") (string "g")) \} \\
& \text{ (symbol "g")))}) \\
& (\%o12) \\
& \text{def g(x, v = v):} \\
& \text{ v = Stack(\{\}, v) \\
& \text{ v.ins({"x" : x})} \\
& \text{ return(f["pow"](v["x"], 2))} \\
& \text{ f["g"] = g)
\end{align*}
\]

Here, mdefine, which is the type of expr is present in *maxima-special-ir-map*, which returns func-def-to-ir as handler function, which is then called with expr
to generate the IR.

To define/modify translation for a type, add an entry to *maxima-direct-ir-map* if only a part of the IR needs to be generated and the rest can be appended, otherwise, for complete handling of expr, add an entry to *maxima-special-ir-map* and define a function with the name defined in *maxima-special-ir-map* which returns the IR for the form. The function naming convention for ir generators is (type)-to-ir, where type is the (caar expr) for expression(mdefine -> func-def-to-ir). The function must return a valid IR for the specific type.

- After the generation of IR, the function ir-to-python is called with the generated ir as an argument, which performs the codegen in a recursive manner.
  - ir-to-python looks for lisp (car ir) in the hash-table *ir-python-direct-templates*, which maps IR type to function handlers and calls the function returned with ir as an argument.

- To extend the IR of pytranslate, define a function with the naming convention (type)-to-python and add the name to *ir-python-direct-templates*.
83 quantum_computing-pkg

83.1 Package quantum_computing

The quantum_computing package provides several functions to simulate quantum computing circuits. The state of a system of \( n \) qubits is represented by a list of \( 2^n \) complex numbers and an operator acting on \( m \) qubits is represented by a \( 2^m \) by \( 2^m \) matrix. A hash array \textit{qmatrix} is defined with 6 common one-qubit matrices: the identity, the Pauli matrices, the Hadamard matrix and the phase matrix.

The major disadvantage compared to a real quantum computer is very slow computing times even with a few qubits. An advantage is that, unlike a quantum computer, in this simulator a quantum state can be cloned using \texttt{copylist}.

This is an additional package that must be loaded with \texttt{load("quantum_computing")} in order to use it.

83.2 Functions and Variables for Quantum_Computing

\textbf{binlist} \[ \text{[Function]} \]

\begin{verbatim}
binlist (k)
binlist (k, n)
\end{verbatim}

\texttt{binlist} \((k)\), where \( k \) must be a natural number, returns a list of binary digits 0 or 1 corresponding to the digits of \( k \) in binary representation. \texttt{binlist}(\(k, n\)) does the same but returns a list of length \( n \), with leading zeros as necessary. Notice that for the result to represent a possible state of \( m \) qubits, \( n \) should be equal to \( 2^m \) and \( k \) should be between 0 and \( 2^m-1 \).

\textbf{binlist2dec (lst)} \[ \text{[Function]} \]

Given a list \( \text{lst} \) with \( n \) binary digits, it returns the decimal number it represents.

\textbf{CNOT} \((q, i, j)\) \[ \text{[Function]} \]

Changes the value of the \( j \)'th qubit, in a state \( q \) of \( m \) qubits, when the value of the \( i \)'th qubit equals 1. It modifies the list \( q \) and returns its modified value.

\textbf{controlled} \((U, q, c, i)\) \[ \text{[Function]} \]

Applies a matrix \( U \), acting on \( m \) qubits, on qubits \( i \) through \( i+m-1 \) of the state \( q \) of \( n \) qubits \((n > m)\), when the value of the \( c \)'th qubit in \( q \) equals 1. \( i \) should be an integer between 1 and \( n+1-m \) and \( c \) should be an integer between 1 and \( n \), excluding the qubits to be modified \((i \) through \( i+m-1)\).

\( U \) can be one of the indices of the array of common matrices \textit{qmatrix} (see \textit{qmatrix}). The state \( q \) is modified and shown in the output.

\textbf{gate} \[ \text{[Function]} \]

\begin{verbatim}
gate (U, q)
gate (U, q, i)
gate (U, q, i1, \ldots, im)
\end{verbatim}

\( U \) must be a matrix acting on states of \( m \) qubits; \( q \) a list corresponding to a state of \( n \) qubits \((n >= m)\); \( i \) and the \( m \) numbers \( i1, \ldots, im \) must be different integers between 1 and \( n \).
gate(U, q) applies matrix U to each qubit of q, when m equals 1, or to the first m qubits of q when m is bigger than 1.
gate(U, q, i) applies matrix U to the qubits i through i+m-1 of q.
gate(U, q, il, ..., in) applies matrix U to the in the positions il, ..., im.
U can be one of the indices of the array of common matrices qmatrix (see qmatrix). The state q is modified and shown in the output.

**gate_matrix**

- **gate_matrix(U, n)**
  - **gate_matrix(U, n, i1, ..., im)**
  
  U must be a 2 by 2 matrix or one of the indices of the array of common matrices qmatrix (see qmatrix). gate_matrix(U, n) returns the matrix corresponding to the action of U on each qubit in a state of n qubits.
gate_matrix(U, n, il, ..., im) returns the matrix corresponding to the action of U on qubits il, ..., im of a state of n qubits, where il, ..., im are different integers between 1 and n.

**linsert(e, lst, p)**

  - Inserts the expression or list e into the list lst at position p. The list can be empty and p must be an integer between 1 and the length of lst plus 1.

**lreplace(e, lst, p)**

  - If e is a list of length n, the elements in the positions p, p+1, ..., p+n-1 of the list lst are replaced by e, or the first elements of e if the end of lst is reached. If e is an expression, the element in position p of list lst is replaced by that expression. p must be an integer between 1 and the length of lst.

**normalize(q)**

  - Returns the normalized version of a quantum state given as a list q.

**qdisplay(q)**

  - Represents the state q of a system of n qubits as a linear combination of the computational states with n binary digits. It returns an expression including strings and symbols.

**qmatrix**

  - This variable is a predefined hash array of two by two matrices with the standard matrices: identity, Pauli matrices, Hadamard matrix and the phase matrix. The six possible indices are I, X, Y, Z, H, S. qmatrix[I] is the identity matrix, qmatrix[X] the Pauli x matrix, qmatrix[Y] the Pauli y matrix, qmatrix[Z] the Pauli z matrix, qmatrix[H] the Hadamard matrix and qmatrix[S] the phase matrix.

**qmeasure**

  - qmeasure(q)
  - qmeasure(q, il, ..., im)
  
  Measures the value of one or more qubits in a system of n qubits with state q. The m positive integers il, ..., im are the positions of the qubits to be measured. It requires 1 or more arguments. The first argument must be the state q. If the only argument given is q, all the n qubits will be measured.
It returns a list with the values of the qubits measured (either 0 or 1), in the same order they were requested or in ascending order if the only argument given was q. It modifies the list q, reflecting the collapse of the quantum state after the measurement.

\textbf{qubits}

\begin{verbatim}
qubits (n)
qubits (i1, ..., in)
qubits(n) returns a list representing the ground state of a system of n qubits.
qubits(i1, ..., in) returns a list with representing the state of n qubits with values i1, ..., in.
\end{verbatim}

\textbf{qswap} (q, i, j)

Interchanges the states of qubits i and j in the state q of a system of several qubits. It modifies the list q and returns its modified value.

\textbf{Rx} (a)

Returns the 2 by two matrix (acting on one qubit) corresponding to a rotation of with an angle of a radians around the x axis.

\textbf{Ry} (a)

Returns the 2 by two matrix (acting on one qubit) corresponding to a rotation of with an angle of a radians around the y axis.

\textbf{Rz} (a)

Returns the 2 by two matrix (acting on one qubit) corresponding to a rotation of with an angle of a radians around the z axis.

\textbf{tprod} (o1, ..., on)

Returns the tensor product of the n matrices or lists o1, ..., on.

\textbf{toffoli} (q, i, (j, (k))

Changes the value of the k’th qubit, in the state q of n qubits, if the values of the i’th and j’th qubits are equal to 1. It modifies the list q and returns its new value.
84 ratpow

The package ratpow provides functions that return the coefficients of the numerator of a CRE polynomial in a given variable.

For example,

- \( \text{ratp_coeffs}(5x^7-3x^2+4,x) \) returns \([7,5],[2,-3],[0,4]\), which omits zero terms;
- \( \text{ratp_dense_coeffs}(5x^7-y*x^2+4,x) \) returns \([5,0,0,0,-y,0,4]\), which includes zero terms;
- \( \text{ratp_dense_coeffs}(x^4-y^4)/(x-y),x) \) returns \([1,y^2,y^3]\), because CRE simplifies the expression to \(x^3+y*x^2+y^2*x+y^3\);
- \( \text{ratp_dense_coeffs}(x+\sqrt{x}),x) \) returns \([1,\sqrt{x}]\) while \( \text{ratp_dense_coeffs}(x+\sqrt{x},\sqrt{x}) \) returns \([1,x]\): in CRE form, \(x\) and \(\sqrt{x}\) are treated as independent variables.

The returned coefficients are in CRE form except for numbers.

For the list of vars of a CRE polynomial, use showratvars. For the denominator of a CRE polynomial, use ratdenom.

For information about CREs see also rat, ratdisrep and showratvars.

84.1 Functions and Variables for ratpow

ratp_hipow (expr, x)  \[\text{Function}\]

Returns the highest power of \(x\) in \(\text{ratnumer}(\text{expr})\)

\[
\begin{align*}
(\%i1) \ & \text{load("ratpow")}$
(\%i2) \ & \text{ratp_hipow( }x^{(5/2)} + x^{2}, \ , x) ;
(\%o2) \ & \ 2
(\%i3) \ & \text{ratp_hipow( }x^{(5/2)} + x^{2}, \sqrt{x}) ;
(\%o3) \ & \ 5
\end{align*}
\]

ratp_lopow (expr, x)  \[\text{Function}\]

Returns the lowest power of \(x\) in \(\text{ratnumer}(\text{expr})\)

\[
\begin{align*}
(\%i1) \ & \text{load("ratpow")}$
(\%i2) \ & \text{ratp_lopow( }x^5 + x^2 , x) ;
(\%o2) \ & \ 2
(\%i2) \ & \text{ratp_lopow( }x^5 + x^2 + 1 , x) ;
(\%o2) \ & \ 0
\end{align*}
\]

The following example returns 0 since 1 equals \(x^0\):

\[
\begin{align*}
(\%i1) \ & \text{load("ratpow")}$
(\%i2) \ & \text{ratp_lopow( }x^5 + x^2 + 1 , x) ;
(\%o2) \ & \ 0
\end{align*}
\]

The CRE form of the following equation contains \(\sqrt{x}\) and \(x\). Since they are interpreted as independent variables, \(\text{ratp_lopow}\) returns 0:

\[
\begin{align*}
(\%i1) \ & \text{load("ratpow")}$
(\%i2) \ & \text{g:}\sqrt{x}^5 + \sqrt{x}^2 ;
(\%o2) \ & \ 5/2
(\%o2) \ & \ x + x
\end{align*}
\]

\(\sqrt{x}\)
(%i3) showratvars(g);
   1/2
(%o3) [x, x]
(%i4) ratp_lopow( g, x);
(%o4) 0
(%i5) ratp_lopow( g, sqrt(x));
(%o5) 0

ratp_coeffs (expr, x) [Function]
Returns the powers and coefficients of x in ratnumer(expr) as a list of length-2 lists; returned coefficients are in CRE form except for numbers.

ratp_coeffs (expr, x)
(%i1) load("ratpow")$
(%i2) ratp_coeffs( 4*x^3 + x + sqrt(x), x);
(%o2)/R/ [[3, 4], [1, 1], [0, sqrt(x)]]

ratp_dense_coeffs (expr, x) [Function]
Returns the coefficients of powers of x in ratnumer(expr) from highest to lowest; returned coefficients are in CRE form except for numbers.

ratp_dense_coeffs (expr, x)
(%i1) load("ratpow")$
(%i2) ratp_dense_coeffs( 4*x^3 + x + sqrt(x), x);
(%o2)/R/ [4, 0, 1, sqrt(x)]
85 romberg

85.1 Functions and Variables for romberg

romberg

romberg (expr, x, a, b)
romberg (F, a, b)

Computes a numerical integration by Romberg’s method.

romberg(expr, x, a, b) returns an estimate of the integral integrate(expr, x, a, b). expr must be an expression which evaluates to a floating point value when x is bound to a floating point value.

romberg(F, a, b) returns an estimate of the integral integrate(F(x), x, a, b) where x represents the unnamed, sole argument of F; the actual argument is not named x. F must be a Maxima or Lisp function which returns a floating point value when the argument is a floating point value. F may name a translated or compiled Maxima function.

The accuracy of romberg is governed by the global variables rombergabs and rombergtol. romberg terminates successfully when the absolute difference between successive approximations is less than rombergabs, or the relative difference in successive approximations is less than rombergtol. Thus when rombergabs is 0.0 (the default) only the relative error test has any effect on romberg.

romberg halves the stepsize at most rombergit times before it gives up; the maximum number of function evaluations is therefore 2^rombergit. If the error criterion established by rombergabs and rombergtol is not satisfied, romberg prints an error message. romberg always makes at least rombergmin iterations; this is a heuristic intended to prevent spurious termination when the integrand is oscillatory.

romberg repeatedly evaluates the integrand after binding the variable of integration to a specific value (and not before). This evaluation policy makes it possible to nest calls to romberg, to compute multidimensional integrals. However, the error calculations do not take the errors of nested integrations into account, so errors may be underestimated. Also, methods devised especially for multidimensional problems may yield the same accuracy with fewer function evaluations.

See also Section 19.3 Introduction to QUADPACK, a collection of numerical integration functions.

Examples:

A 1-dimensional integration.

(%i1) f(x) := 1/((x - 1)^2 + 1/100) + 1/((x - 2)^2 + 1/1000) + 1/((x - 3)^2 + 1/200);

1 1 1
(%o1) f(x) := ---------------- + ---------------- + ----------------
2 1 2 1 2 1
(x - 1) + --- (x - 2) + ---- (x - 3) + ----
100 1000 200
(%i2) rombergtol : 1e-6;
(%o2) 9.999999999999999e-7
(%i3) rombergit : 15;
(%o3) 15
(%i4) estimate : romberg (f(x), x, -5, 5);
(%o4) 173.6730736617464
(%i5) exact : integrate (f(x), x, -5, 5);
3/2 3/2 3/2 3/2
(%o5) 10 atan(7 10 ) + 10 atan(3 10 )
3/2 9/2 3/2 5/2
+ 5 2 atan(5 2 ) + 5 2 atan(5 2 ) + 10 atan(60)
+ 10 atan(40)
(%i6) abs (estimate - exact) / exact, numer;
(%o6) 7.552722451569877e-11

A 2-dimensional integration, implemented by nested calls to romberg.
(%i1) g(x, y) := x*y / (x + y);
(%o1) g(x, y) := ------
x + y
(%i2) rombergtol : 1e-6;
(%o2) 9.999999999999999e-7
(%i3) estimate : romberg (romberg (g(x, y), y, 0, x/2), x, 1, 3);
(%o3) 0.8193023962835647
(%i4) assume (x > 0);
(%o4) [x > 0]
(%i5) integrate (integrate (g(x, y), y, 0, x/2), x, 1, 3);
3
2 log(-) - 1
9 2 9
(%o5) (- 9 log(-)) + 9 log(3) + ------------ + -
2 6 2
(%i6) exact : radcan (%);
26 log(3) - 26 log(2) - 13
(%o6) - ----------------------
3
(%i7) abs (estimate - exact) / exact, numer;
(%o7) 1.371197987185102e-10

rombergabs

[Option variable]
Default value: 0.0

The accuracy of romberg is governed by the global variables rombergabs and
rombergtol. romberg terminates successfully when the absolute difference between
successive approximations is less than rombergabs, or the relative difference in
successive approximations is less than rombergtol. Thus when rombergabs is 0.0
(the default) only the relative error test has any effect on romberg.
See also rombergit and rombergmin.
rombergit  
[Option variable]  
Default value: 11  
romberg halves the stepsize at most rombergit times before it gives up; the maximum number of function evaluations is therefore $2^{\text{rombergit}}$. romberg always makes at least rombergmin iterations; this is a heuristic intended to prevent spurious termination when the integrand is oscillatory.  
See also rombergabs and rombergtol.

rombergmin  
[Option variable]  
Default value: 0  
romberg always makes at least rombergmin iterations; this is a heuristic intended to prevent spurious termination when the integrand is oscillatory.  
See also rombergit, rombergabs, and rombergtol.

rombergtol  
[Option variable]  
Default value: 1e-4  
The accuracy of romberg is governed by the global variables rombergabs and rombergtol. romberg terminates successfully when the absolute difference between successive approximations is less than rombergabs, or the relative difference in successive approximations is less than rombergtol. Thus when rombergabs is 0.0 (the default) only the relative error test has any effect on romberg.  
See also rombergit and rombergmin.
86 simplex

86.1 Introduction to simplex

*simplex* is a package for linear optimization using the simplex algorithm.

Example:

```
(%i1) load("simplex")
(%i2) minimize_lp(x+y, [3*x+2*y>2, x+4*y>3]);
   9   7   1
---, [y = --, x = -]
10  10  5
```

86.1.1 Tests for simplex

There are some tests in the directory `share/simplex/Tests`.

86.1.1.1 klee_minty

The function *klee_minty* produces input for *linear_program*, for which exponential time for solving is required without scaling.

Example:

```
load("klee_minty")
apply(linear_program, klee_minty(6));
```

A better approach:

```
epsilon_sx : 0$
scale_sx : true$
apply(linear_program, klee_minty(10));
```

86.1.1.2 NETLIB

Some smaller problems from netlib (https://www.netlib.org/lp/data/) test suite are converted to a format, readable by Maxima. Problems are `adlittle`, `afiro`, `kb2`, `sc50a` and `share2b`. Each problem has three input files in CSV format for matrix $A$ and vectors $b$ and $c$.

Example:

```
A : read_matrix("adlittle_A.csv", 'csv)$
b : read_list("adlittle_b.csv", 'csv)$
c : read_list("adlittle_c.csv", 'csv)$
linear_program(A, b, c)$
```

Results:

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>MINIMUM</th>
<th>SCALING</th>
</tr>
</thead>
<tbody>
<tr>
<td>adlittle</td>
<td>+2.2549496316E+05</td>
<td>false</td>
</tr>
<tr>
<td>afiro</td>
<td>-4.6475314286E+02</td>
<td>false</td>
</tr>
<tr>
<td>kb2</td>
<td>-1.7499001299E+03</td>
<td>true</td>
</tr>
<tr>
<td>sc50a</td>
<td>-6.4575077059E+01</td>
<td>false</td>
</tr>
</tbody>
</table>
The Netlib website [https://www.netlib.org/lp/data/readme](https://www.netlib.org/lp/data/readme) lists the values as:

- **PROBLEM**  **MINIMUM**
  - **adlittle**  +2.2549496316E+05
  - **afiro**  -4.6475314286E+02
  - **kb2**  -1.7499001299E+03
  - **sc50a**  -6.4575077059E+01
  - **share2b**  -4.1573224074E+02

**86.2 Functions and Variables for simplex**

**epsilon_lp**  
Default value: $10^{-8}$

Epsilon used for numerical computations in `linear_program`; it is set to 0 in `linear_program` when all inputs are rational.

Example:

```
(%i1) load("simplex")

(%i2) minimize_lp(-x, [1e-9*x + y <= 1], [x,y]);
Warning: linear_program(A,b,c): non-rat inputs found, epsilon_lp= 1.0e-8
Warning: Solution may be incorrect.
(%o2) Problem not bounded!
(%i3) minimize_lp(-x, [10^-9*x + y <= 1], [x,y]);
(%o3) [- 10000000, [y = 0, x = 10000000]]
(%i4) minimize_lp(-x, [1e-9*x + y <= 1], [x,y]), epsilon_lp=0;
(%o4) [- 9.999999999999999e+8, [y = 0, x = 9.999999999999999e+8]]
```

See also: `linear_program`, `ratnump`.

**linear_program(A, b, c)**  
`linear_program` is an implementation of the simplex algorithm. `linear_program(A, b, c)` computes a vector $x$ for which $c.x$ is minimum possible among vectors for which $A.x = b$ and $x \geq 0$. Argument $A$ is a matrix and arguments $b$ and $c$ are lists.

`linear_program` returns a list which contains the minimizing vector $x$ and the minimum value $c.x$. If the problem is not bounded, it returns "Problem not bounded!" and if the problem is not feasible, it returns "Problem not feasible!".

To use this function first load the `simplex` package with `load("simplex")`.

Example:

```
(%i2) A: matrix([1,1,-1,0], [2,-3,0,-1], [4,-5,0,0])
(%i3) b: [1,1,6]
(%i4) c: [1,-2,0,0]
(%i5) linear_program(A, b, c);
13 19 3
[[-4, -4, --, 0], --] 2 2 2
```

See also: `minimize_lp`, `scale_lp`, and `epsilon_lp`.
maximize_lp (obj, cond, [pos])  [Function]
Maximizes linear objective function obj subject to some linear constraints cond. See minimize_lp for detailed description of arguments and return value.
See also: minimize_lp.

minimize_lp (obj, cond, [pos])  [Function]
Minimizes a linear objective function obj subject to some linear constraints cond.
cond a list of linear equations or inequalities. In strict inequalities > is replaced by >= and < by <=. The optional argument pos is a list of decision variables which are assumed to be positive.

If the minimum exists, minimize_lp returns a list which contains the minimum value of the objective function and a list of decision variable values for which the minimum is attained. If the problem is not bounded, minimize_lp returns "Problem not bounded!" and if the problem is not feasible, it returns "Problem not feasible!".

The decision variables are not assumed to be non-negative by default. If all decision variables are non-negative, set nonnegative_lp to true or include all in the optional argument pos. If only some of decision variables are positive, list them in the optional argument pos (note that this is more efficient than adding constraints).

minimize_lp uses the simplex algorithm which is implemented in maxima linear_program function.
To use this function first load the simplex package with load("simplex");.
Examples:

(%i1) minimize_lp(x+y, [3*x+y=0, x+2*y>2]);
   4  6  2
(%o1) [-, [y = -], [x = -]]
   5  5  5

(%i2) minimize_lp(x+y, [3*x+y>0, x+2*y>2]), nonnegative_lp=true;
(%o2) [1, [y = 1, x = 0]]

(%i3) minimize_lp(x+y, [3*x+y>0, x+2*y>2], all);
(%o3) [1, [y = 1, x = 0]]

(%i4) minimize_lp(x+y, [3*x+y=0, x+2*y>2]), nonnegative_lp=true;
(%o4) Problem not feasible!

(%i5) minimize_lp(x+y, [3*x+y>0]);
(%o5) Problem not bounded!

There is also a limited ability to solve linear programs with symbolic constants.

(%i1) declare(c,constant)$
(%i2) maximize_lp(x+y, [y<=-x/c+3, y<=-x+4], [x, y]), epsilon_lp=0;
Is (c-1)*c positive, negative or zero?
p;
Is c*(2*c-1) positive, negative or zero?
p;
Is c positive or negative?
p;
Is c-1 positive, negative or zero?
\texttt{p;}
Is 2*c-1 positive, negative or zero?
\texttt{p;}
Is 3*c-4 positive, negative or zero?
\texttt{p;}
\begin{verbatim}
        1  1
[4, [x = ----, y = 3 - ----------]]
        1  1
        1 - - (1 - -) c
        c  c
\end{verbatim}
\texttt{(\%i1) \texttt{(assume(c>4/3), declare(c,constant))}}$
\texttt{(\%i2) maximize_lp(x+y, [y<=-x/c+3, y<=-x+4], [x, y]), epsilon_lp=0;}
\begin{verbatim}
        1  1
[4, [x = ----, y = 3 - ----------]]
        1  1
        1 - - (1 - -) c
        c  c
\end{verbatim}
\texttt{See also: maximize_lp, nonnegative_lp, epsilon_lp.}

\texttt{nonnegative_lp} \hfill [Option variable]
\texttt{nongative_lp} \hfill [Option variable]
Default value: \texttt{false}
If \texttt{nonnegative_lp} is true all decision variables to \texttt{minimize_lp} and \texttt{maximize_lp} are assumed to be non-negative. \texttt{nongative_lp} is a deprecated alias.
See also: \texttt{minimize_lp}.

\texttt{scale_lp} \hfill [Option variable]
Default value: \texttt{false}
When \texttt{scale_lp} is true, \texttt{linear_program} scales its input so that the maximum absolute value in each row or column is 1.

\texttt{pivot_count_sx} \hfill [Variable]
After \texttt{linear_program} returns, \texttt{pivot_count_sx} is the number of pivots in last computation.

\texttt{pivot_max_sx} \hfill [Variable]
\texttt{pivot_max_sx} is the maximum number of pivots allowed by \texttt{linear_program}. 
87 simplification

87.1 Introduction to simplification

The directory maxima/share/simplification contains several scripts which implement simplification rules and functions, and also some functions not related to simplification.

87.2 Package absimp

The absimp package contains pattern-matching rules that extend the built-in simplification rules for the abs and signum functions. absimp respects relations established with the built-in assume function and by declarations such as mode_declare (m, even, n, odd) for even or odd integers.

absimp defines unitramp and unitstep functions in terms of abs and signum.

load ("absimp") loads this package. demo ("absimp") shows a demonstration of this package.

Examples:

(%i1) load ("absimp")$

(%i2) (abs (x))^2;

2

(%o2) x

(%i3) diff (abs (x), x);

x

(%o3) ------

abs(x)

(%i4) cosh (abs (x));

(%o4) cosh(x)

87.3 Package facexp

The facexp package contains several related functions that provide the user with the ability to structure expressions by controlled expansion. This capability is especially useful when the expression contains variables that have physical meaning, because it is often true that the most economical form of such an expression can be obtained by fully expanding the expression with respect to those variables, and then factoring their coefficients. While it is true that this procedure is not difficult to carry out using standard Maxima functions, additional fine-tuning may also be desirable, and these finishing touches can be more difficult to apply.

The function facsum and its related forms provide a convenient means for controlling the structure of expressions in this way. Another function, collectterms, can be used to add two or more expressions that have already been simplified to this form, without resimplifying the whole expression again. This function may be useful when the expressions are very large.

load ("facexp") loads this package. demo ("facexp") shows a demonstration of this package.
facsum (expr, arg_1, ..., arg_n)  [Function]

Returns a form of expr which depends on the arguments arg_1, ..., arg_n. The arguments can be any form suitable for ratvars, or they can be lists of such forms. If the arguments are not lists, then the form returned is fully expanded with respect to the arguments, and the coefficients of the arguments are factored. These coefficients are free of the arguments, except perhaps in a non-rational sense.

If any of the arguments are lists, then all such lists are combined into a single list, and instead of calling factor on the coefficients of the arguments, facsum calls itself on these coefficients, using this newly constructed single list as the new argument list for this recursive call. This process can be repeated to arbitrary depth by nesting the desired elements in lists.

It is possible that one may wish to facsum with respect to more complicated subexpressions, such as log (x + y). Such arguments are also permissible.

Occasionally the user may wish to obtain any of the above forms for expressions which are specified only by their leading operators. For example, one may wish to facsum with respect to all log’s. In this situation, one may include among the arguments either the specific log’s which are to be treated in this way, or alternatively, either the expression operator (log) or 'operator (log). If one wished to facsum the expression expr with respect to the operators op_1, ..., op_n, one would evaluate facsum (expr, operator (op_1, ..., op_n)). The operator form may also appear inside list arguments.

In addition, the setting of the switches facsum_combine and nextlayerfactor may affect the result of facsum.

default of nextlayerfactor: false

When nextlayerfactor is true, recursive calls of facsum are applied to the factors of the factored form of the coefficients of the arguments.

When false, facsum is applied to each coefficient as a whole whenever recursive calls to facsum occur.

Inclusion of the atom nextlayerfactor in the argument list of facsum has the effect of nextlayerfactor: true, but for the next level of the expression only. Since nextlayerfactor is always bound to either true or false, it must be presented single-quoted whenever it appears in the argument list of facsum.

default of facsum_combine: true

facsum_combine controls the form of the final result returned by facsum when its argument is a quotient of polynomials. If facsum_combine is false then the form will be returned as a fully expanded sum as described above, but if true, then the expression returned is a ratio of polynomials, with each polynomial in the form described above.

The true setting of this switch is useful when one wants to facsum both the numerator and denominator of a rational expression, but does not want the denominator to be multiplied through the terms of the numerator.
factorfacsum (expr, arg_1, ... arg_n)  
Returns a form of expr which is obtained by calling facsum on the factors of expr with arg_1, ... arg_n as arguments. If any of the factors of expr is raised to a power, both the factor and the exponent will be processed in this way.

collectterms (expr, arg_1, ..., arg_n)  
Collects all terms that contain arg_1 ... arg_n. If several expressions have been simplified with the following functions facsum, factorfacsum, factenexpand, facexpten or factorfacexpten, and they are to be added together, it may be desirable to combine them using the function collectterms. collectterms can take as arguments all of the arguments that can be given to these other associated functions with the exception of nextlayerfactor, which has no effect on collectterms. The advantage of collectterms is that it returns a form similar to facsum, but since it is adding forms that have already been processed by facsum, it does not need to repeat that effort. This capability is especially useful when the expressions to be summed are very large. 

See also factor.

Example:

```lisp
(%i1) (exp(x)+2)*x+exp(x);
 x  x
(%o1)  x (%e + 2) + %e
(%i2) collectterms(expand(%),exp(x));
 x
(%o2)  (x + 1) %e + 2 x
```

87.4 Package functs

rempart (expr, n)  
Removes part n from the expression expr.

If n is a list of the form [l, m] then parts l thru m are removed.

To use this function write first load("functs").

wronskian ([f_1, ..., f_n], x)  
Returns the Wronskian matrix of the list of expressions [f_1, ..., f_n] in the variable x. The determinant of the Wronskian matrix is the Wronskian determinant of the list of expressions.

To use wronskian, first load("functs"). Example:

```lisp
(%i1) load ("functs")$
(%i2) wronskian([f(x), g(x)],x);

[ f(x)  g(x) ]
[             ]
[ d         d ]
[ -- (f(x)) -- (g(x))]   
[ dx        dx]

(%o2)
```

tracematrix (M)  
Returns the trace (sum of the diagonal elements) of matrix M.
To use this function write first `load("functs")`.

**rational (z)**

[Function]

Multiplies numerator and denominator of z by the complex conjugate of denominator, thus rationalizing the denominator. Returns canonical rational expression (CRE) form if given one, else returns general form.

To use this function write first `load("functs")`.

**nonzeroandfreeof (x, expr)**

[Function]

Returns `true` if `expr` is nonzero and `freeof (x, expr)` returns `true`. Returns `false` otherwise.

To use this function write first `load("functs")`.

**linear (expr, x)**

[Function]

When `expr` is an expression of the form `a*x + b` where `a` is nonzero, and `a` and `b` are free of `x`, `linear` returns a list of three equations, one for each of the three formal variables `b`, `a`, and `x`. Otherwise, `linear` returns `false`.

`load("antid")` loads this function.

Example:

```
(%i1) load("antid");
(%i2) linear ((1 - w)*(1 - x)*z, z);
(%o2) [bargumentb = 0, aargumenta = (w - 1) x - w + 1, xargumentx = z]
(%i3) linear (cos(u - v) + cos(u + v), u);
(%o3) false
```

**gcdivide (p, q)**

[Function]

When the option variable `takegcd` is `true` which is the default, `gcdivide` divides the polynomials `p` and `q` by their greatest common divisor and returns the ratio of the results. `gcdivide` calls the function `ezgcd` to divide the polynomials by the greatest common divisor.

When `takegcd` is `false`, `gcdivide` returns the ratio `p/q`.

To use this function write first `load("functs")`.

See also `ezgcd`, `gcd`, `gcdex`, and `poly_gcd`.

Example:

```
(%i1) load("functs")$
(%i2) p1:6*x^3+19*x^2+19*x+6;
3 2
(%o2) 6 x + 19 x + 19 x + 6
(%i3) p2:6*x^5+13*x^4+12*x^3+13*x^2+6*x;
5 4 3 2
(%o3) 6 x + 13 x + 12 x + 13 x + 6 x
(%i4) gcdivide(p1, p2);
   2
x + 1
```
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(%i4) ------
3
x + x

(%i5) takegcd:false;
(%o5) false

(%i6) gcdivide(p1, p2);

\[
\frac{3 \cdot 6 x + 19 x + 19 x + 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 6 x + 13 x + 12 x + 13 x + 6 x}
\]

(%i7) ratsimp(%);

\[
\frac{x + 1}{3 \cdot x + x}
\]

**arithmetic** \(a, d, n\)  
Returns the \(n\)-th term of the arithmetic series \(a, a + d, a + 2d, \ldots, a + (n - 1)d\).

To use this function write first `load("functs")`.

**geometric** \(a, r, n\)  
Returns the \(n\)-th term of the geometric series \(a, a*r, a*r^2, \ldots, a*r^{(n - 1)}\).

To use this function write first `load("functs")`.

**harmonic** \(a, b, c, n\)  
Returns the \(n\)-th term of the harmonic series \(a/b, a/(b + c), a/(b + 2*c), \ldots, a/(b + (n - 1)*c)\).

To use this function write first `load("functs")`.

**arithsum** \(a, d, n\)  
Returns the sum of the arithmetic series from 1 to \(n\).

To use this function write first `load("functs")`.

**geosum** \(a, r, n\)  
Returns the sum of the geometric series from 1 to \(n\). If \(n\) is infinity (\(\text{inf}\)) then a sum is finite only if the absolute value of \(r\) is less than 1.

To use this function write first `load("functs")`.

**gaussprob** \(x\)  
Returns the Gaussian probability function \(\text{e}^{-x^2/2} / \sqrt{2*\pi}\).

To use this function write first `load("functs")`.

**gd** \(x\)  
Returns the Gudermannian function \(2*\text{atan}(%e^x)-\pi/2\).

To use this function write first `load("functs")`. 
agd (x) [Function]
Returns the inverse Gudermannian function \( \log(\tan(\pi/4 + x/2)) \).
To use this function write first \texttt{load("functs")}.

vers (x) [Function]
Returns the versed sine \( 1 - \cos(x) \).
To use this function write first \texttt{load("functs")}.

covers (x) [Function]
Returns the coversed sine \( 1 - \sin(x) \).
To use this function write first \texttt{load("functs")}.

exsec (x) [Function]
Returns the exsecant \( \sec(x) - 1 \).
To use this function write first \texttt{load("functs")}.

hav (x) [Function]
Returns the haversine \( (1 - \cos(x))/2 \).
To use this function write first \texttt{load("functs")}.

combination (n, r) [Function]
Returns the number of combinations of \( n \) objects taken \( r \) at a time.
To use this function write first \texttt{load("functs")}.

permutation (n, r) [Function]
Returns the number of permutations of \( r \) objects selected from a set of \( n \) objects.
To use this function write first \texttt{load("functs")}.

87.5 Package ineq

The \texttt{ineq} package contains simplification rules for inequalities.

Example session:

```
(\%i1) load("ineq")$
tellsimp: warning: rule will treat '+'
   as noncommutative and nonassociative.
tellsimp: warning: rule will treat '+'
   as noncommutative and nonassociative.
tellsimp: warning: rule will treat '+'
   as noncommutative and nonassociative.
tellsimp: warning: rule will treat '+'
   as noncommutative and nonassociative.
tellsimp: warning: rule will treat '+'
   as noncommutative and nonassociative.
tellsimp: warning: rule will treat '+'
   as noncommutative and nonassociative.
tellsimp: warning: rule will treat '+'
   as noncommutative and nonassociative.
tellsimp: warning: rule will treat '+'
   as noncommutative and nonassociative.
```
tellsimp: warning: rule will treat '+' as noncommutative and nonassociative.

(%i2) a>=4; /* a sample inequality */
(%o2) a >= 4
(%o3) b + a > c + 4
(%o4) 7 x < 7 y
(%o5) - 2 x <= - 6 z

2
(%o6) 1 <= a + 1
(%o8) 2 x < 3 x
(%o9) a >= b
(%o10) a + 3 >= b + 3
(%o11) a >= b
(%o12) a >= c - b
(%o13) b + a >= c
(%o14) (- c) + b + a >= 0
(%o15) c - b - a <= 0

2
(%o16) (z - 1) > - 2 z

2
(%o17) z + 1 > 0
(%o18) true
(%i19) (b>c)+%; /* add a second, strict inequality */

Be careful about using parentheses around the inequalities: when the user types in (A > B) + (C = 5) the result is A + C > B + 5, but A > B + C = 5 is a syntax error, and (A > B + C) = 5 is something else entirely.

Do disprule (all) to see a complete listing of the rule definitions.

The user will be queried if Maxima is unable to decide the sign of a quantity multiplying an inequality.

The most common mis-feature is illustrated by:

(%i1) eq: a > b;
(%o1)
(%i2) 2*eq;
(%o2)
(%i3) % - eq;
(%o3)

Another problem is 0 times an inequality; the default to have this turn into 0 has been left alone. However, if you type X*some_inequality and Maxima asks about the sign of X and you respond zero (or z), the program returns X*some_inequality and not use the information that X is 0. You should do ev (%, x: 0) in such a case, as the database will only be used for comparison purposes in decisions, and not for the purpose of evaluating X.

The user may note a slower response when this package is loaded, as the simplifier is forced to examine more rules than without the package, so you might wish to remove the rules after making use of them. Do kill (rules) to eliminate all of the rules (including
any that you might have defined); or you may be more selective by killing only some of
them; or use `remrule` on a specific rule.

Note that if you load this package after defining your own rules you will clobber your
rules that have the same name. The rules in this package are: `*rule1`, ..., `*rule8`, `+rule1`,
..., `+rule18`, and you must enclose the rulename in quotes to refer to it, as in `remrule
("+", "+rule1")` to specifically remove the first rule on "+" or `disprule ("*rule2")` to
display the definition of the second multiplicative rule.

### 87.6 Package rducon

`reduce_consts (expr)`

Replaces constant subexpressions of `expr` with constructed constant atoms, saving the
definition of all these constructed constants in the list of equations `const_eqns`, and
returning the modified `expr`. Those parts of `expr` are constant which return `true`
when operated on by the function `constantp`. Hence, before invoking `reduce_consts`, one
should do

```
declare ([objects to be given the constant property], constant)
```
to set up a database of the constant quantities occurring in your expressions.

If you are planning to generate Fortran output after these symbolic calculations, one
of the first code sections should be the calculation of all constants. To generate this
code segment, do

```
map ('fortran, const_eqns)
```

Variables besides `const_eqns` which affect `reduce_consts` are:

- `const_prefix` (default value: `xx`) is the string of characters used to prefix all symbols
  generated by `reduce_consts` to represent constant subexpressions.
- `const_counter` (default value: 1) is the integer index used to generate unique symbols
to represent each constant subexpression found by `reduce_consts`.

`load ("rducon")` loads this function. `demo ("rducon")` shows a demonstration of
this function.

### 87.7 Package scifac

`gcfac (expr)`

`gcfac` is a factoring function that attempts to apply the same heuristics which scient-
ists apply in trying to make expressions simpler. `gcfac` is limited to monomial-type
factoring. For a sum, `gcfac` does the following:

1. Factors over the integers.
2. Factors out the largest powers of terms occurring as coefficients, regardless of the
   complexity of the terms.
3. Uses (1) and (2) in factoring adjacent pairs of terms.
4. Repeatedly and recursively applies these techniques until the expression no longer
   changes.
Item (3) does not necessarily do an optimal job of pairwise factoring because of the combinatorially-difficult nature of finding which of all possible rearrangements of the pairs yields the most compact pair-factored result.

\texttt{load ("scifac")} loads this function. \texttt{demo ("scifac")} shows a demonstration of this function.
88 solve_rec

88.1 Introduction to solve_rec

solve_rec is a package for solving linear recurrences with polynomial coefficients.
A demo is available with demo("solve_rec");

Example:

(%i1) load("solve_rec")$
(%i2) solve_rec((n+4)*s[n+2] + s[n+1] - (n+1)*s[n], s[n]);

\[ s(n) = \frac{(-1)^k (2n + 3)}{n(n + 1)(n + 2)} + \frac{k}{2} \]

88.2 Functions and Variables for solve_rec

reduce_order (rec, sol, var) [Function]

Reduces the order of linear recurrence rec when a particular solution sol is known.
The reduced recurrence can be used to get other solutions.

Example:

(%i3) rec: x[n+2] = x[n+1] + x[n]/n;
(%o3) x = x + \frac{x}{n + 1}

(%i4) solve_rec(rec, x[n]);

WARNING: found some hypergeometrical solutions!

(%o4) x = %k n

(%i5) reduce_order(rec, n, x[n]);

(%t5)

\[ x = n \frac{(-n - 2) %u - %u}{n + 1} \]
\begin{verbatim}
(%i6) solve_rec((n+2)*%u[n+1] + %u[n], %u[n]);

\[
\frac{-1}{n(n+1)!}
\]

So the general solution is

\[
\sum_{j=0}^{n-1} \frac{(-1)^j}{(j+1)!} + \frac{\%k \cdot n}{2}
\]

\textbf{simplify_products} \\
[Option variable] \\
Default value: true

If \texttt{simplify_products} is true, \texttt{solve_rec} will try to simplify products in result.
See also: \texttt{solve_rec}.

\textbf{simplify_sum (expr)} \\
[Function] \\
Tries to simplify all sums appearing in \texttt{expr} to a closed form.
To use this function first load the \texttt{simplify_sum} package with \texttt{load("simplify_sum")}.
Example:

\begin{verbatim}
(%i1) load("simplify_sum")$
(%i2) sum(binomial(n+k,k)/2^k,k,1,n)+sum(binomial(2*n,2*k),k,1,n);

\[
\sum_{k=1}^{n} \binom{n+k}{k} \frac{1}{2^k} + \sum_{k=1}^{n} \binom{2n}{2k}
\]

(%o2) \texttt{2 n - 1 n} \texttt{2} + \texttt{2} - \texttt{2}
\end{verbatim}

\textbf{solve_rec (eqn, var, [init])} \\
[Function] \\
Solves for hypergeometrical solutions to linear recurrence \texttt{eqn} with polynomials coefficient in variable \texttt{var}. Optional arguments \texttt{init} are initial conditions.
\texttt{solve_rec} can solve linear recurrences with constant coefficients, finds hypergeometrical solutions to homogeneous linear recurrences with polynomial coefficients, rational
\end{verbatim}
Chapter 88: solve_rec

solutions to linear recurrences with polynomial coefficients and can solve Ricatti type recurrences.

Note that the running time of the algorithm used to find hypergeometrical solutions is exponential in the degree of the leading and trailing coefficient.

To use this function first load the solve_rec package with load("solve_rec");

Example of linear recurrence with constant coefficients:

```
(%i2) solve_rec(a[n]=a[n-1]+a[n-2]+n/2^n, a[n]);
```

```
(n (sqrt(5) - 1) %k (- 1)
1 n
-- --- - ----
2 5 2
```

```
(%o2) a = ------------------------- - ----
2 5 2
(n (sqrt(5) + 1) %k
2 2
+ ------------------ - ----
2 5 2
```

Example of linear recurrence with polynomial coefficients:

```
(%i7) 2*x*(x+1)*y[x] - (x^2+3*x-2)*y[x+1] + (x-1)*y[x+2];
```

```
2
(%o7) (x - 1) y - (x + 3 x - 2) y + 2 x (x + 1) y
```

```
(%i8) solve_rec(%, y[x], y[1]=1, y[3]=3);
```

```
x
3 2 x!
-- --- - --
x 4 2
```

Example of Ricatti type recurrence:

```
(%i2) x*y[x+1]*y[x] - y[x+1]/(x+2) + y[x]/(x-1) = 0;
```

```
y
x + 1 x
%o2
y - ------- + ------- = 0
x x + 1 x + 2 x - 1
```

```
(%i3) solve_rec(%), y[x], y[3]=5)$
```

```
(%i4) ratsimp(minfactorial(factcomb(%)));
```

```
3
30 x - 30 x
```

See also: solve_rec_rat, simplify_products and product_use.gamma.
solve_rec_rat (eqn, var, [init])  
Solves for rational solutions to linear recurrences. See solve_rec for description of arguments.

To use this function first load the solve_rec package with load("solve_rec");

Example:

(%i1) (x+4)*a[x+3] + (x+3)*a[x+2] - x*a[x+1] + (x^2-1)*a[x];
(%o1) (x + 4) a + (x + 3) a - x a
    x + 3 x + 2 x + 1
       2
    + (x - 1) a
    x

(%i2) solve_rec_rat(% = (x+2)/(x+1), a[x]);
   1
(%o2) a = ---------------------
    x (x - 1) (x + 1)

See also: solve_rec.

product_use_gamma  
[Option variable]
Default value: true

When simplifying products, solve_rec introduces gamma function into the expression if product_use_gamma is true.

See also: simplify_products, solve_rec.

summand_to_rec  
[Function]

summand_to_rec (summand, k, n)
summand_to_rec (summand, [k, lo, hi], n)

Returns the recurrence satisfied by the sum

\[
\frac{\text{hi} \cdot \text{summand}}{\text{k} = \text{lo}}
\]

where summand is hypergeometrical in k and n. If lo and hi are omitted, they are assumed to be lo = -inf and hi = inf.

To use this function first load the simplify_sum package with load("simplify_sum").

Example:

(%i1) load("simplify_sum")$
(%i2) summand: binom(n,k);
(%o2) \text{binomial}(n, k)
(%i3) summand_to_rec(summand,k,n);
(%o3) 2 \text{sm} - \text{sm} = 0
\text{n} \text{n + 1}
summand: \( \frac{\binom{n}{k}}{k+1} \);

\[
\binom{n}{k} \quad \frac{1}{k+1}
\]

summand_to_rec(summand, [k, 0, n], n);

\[
2 \frac{(n+1) \text{sm}}{n} - \frac{(n+2) \text{sm}}{n+1} = -1
\]
89 stats

89.1 Introduction to stats

Package stats contains a set of classical statistical inference and hypothesis testing procedures.

All these functions return an inference_result Maxima object which contains the necessary results for population inferences and decision making.

Global variable stats_numer controls whether results are given in floating point or symbolic and rational format; its default value is true and results are returned in floating point format.

Package descriptive contains some utilities to manipulate data structures (lists and matrices); for example, to extract subsamples. It also contains some examples on how to use package numericalio to read data from plain text files. See descriptive and numericalio for more details.

Package stats loads packages descriptive, distrib and inference_result.

For comments, bugs or suggestions, please contact the author at 'mario AT edu DOT xunta DOT es'.

89.2 Functions and Variables for inference_result

inference_result (title, values, numbers) [Function]

Constructs an inference_result object of the type returned by the stats functions.

Argument title is a string with the name of the procedure; values is a list with elements of the form symbol = value and numbers is a list with positive integer numbers ranging from one to length(values), indicating which values will be shown by default.

Example:

This is a simple example showing results concerning a rectangle. The title of this object is the string "Rectangle", it stores five results, named 'base, 'height, 'diagonal, 'area, and 'perimeter, but only the first, second, fifth, and fourth will be displayed. The 'diagonal is stored in this object, but it is not displayed; to access its value, make use of function take_inference.

(%i1) load("inference_result")$
(%i2) b: 3$ h: 2$
(%i3) inference_result("Rectangle",
['base=b,
 'height=h,
 'diagonal=sqrt(b^2+h^2),
 'area=b*h,
 'perimeter=2*(b+h)],
[1,2,5,4] );
| Rectangle
Maxima Manual

| base = 3 |
| height = 2 |
| perimeter = 10 |
| area = 6 |

```lisp
(%i3) take_inference('diagonal,%);
(%o4) sqrt(13)
```

See also `take_inference`.

### inferencep

**[Function]**

`inferencep(obj)`

Returns `true` or `false`, depending on whether `obj` is an `inference_result` object or not.

### items_inference

**[Function]**

`items_inference(obj)`

Returns a list with the names of the items stored in `obj`, which must be an `inference_result` object.

**Example:**

The `inference_result` object stores two values, named `pi` and `e`, but only the second is displayed. The `items_inference` function returns the names of all items, no matter they are displayed or not.

```lisp
(%i1) load("inference_result")$
(%i2) inference_result("Hi", [\[pi=\pi, 'e=%e], [2]]);
   | Hi
   | e = %e
(%o2)
(%i3) items_inference(%);
(%o3) \[pi, e\]
```

### take_inference

**[Function]**

`take_inference(n, obj)`

`take_inference(name, obj)`

`take_inference(list, obj)`

Returns the `n`-th value stored in `obj` if `n` is a positive integer, or the item named `name` if this is the name of an item. If the first argument is a list of numbers and/or symbols, function `take_inference` returns a list with the corresponding results.

**Example:**

Given an `inference_result` object, function `take_inference` is called in order to extract some information stored in it.

```lisp
(%i1) load("inference_result")$
(%i2) b: 3$ h: 2$
(%i3) sol: inference_result("Rectangle",
   ['base=b, 
    'height=h, 
    'diagonal=sqrt(b^2+h^2),
```
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\[
\begin{array}{l}
\text{'area=b*h,}
\text{'perimeter=2*(b+h)],}
\text{[1,2,5,4] );}
\end{array}
\]
\[
\begin{array}{l}
\text{Rectangle}
\end{array}
\]
\[
\begin{array}{l}
\text{base = 3}
\end{array}
\]
\[
\begin{array}{l}
\text{(%o3) height = 2}
\end{array}
\]
\[
\begin{array}{l}
\text{perimeter = 10}
\end{array}
\]
\[
\begin{array}{l}
\text{area = 6}
\end{array}
\]
\[
\begin{array}{l}
\text{(%i4) take_inference('base,sol);} \\
\text{(%o4) 3}
\end{array}
\]
\[
\begin{array}{l}
\text{(%i5) take_inference(5,sol);} \\
\text{(%o5) 10}
\end{array}
\]
\[
\begin{array}{l}
\text{(%i6) take_inference([1,'diagonal],sol);} \\
\text{(%o6) [3, sqrt(13)]}
\end{array}
\]
\[
\begin{array}{l}
\text{(%i7) take_inference(items_inference(sol),sol);} \\
\text{(%o7) [3, 2, sqrt(13), 6, 10]}
\end{array}
\]

See also inference_result, and take_inference.

89.3 Functions and Variables for stats

\text{stats_numer} \quad [\text{Option variable}]

Default value: true

If \text{stats_numer} is true, inference statistical functions return their results in floating point numbers. If it is false, results are given in symbolic and rational format.

\text{test_mean} \quad [\text{Function}]

\text{test_mean (x)} \\
\text{test_mean (x, options ...)}

This is the mean t-test. Argument x is a list or a column matrix containing an one dimensional sample. It also performs an asymptotic test based on the Central Limit Theorem if option 'asymptotic is true.

Options:

- 'mean, default 0, is the mean value to be checked.
- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.
- 'dev, default 'unknown, this is the value of the standard deviation when it is known; valid values are: 'unknown or a positive expression.
- 'conflevel, default 95/100, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).
- 'asymptotic, default false, indicates whether it performs an exact t-test or an asymptotic one based on the Central Limit Theorem; valid values are true and false.
The output of function `test_mean` is an **inference_result** Maxima object showing the following results:

1. 'mean_estimate: the sample mean.
2. 'conf_level: confidence level selected by the user.
3. 'conf_interval: confidence interval for the population mean.
4. 'method: inference procedure.
5. 'hypotheses: null and alternative hypotheses to be tested.
6. 'statistic: value of the sample statistic used for testing the null hypothesis.
7. 'distribution: distribution of the sample statistic, together with its parameter(s).
8. 'p_value: p-value of the test.

Examples:

Performs an exact $t$-test with unknown variance. The null hypothesis is $H_0 : \text{mean} = 50$ against the one sided alternative $H_1 : \text{mean} < 50$; according to the results, the $p$-value is too great, there are no evidence for rejecting $H_0$.

```
(%i1) load("stats")$
(%i2) data: [78, 64, 35, 45, 45, 75, 43, 74, 42, 42]$  
(%i3) test_mean(data,'conflevel=0.9,'alternative='less,'mean=50);  
    | MEAN TEST  
    | mean_estimate = 54.3 
    | conf_level = 0.9 
    | conf_interval = [minf, 61.51314273502712]  
    | method = Exact t-test. Unknown variance.  
    | hypotheses = H0: mean = 50 , H1: mean < 50  
    | statistic = 0.8244705235071678  
    | distribution = [student_t, 9]  
    | p_value = 0.7845100411786889
```

This time Maxima performs an asymptotic test, based on the **Central Limit Theorem**. The null hypothesis is $H_0 : \text{equal}(\text{mean}, 50)$ against the two sided alternative $H_1 : \text{notequal}(\text{mean}, 50)$; according to the results, the $p$-value is very small, $H_0$ should be rejected in favor of the alternative $H_1$. Note that, as indicated by the **Method** component, this procedure should be applied to large samples.

```
(%i1) load("stats")$
(%i2) test_mean([36, 118, 52, 87, 35, 256, 56, 178, 57, 57, 89, 34, 25, 98, 35,  
98, 41, 45, 198, 54, 79, 63, 35, 45, 44, 75, 42, 75, 45, 45,  
45, 51, 123, 54, 151],
```
test_means_difference

This is the difference of means $t$-test for two samples. Arguments $x1$ and $x2$ are lists or column matrices containing two independent samples. In case of different unknown variances (see options 'dev1', 'dev2' and 'varequal' below), the degrees of freedom are computed by means of the Welch approximation. It also performs an asymptotic test based on the Central Limit Theorem if option 'asymptotic' is set to true.

Options:

- 'alternative', default 'twosided', is the alternative hypothesis; valid values are: 'twosided', 'greater' and 'less'.
- 'dev1', default 'unknown', this is the value of the standard deviation of the $x1$ sample when it is known; valid values are: 'unknown' or a positive expression.
- 'dev2', default 'unknown', this is the value of the standard deviation of the $x2$ sample when it is known; valid values are: 'unknown' or a positive expression.
- 'varequal', default false, whether variances should be considered to be equal or not; this option takes effect only when 'dev1' and/or 'dev2' are 'unknown'.
- 'conflevel', default 95/100, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).
- 'asymptotic', default false, indicates whether it performs an exact $t$-test or an asymptotic one based on the Central Limit Theorem; valid values are true and false.

The output of function test_means_difference is an inference_result Maxima object showing the following results:

1. 'diff_estimate': the difference of means estimate.
2. 'conf_level: confidence level selected by the user.
3. 'conf_interval: confidence interval for the difference of means.
4. 'method: inference procedure.
5. 'hypotheses: null and alternative hypotheses to be tested.
6. 'statistic: value of the sample statistic used for testing the null hypothesis.
7. 'distribution: distribution of the sample statistic, together with its parameter(s).
8. 'p_value: p-value of the test.

Examples:
The equality of means is tested with two small samples $x$ and $y$, against the alternative $H_1: m_1 > m_2$, being $m_1$ and $m_2$ the populations means; variances are unknown and supposed to be different.

(%i1) load("stats")
(%i2) x: [20.4,62.5,61.3,44.2,11.1,23.7]
(%i3) y: [1.2,6.9,38.7,20.4,17.2]
(%i4) test_means_difference(x,y,'alternative='greater);

<table>
<thead>
<tr>
<th>DIFFERENCE OF MEANS TEST</th>
</tr>
</thead>
<tbody>
<tr>
<td>diff_estimate = 20.31999999999999</td>
</tr>
<tr>
<td>conf_level = 0.95</td>
</tr>
<tr>
<td>conf_interval = [- .04597417812882298, inf]</td>
</tr>
</tbody>
</table>

(%o4) method = Exact t-test. Welch approx.

| hypotheses = H0: mean1 = mean2 , H1: mean1 > mean2 |
| statistic = 1.838004300728477 |
| distribution = [student_t, 8.62758740184604] |
| p_value = .05032746527991905 |

The same test as before, but now variances are supposed to be equal.

(%i1) load("stats")
(%i2) x: [20.4,62.5,61.3,44.2,11.1,23.7]
(%i3) y: matrix([1.2],[6.9],[38.7],[20.4],[17.2])
(%i4) test_means_difference(x,y,'alternative='greater, 'varequal=true);

<table>
<thead>
<tr>
<th>DIFFERENCE OF MEANS TEST</th>
</tr>
</thead>
<tbody>
<tr>
<td>diff_estimate = 20.31999999999999</td>
</tr>
<tr>
<td>conf_level = 0.95</td>
</tr>
</tbody>
</table>
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| conf_interval = [- .7722627696897568, inf] |
| method = Exact t-test. Unknown equal variances |
| hypotheses = H0: mean1 = mean2 , H1: mean1 > mean2 |
| statistic = 1.765996124515009 |
| distribution = [student_t, 9] |
| p_value = .05560320992529344 |

\[
\text{test\_variance}(x) \quad \text{[Function]} \\
\text{test\_variance}(x, \text{options}, ...) \\
\]

This is the variance chi^2-test. Argument \(x\) is a list or a column matrix containing an one dimensional sample taken from a normal population.

Options:

- 'mean, default 'unknown, is the population’s mean, when it is known.
- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.
- 'variance, default 1, this is the variance value (positive) to be checked.
- 'conflevel, default 95/100, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).

The output of function \text{test\_variance} is an \text{inference\_result} Maxima object showing the following results:

1. 'var\_estimate: the sample variance.
2. 'conf\_level: confidence level selected by the user.
3. 'conf\_interval: confidence interval for the population variance.
4. 'method: inference procedure.
5. 'hypotheses: null and alternative hypotheses to be tested.
6. 'statistic: value of the sample statistic used for testing the null hypothesis.
7. 'distribution: distribution of the sample statistic, together with its parameter.
8. 'p\_value: \(p\)-value of the test.

Examples:

It is tested whether the variance of a population with unknown mean is equal to or greater than 200.

\[
(\%i1) \quad \text{load("stats")}\$ \\
(\%i2) \quad x: [203,229,215,220,223,233,208,228,209]\$ \\
(\%i3) \quad \text{test\_variance}(x,'alternative='greater,'variance=200); \\
\quad \text{VARIANCE TEST} \\
\]
\[
\begin{align*}
\text{var_estimate} &= 110.75 \\
\text{conf_level} &= 0.95 \\
\text{conf_interval} &= [57.1343376937479, \infty] \\
\text{method} &= \text{Variance Chi-square test. Unknown mean.} \\
\text{hypotheses} &= H_0: \text{var} = 200, H_1: \text{var} > 200 \\
\text{statistic} &= 4.43 \\
\text{distribution} &= [\text{chi2}, 8] \\
\text{p\_value} &= .8163948512777689 \\
\end{align*}
\]

\texttt{test\_variance\_ratio}  \hspace{1cm} \text{[Function]}

\texttt{test\_variance\_ratio (x1, x2)}
\texttt{test\_variance\_ratio (x1, x2, options ...)}

This is the variance ratio \(F\)-test for two normal populations. Arguments \(x1\) and \(x2\) are lists or column matrices containing two independent samples.

Options:
- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.
- 'mean1, default 'unknown, when it is known, this is the mean of the population from which \(x1\) was taken.
- 'mean2, default 'unknown, when it is known, this is the mean of the population from which \(x2\) was taken.
- 'conflevel, default 95/100, confidence level for the confidence interval of the ratio; it must be an expression which takes a value in \((0,1)\).

The output of function \texttt{test\_variance\_ratio} is an \texttt{inference\_result} Maxima object showing the following results:

1. 'ratio\_estimate: the sample variance ratio.
2. 'conf\_level: confidence level selected by the user.
3. 'conf\_interval: confidence interval for the variance ratio.
4. 'method: inference procedure.
5. 'hypotheses: null and alternative hypotheses to be tested.
6. 'statistic: value of the sample statistic used for testing the null hypothesis.
7. 'distribution: distribution of the sample statistic, together with its parameters.
8. 'p\_value: \(p\)-value of the test.

Examples:
The equality of the variances of two normal populations is checked against the alternative that the first is greater than the second.

(%i1) load("stats")$
(%i2) x: [20.4,62.5,61.3,44.2,11.1,23.7]$
(%i3) y: [1.2,6.9,38.7,20.4,17.2]$
(%i4) test_variance_ratio(x,y,'alternative='greater);

<table>
<thead>
<tr>
<th>VARIANCE RATIO TEST</th>
</tr>
</thead>
<tbody>
<tr>
<td>ratio_estimate = 2.31693391522034</td>
</tr>
<tr>
<td>conf_level = 0.95</td>
</tr>
<tr>
<td>conf_interval = [.3703504689507268, inf]</td>
</tr>
</tbody>
</table>
(%o4) | method = Variance ratio F-test. Unknown means. |
| hypotheses = H0: var1 = var2 , H1: var1 > var2 |
| statistic = 2.31693391522034 |
| distribution = [f, 5, 4] |
| p_value = .2179269692254457 |

**test_proportion**

*Function*

test_proportion (x, n)
test_proportion (x, n, options ...)

Inferences on a proportion. Argument x is the number of successes in n trials in a Bernoulli experiment with unknown probability.

Options:
- 'proportion, default 1/2, is the value of the proportion to be checked.
- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.
- 'conflevel, default 95/100, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).
- 'asymptotic, default false, indicates whether it performs an exact test based on the binomial distribution, or an asymptotic one based on the Central Limit Theorem; valid values are true and false.
- 'correct, default true, indicates whether Yates correction is applied or not.

The output of function test_proportion is an inference_result Maxima object showing the following results:
1. 'sample_proportion: the sample proportion.
2. 'conf_level: confidence level selected by the user.
3. 'conf_interval: Wilson confidence interval for the proportion.
4. 'method: inference procedure.
5. 'hypotheses: null and alternative hypotheses to be tested.
6. 'statistic: value of the sample statistic used for testing the null hypothesis.
7. 'distribution: distribution of the sample statistic, together with its parameters.
8. 'p_value: p-value of the test.

Examples:
Performs an exact test. The null hypothesis is $H_0: p = 1/2$ against the one sided alternative $H_1: p < 1/2$.

```maxima
(%i1) load("stats")$
(%i2) test_proportion(45, 103, alternative = less);
```
```
| PROPORTION TEST |
| sample_proportion = .4368932038834951 |
| conf_level = 0.95 |
| conf_interval = [0, 0.522714149150231] |

(%o2) method = Exact binomial test.
| hypotheses = H0: p = 0.5 , H1: p < 0.5 |
| statistic = 45 |
| distribution = [binomial, 103, 0.5] |
| p_value = .1184509388901454 |
```
A two sided asymptotic test. Confidence level is 99/100.

```maxima
(%i1) load("stats")$
(%i2) fpprintprec:7$
(%i3) test_proportion(45, 103, conflevel = 99/100, asymptotic=true);
```
```
| PROPORTION TEST |
| sample_proportion = .43689 |
| conf_level = 0.99 |
| conf_interval = [.31422, .56749] |

(%o3) method = Asymptotic test with Yates correction.
| hypotheses = H0: p = 0.5 , H1: p # 0.5 |
| statistic = .43689  |
| distribution = [normal, 0.5, .048872] |
| p_value = .19662  |

\textbf{test\_proportions\_difference}

\texttt{test\_proportions\_difference (x1, n1, x2, n2)}

\texttt{test\_proportions\_difference (x1, n1, x2, n2, options ...)}

Inferences on the difference of two proportions. Argument \textit{x1} is the number of successes in \textit{n1} trials in a Bernoulli experiment in the first population, and \textit{x2} and \textit{n2} are the corresponding values in the second population. Samples are independent and the test is asymptotic.

Options:

- 'alternative', default 'twosided, is the alternative hypothesis; valid values are: 'twosided (p1 \neq p2), 'greater (p1 > p2) and 'less (p1 < p2).
- 'conflevel, default 95/100, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).
- 'correct, default true, indicates whether Yates correction is applied or not.

The output of function \texttt{test\_proportions\_difference} is an \texttt{inference\_result} Maxima object showing the following results:

1. 'proportions: list with the two sample proportions.
2. 'conf\_level: confidence level selected by the user.
3. 'conf\_interval: Confidence interval for the difference of proportions \( p_1 - p_2 \).
4. 'method: inference procedure and warning message in case of any of the samples sizes is less than 10.
5. 'hypotheses: null and alternative hypotheses to be tested.
6. 'statistic: value of the sample statistic used for testing the null hypothesis.
7. 'distribution: distribution of the sample statistic, together with its parameters.
8. 'p\_value: \( p \)-value of the test.

Examples:

A machine produced 10 defective articles in a batch of 250. After some maintenance work, it produces 4 defective in a batch of 150. In order to know if the machine has improved, we test the null hypothesis \( H_0: p_1 = p_2 \), against the alternative \( H_0: p_1 > p_2 \), where \( p_1 \) and \( p_2 \) are the probabilities for one produced article to be defective before and after maintenance. According to the \( p \)-value, there is not enough evidence to accept the alternative.

\begin{verbatim}
(%i1) load("stats")$
(%i2) fpprintprec:7$
(%i3) test\_proportions\_difference(10, 250, 4, 150, alternative = greater);

\textbf{DIFFERENCE OF PROPORTIONS TEST}
\end{verbatim}
proportions = [0.04, 0.0266667]  
conf_level = 0.95  
conf_interval = [-0.02172761, 1]  

(%o3)  method = Asymptotic test. Yates correction.  

hypotheses = H0: p1 = p2, H1: p1 > p2  

statistic = 0.0133333  

distribution = [normal, 0, 0.01898069]  

p_value = 0.2411936  

Exact standard deviation of the asymptotic normal distribution when the data are unknown.

(%i1) load("stats")$  
(%i2) stats_numer: false$  
(%i3) sol: test_proportions_difference(x1,n1,x2,n2)$  
(%i4) last(take_inference('distribution,sol));  

\[
\frac{1}{n_2} \frac{1}{n_1} \frac{x_2 + x_1}{(x_2 + x_1)(1 - \frac{n_2 + n_1}{n_2 + n_1})}  
\]

\[
\sqrt{\frac{n_2 + n_1}{n_2 + n_1}}  
\]

test_sign

\[ \text{test_sign (x)} \]
\[ \text{test_sign (x, options ...)} \]

This is the non parametric sign test for the median of a continuous population. Argument x is a list or a column matrix containing an one dimensional sample.

Options:
- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.
- 'median, default 0, is the median value to be checked.

The output of function test_sign is an inference_result Maxima object showing the following results:
1. 'med_estimate: the sample median.
2. 'method: inference procedure.
3. 'hypotheses: null and alternative hypotheses to be tested.
4. 'statistic: value of the sample statistic used for testing the null hypothesis.
5. 'distribution: distribution of the sample statistic, together with its parameter(s).
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6. 'p_value: p-value of the test.

Examples:
Checks whether the population from which the sample was taken has median 6, against the alternative \( H_1: median > 6 \).

\[
\begin{align*}
\text{(i1)} & \text{ load("stats")}$
\text{(i2)} & x: [2,0.1,7,1.8,4,2.3,5.6,7.4,5.1,6.1,6]$ \\
\text{(i3)} & \text{ test_sign(x,'median=6,'alternative='greater);} \\
& \text{ SIGN TEST} \\
& \text{| med_estimate = 5.1} \\
& \text{| method = Non parametric sign test.} \\
\text{(o3)} & \text{ hypotheses = H0: median = 6 , H1: median > 6} \\
& \text{| statistic = 7} \\
& \text{| distribution = [binomial, 10, 0.5]} \\
& \text{| p_value = .0546874999999989}
\end{align*}
\]

test_signed_rank

This is the Wilcoxon signed rank test to make inferences about the median of a continuous population. Argument \( x \) is a list or a column matrix containing an one dimensional sample. Performs normal approximation if the sample size is greater than 20, or if there are zeroes or ties.

See also pdf_rank_test and cdf_rank_test

Options:
- 'median, default 0, is the median value to be checked.
- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.

The output of function test_signed_rank is an \text{inference_result} Maxima object with the following results:
1. 'med_estimate: the sample median.
2. 'method: inference procedure.
3. 'hypotheses: null and alternative hypotheses to be tested.
4. 'statistic: value of the sample statistic used for testing the null hypothesis.
5. 'distribution: distribution of the sample statistic, together with its parameter(s).
6. 'p_value: p-value of the test.

Examples:
Checks the null hypothesis \( H_0 : \text{median} = 15 \) against the alternative \( H_1 : \text{median} > 15 \). This is an exact test, since there are no ties.

\[
\begin{align*}
(\%i1) \ & \ \text{load("stats")}\$ \\
(\%i2) \ & \ x: [17.1, 15.9, 13.7, 13.4, 15.5, 17.6]$
(\%i3) \ & \ \text{test_signed_rank(x,median=15,alternative=greater);} \\
\end{align*}
\]

<table>
<thead>
<tr>
<th>SIGNED RANK TEST</th>
</tr>
</thead>
<tbody>
<tr>
<td>med_estimate = 15.7</td>
</tr>
<tr>
<td>method = Exact test</td>
</tr>
</tbody>
</table>

(\%o3) hypotheses = H0: med = 15 , H1: med > 15

| statistic = 14 |
| distribution = [signed_rank, 6] |
| p_value = 0.28125 |

Checks the null hypothesis \( H_0 : \text{equal}(\text{median}, 2.5) \) against the alternative \( H_1 : \text{notequal}(\text{median}, 2.5) \). This is an approximated test, since there are ties.

\[
\begin{align*}
(\%i1) \ & \ \text{load("stats")}\$ \\
(\%i2) \ & \ y:[1.9,2.3,2.6,1.9,1.6,3.3,4.2,2.4,2.9,1.5,3.2.9,4.2,3.1]$ \\
(\%i3) \ & \ \text{test_signed_rank(y,median=2.5);} \\
\end{align*}
\]

<table>
<thead>
<tr>
<th>SIGNED RANK TEST</th>
</tr>
</thead>
<tbody>
<tr>
<td>med_estimate = 2.9</td>
</tr>
<tr>
<td>method = Asymptotic test. Ties</td>
</tr>
</tbody>
</table>

(\%o3) hypotheses = H0: med = 2.5 , H1: med # 2.5

| statistic = 76.5 |
| distribution = [normal, 60.5, 17.58195097251724] |
| p_value = .3628097734643669 |

test_rank_sum

\[\text{test_rank_sum (x1, x2)}\]
\[\text{test_rank_sum (x1, x2, option)}\]

This is the Wilcoxon-Mann-Whitney test for comparing the medians of two continuous populations. The first two arguments \( x1 \) and \( x2 \) are lists or column matrices with the data of two independent samples. Performs normal approximation if any of the sample sizes is greater than 10, or if there are ties.

Option:
• 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.

The output of function test_rank_sum is an inference_result Maxima object with the following results:
1. 'method: inference procedure.
2. 'hypotheses: null and alternative hypotheses to be tested.
3. 'statistic: value of the sample statistic used for testing the null hypothesis.
4. 'distribution: distribution of the sample statistic, together with its parameters.
5. 'p_value: p-value of the test.

Examples:
Checks whether populations have similar medians. Samples sizes are small and an exact test is made.

(%i1) load("stats")$
(%i2) x: [12,15,17,38,42,10,23,35,28]$  
(%i3) y: [21,18,25,14,52,65,40,43]$  
(%i4) test_rank_sum(x,y);

| RANK SUM TEST |
| method = Exact test |
| hypotheses = H0: med1 = med2, H1: med1 # med2 |
| statistic = 22 |
| distribution = [rank_sum, 9, 8] |
| p_value = .199588646474702 |

Now, with greater samples and ties, the procedure makes normal approximation. The alternative hypothesis is $H_1: median1 < median2$.

(%i1) load("stats")$
(%i2) x: [39,42,35,13,10,23,15,20,17,27]$  
(%i3) y: [20,52,66,19,41,32,44,25,14,39,43,35,19,56,27,15]$  
(%i4) test_rank_sum(x,y,'alternative='less);

| RANK SUM TEST |
| method = Asymptotic test. Ties |
| hypotheses = H0: med1 = med2, H1: med1 < med2 |
| statistic = 48.5 |
| distribution = [normal, 79.5, 18.95419580097078] |
| p_value = .0509698566659841 |
test_normality (x) [Function]
Shapiro-Wilk test for normality. Argument x is a list of numbers, and sample size must be greater than 2 and less or equal than 5000, otherwise, function test_normality signals an error message.

Reference:

The output of function test_normality is an inference_result Maxima object with the following results:
1. 'statistic: value of the W statistic.
2. 'p_value: p-value under normal assumption.

Examples:
Checks for the normality of a population, based on a sample of size 9.

(%i1) load("stats")$
(%i2) x:[12,15,17,38,42,10,23,35,28]$
(%i3) test_normality(x);

| SHAPIRO - WILK TEST |
| statistic = .9251055695162436 |
| p_value = .4361763918860381 |

linear_regression [Function]
linear_regression (x) linear_regression (x option)
Multivariate linear regression, \( y_i = b0 + b1 * x_{1i} + b2 * x_{2i} + ... + bk * x_{ki} + u_i \), where \( u_i \) are \( N(0, \sigma) \) independent random variables. Argument x must be a matrix with more than one column. The last column is considered as the responses \( (y_i) \).

Option:
- 'conflevel, default 95/100, confidence level for the confidence intervals; it must be an expression which takes a value in \((0,1)\).

The output of function linear_regression is an inference_result Maxima object with the following results:
1. 'b_estimation: regression coefficients estimates.
2. 'b_covariances: covariance matrix of the regression coefficients estimates.
3. b_conf_int: confidence intervals of the regression coefficients.
4. b_statistics: statistics for testing coefficient.
5. b_p_values: p-values for coefficient tests.
6. b_distribution: probability distribution for coefficient tests.
7. v_estimation: unbiased variance estimator.
8. v_conf_int: variance confidence interval.
10. residuals: residuals.
11. **adc**: adjusted determination coefficient.
12. **aic**: Akaike’s information criterion.
13. **bic**: Bayes’s information criterion.

Only items 1, 4, 5, 6, 7, 8, 9 and 11 above, in this order, are shown by default. The rest remain hidden until the user makes use of functions `items_inference` and `take_inference`.

Example:

Fitting a linear model to a trivariate sample. The last column is considered as the responses ($y_i$).

```lisp
(%i2) load("stats")
(%i3) X:matrix(
     [58,111,64],
     [84,131,78],
     [78,158,83],
     [81,147,88],
     [82,121,89],
     [102,165,99],
     [85,174,101],
     [102,169,102])
(%i4) fpprintprec: 4
(%i5) res: linear_regression(X);
   LINEAR REGRESSION MODEL
   b_estimation = [9.054, .5203, .2397]
   b_statistics = [.6051, 2.246, 1.74]
   b_p_values = [.5715, .07466, .1423]
   b_distribution = [student_t, 5]
   v_estimation = 35.27
   v_conf_int = [13.74, 212.2]
   v_distribution = [chi2, 5]
   adc = .7922
(%i6) items_inference(res);
(%o6) [b_estimation, b_covariances, b_conf_int, b_statistics,
     b_p_values, b_distribution, v_estimation, v_conf_int,
     v_distribution, residuals, adc, aic, bic]
(%i7) take_inference('b_covariances, res);
   [ 223.9  - 1.12  -.8532 ]
   [                   ]
   [ - 1.12  .05367  -.02305 ]
   [                   ]
   [ -.8532  .02305  .01898 ]
(%i8) take_inference('bic, res);
(%o8) 30.98
(%i9) load("draw")$
(%i10) draw2d(
    points_joined = true,
    grid = true,
    points(take_inference('residuals, res)) )$

89.4 Functions and Variables for special distributions

pdf_signed_rank (x, n)  [Function]
    Probability density function of the exact distribution of the signed rank statistic.
    Argument x is a real number and n a positive integer.
    See also test_signed_rank.

cdf_signed_rank (x, n)  [Function]
    Cumulative density function of the exact distribution of the signed rank statistic.
    Argument x is a real number and n a positive integer.
    See also test_signed_rank.

pdf_rank_sum (x, n, m)  [Function]
    Probability density function of the exact distribution of the rank sum statistic.
    Argument x is a real number and n and m are both positive integers.
    See also test_rank_sum.

cdf_rank_sum (x, n, m)  [Function]
    Cumulative density function of the exact distribution of the rank sum statistic.
    Argument x is a real number and n and m are both positive integers.
    See also test_rank_sum.
90 stirling

90.1 Functions and Variables for stirling

stirling

stirling (z,n)
stirling (z,n,pred)

Replace gamma(x) with the $O(1/x^{2n-1})$ Stirling formula when $n$ isn't a nonnegative integer, signal an error. With the optional third argument pred, the Stirling formula is applied only when pred is true.


Examples:

(%i1) load ("stirling")$

(%i2) stirling(gamma(%alpha+x)/gamma(x),1);

1/2 - x x + %alpha - 1/2
x (x + %alpha)

1 1
---------- - ---- - %alpha
12 (x + %alpha) 12 x
%e

(%i3) taylor(%,x,inf,1);

%alpha 2 %alpha
%alpha x %alpha - x %alpha
2 x + -------------- + . . .

(%i4) map('factor,%);

%alpha - 1
%alpha (%alpha - 1) %alpha x

The function stirling knows the difference between the variable 'gamma' and the
function gamma:

(%i5) stirling(gamma + gamma(x),0);

x - 1/2 - x
%e

(%i6) stirling(gamma(y) + gamma(x),0);

y - 1/2 - y
%e

To apply the Stirling formula only to terms that involve the variable k, use an optional
third argument; for example

(%i7) makegamma(pochhammer(a,k)/pochhammer(b,k));
(%o7)  (gamma(b)*gamma(k+a))/(gamma(a)*gamma(k+b))
(%i8)  stirling(%1, lambda([s], not(freeof(k,s))));
(%o8)  (%e^(b-a)*gamma(b)*(k+a)^(k+a-1/2)*(k+b)^(-k-b+1/2))/gamma(a)

The terms \texttt{gamma(a)} and \texttt{gamma(b)} are free of \texttt{k}, so the Stirling formula was not applied to these two terms.

To use this function write first \texttt{load("stirling").}
91 stringproc

91.1 Introduction to String Processing

The package stringproc contains functions for processing strings and characters including formatting, encoding and data streams. This package is completed by some tools for cryptography, e.g. base64 and hash functions.

It can be directly loaded via load("stringproc") or automatically by using one of its functions.

For questions and bug reports please contact the author. The following command prints his e-mail-address.

printf(true, "~-{a}~@gmail.com", split(sdowncase("Volker van Nek")))$

A string is constructed by typing e.g. "Text". When the option variable stringdisp is set to false, which is the default, the double quotes won't be printed. [stringp], page 1206, is a test, if an object is a string.

(%i1) str: "Text";
(%o1) Text
(%i2) stringp(str);
(%o2) true

Characters are represented by a string of length 1. [charp], page 1198, is the corresponding test.

(%i1) char: "e";
(%o1) e
(%i2) charp(char);
(%o2) true

In Maxima position indices in strings are like in list 1-indexed which results to the following consistency.

(%i1) is(charat("Lisp",1) = charlist("Lisp")[1]);
(%o1) true

A string may contain Maxima expressions. These can be parsed with [parse_string], page 1202.

(%i1) map(parse_string, ["42", "sqrt(2)", "%pi"]);
(%o1) [42, sqrt(2), %pi]
(%i2) map('float, %);
(%o2) [42.0, 1.414213562373095, 3.141592653589793]

Strings can be processed as characters or in binary form as octets. Functions for conversions are [string_to_octets], page 1211, and [octets_to_string], page 1210. Usable encodings depend on the platform, the application and the underlying Lisp. (The following shows Maxima in GNU/Linux, compiled with SBCL.)

(%i1) obase: 16.$
(%i2) string_to_octets("$f€", "cp1252");
(%o2) [24, 0A3, 80]
(%i3) string_to_octets("$f€", "utf-8");
Strings may be written to character streams or as octets to binary streams. The following example demonstrates file input and output of characters.

`[openw]`, page 1192, returns an output stream to a file, `[printf]`, page 1193, writes formatted to that file and by e.g. `[close]`, page 1190, all characters contained in the stream are written to the file.

```lisp
(%i1) s: openw("file.txt");
(%o1) <output stream file.txt>
(%i2) printf(s, "~2tAn atom: "~20t~a~%~2tand a list: "~20t~{~r ~}~%~2t
and an integer: ~20t~d~%~2t", 42, 1.234, sqrt(2), %pi, 1.0e-2, 1.0e-2, 1.0b-2)$
(%i3) close(s)$
```

`[openr]`, page 1192, then returns an input stream from the previously used file and `[readline]`, page 1195, returns the line read as a string. The string may be tokenized by e.g. `[split]`, page 1203, or `[tokens]`, page 1206, and finally parsed by `[parse_string]`, page 1202.

```lisp
(%i4) s: openr("file.txt");
(%o4) <input stream file.txt>
(%i5) readline(s);
(%o5) 42 1.234 sqrt(2) %pi 0.01 1.0E-2 1.0b-2
(%i6) map(parse_string, split(%));
(%o6) [42, 1.234, sqrt(2), %pi, 0.01, 0.01, 1.0b-2]
(%i7) close(s)$
```

### 91.2 String Input and Output

Example: Formatted printing to a file.

```lisp
(%i1) s: openw("file.txt");
(%o1) <output stream file.txt>
(%i2) control: "~2tAn atom: "~20t~a~%~2tand a list: "~20t~{~r ~}~%~2t
and an integer: ~20t~d~%~2t"
(%i3) printf( s,control, 'true,[1,2,3],42 )$
(%o3) false
(%i4) close(s);
(%o4) true
(%i5) s: openr("file.txt");
(%o5) <input stream file.txt>
(%i6) while stringp( tmp:readline(s) ) do print(tmp)$
An atom: true
and a list: one two three
and an integer: 42
(%i7) close(s)$
```

`close (stream)`

Closes `stream` and returns `true` if `stream` had been open.
f\text{length} (stream)

\text{stream} has to be an open stream from or to a file. \text{f\text{length}} then returns the number of bytes which are currently present in this file.

Example: See [writebyte], page 1196, .

\text{flush\textunderscore output} (stream)

Flushes \text{stream} where \text{stream} has to be an output stream to a file.

Example: See [writebyte], page 1196, .

f\text{position} [Function]

\text{f\text{position} (stream)}

\text{f\text{position} (stream, pos)}

Returns the current position in \text{stream}, if \text{pos} is not used. If \text{pos} is used, \text{f\text{position}} sets the position in \text{stream}. \text{stream} has to be a stream from or to a file and \text{pos} has to be a positive number.

Positions in data streams are like in strings or lists 1-indexed, i.e. the first element in \text{stream} is in position 1.

\text{freshline} [Function]

\text{freshline ()}

\text{freshline (stream)}

Writes a new line to the standard output stream if the position is not at the beginning of a line and returns true. Using the optional argument \text{stream} the new line is written to that stream. There are some cases, where \text{freshline()} does not work as expected.

See also [newline], page 1192.

\text{get\textunderscore output\textunderscore stream\textunderscore string} (stream)

Returns a string containing all the characters currently present in \text{stream} which must be an open string-output stream. The returned characters are removed from \text{stream}.

Example: See [make\textunderscore string\textunderscore output\textunderscore stream], page 1191, .

\text{make\textunderscore string\textunderscore input\textunderscore stream} [Function]

\text{make\textunderscore string\textunderscore input\textunderscore stream (string)}

\text{make\textunderscore string\textunderscore input\textunderscore stream (string, start)}

\text{make\textunderscore string\textunderscore input\textunderscore stream (string, start, end)}

Returns an input stream which contains parts of \text{string} and an end of file. Without optional arguments the stream contains the entire string and is positioned in front of the first character. \text{start} and \text{end} define the substring contained in the stream. The first character is available at position 1.

(\%i1) \text{istream : make\textunderscore string\textunderscore input\textunderscore stream("text", 1, 4);}

(\%o1) "<\text{string\textunderscore input\textunderscore stream from "text">}

(\%i2) (while (c : \text{readchar(istream)}) # false do \text{sprint(c), newline()})$

\text{test}$

(\%i3) \text{close(istream)}$

\text{make\textunderscore string\textunderscore output\textunderscore stream} ()

Returns an output stream that accepts characters. Characters currently present in this stream can be retrieved by [get\textunderscore output\textunderscore stream\textunderscore string], page 1191.

(\%i1) \text{ostream : make\textunderscore string\textunderscore output\textunderscore stream();}
(%o1)                  #<string-output stream 09622ea0>
(%i2)  printf(ostream, "foo")$
(%i3)  printf(ostream, "bar")$
(%i4)  string : get_output_stream_string(ostream);  
     foobar
(%i5)  printf(ostream, "baz")$
(%i6)  string : get_output_stream_string(ostream);  
     baz
(%i7)  close(ostream)$

newline [Function]

    newline ()
    newline (stream)

Writes a new line to the standard output stream. Using the optional argument stream the new line is written to that stream. There are some cases, where newline() does not work as expected.

See [sprint], page 1195, for an example of using newline().

openr (file) [Function]

Returns a character input stream to file. If an existing file is opened, openr appends elements at the end of file.

For binary input see Section 76.3 [openr_binary], page 1096, .

openw (file) [Function]

Returns a character output stream to file. If file does not exist, it will be created. If an existing file is opened, openw destructively modifies file.

For binary output see Section 76.3 [openw_binary], page 1096, . See also close and openw.
printf

    printf (dest, string)
    printf (dest, string, expr_1, ..., expr_n)

Produces formatted output by outputting the characters of control-string string and observing that a tilde introduces a directive. The character after the tilde, possibly preceded by prefix parameters and modifiers, specifies what kind of formatting is desired. Most directives use one or more elements of the arguments expr_1, ..., expr_n to create their output.

If dest is a stream or true, then printf returns false. Otherwise, printf returns a string containing the output. By default the streams stdin, stdout and stderr are defined. If Maxima is running as a network client (which is the normal case if Maxima is communicating with a graphical user interface, which must be the server) setup-client will define old_stdout and old_stderr, too.

printf provides the Common Lisp function format in Maxima. The following example illustrates the general relation between these two functions.

(%i1) printf(true, "R~dD~d~%", 2, 2);  
R2D2   
(%o1) false
(%i2) :lisp (format t "R~dD~d~%" 2 2)  
R2D2   
NIL

The following description is limited to a rough sketch of the possibilities of printf. The Lisp function format is described in detail in many reference books. Of good help is e.g. the free available online-manual "Common Lisp the Language" by Guy L. Steele. See chapter 22.3.3 there.

In addition, printf recognizes two format directives which are not known to Lisp format. The format directive ~m indicates Maxima pretty printer output. The format directive ~h indicates a bigfloat number.

~% new line
~& fresh line
~t tab
~$ monetary
~d decimal integer
~b binary integer
~o octal integer
~x hexadecimal integer
~br base-b integer
~r spell an integer
~p plural
~f floating point
~e scientific notation
~g ~f or ~e, depending upon magnitude
~h bigfloat
~a uses Maxima function string
~m Maxima pretty printer output
~s like ~a, but output enclosed in "double quotes"

```
```

~< justification, ~> terminates

~( case conversion, ~) terminates

~[ selection, ~] terminates

~{ iteration, ~} terminates

Note that the directive ~# is not supported.

If dest is a stream or true, then printf returns false. Otherwise, printf returns a string containing the output.

```
(%i1) printf( false, "~a ~a ~4f ~a ~@r",
   "String",sym,bound,sqrt(12),144), bound = 1.234;

(%o1) String sym 1.23 2*sqrt(3) CXLIV

(%i2) printf( false,"~{~a ~}",
   ["one",2,"THREE"]);

(%o2) one 2 THREE

(%i3) printf(true,"~{~9,1f ~}~%~}",mat ),
   mat = args(matrix([1.1,2,3.33],[4,5,6],[7,8.88,9]))$

1.1 2.0 3.3
4.0 5.0 6.0
7.0 8.9 9.0

(%i4) control: "~:~(~r~) bird~p ~[is~;are~] singing."$

(%i5) printf( false,control, n,n,if n=1 then 1 else 2 ), n=2;

(%o5) Two birds are singing.
```

The directive ~h has been introduced to handle bigflots.

```
~w,d,e,x,o,p@H
w : width
d : decimal digits behind floating point
e : minimal exponent digits
x : preferred exponent
o : overflow character
p : padding character
@ : display sign for positive numbers
```

```
(%i11) fpprec : 1000$

(%i12) printf(true, "|~h|~%", 2.b0^-64)
|0.0000000000000000000000054210108624275221700372640043497085712890625|

(%i13) fpprec : 26$

(%i14) printf(true, "|~h|~%", sqrt(2))$
|1.4142135623730950488016887|

(%i15) fpprec : 24$

(%i16) printf(true, "|~h|~%", sqrt(2))$
|1.41421356237309504880169|

(%i17) printf(true, "|~28h|~%", sqrt(2))$
| 1.41421356237309504880169|

(%i18) printf(true, "|~28,,,,'*h|~%", sqrt(2))$
|***1.41421356237309504880169|

(%i19) printf(true, "|~,18h|~%", sqrt(2))$
```
For conversion of objects to strings also see `concat`, `sconcat`, `string` and `simplode`.

**readbyte** *(stream)*  
[Function]  
Removes and returns the first byte in *stream* which must be a binary input stream. If the end of file is encountered *readbyte* returns false.

Example: Read the first 16 bytes from a file encrypted with AES in OpenSSL.

```lisp
(%i1) ibase: obase: 16.$
(%i2) in: openr_binary("msg.bin");
(%o2) #<input stream msg.bin>
(%i3) (L:[]$, thru 16. do push(readbyte(in), L), L:reverse(L));
(%o3) [53, 61, 6C, 74, 65, 64, 5F, 5F, 88, 56, 0DE, 8A, 74, 0FD, 0AD, 0F0]
(%i4) close(in);
(%o4) true
(%i5) map(ascii, rest(L,-8));
(%o5) [S, a, l, t, e, d, _, _]
(%i6) salt: octets_to_number(rest(L,8));
(%o6) 8856de8a74fdadf0
```

**readchar** *(stream)*  
[Function]  
Removes and returns the first character in *stream*. If the end of file is encountered *readchar* returns false.

Example: See `make_string_input_stream`, page 1191.

**readline** *(stream)*  
[Function]  
Returns a string containing all characters starting at the current position in *stream* up to the end of the line or false if the end of the file is encountered.

**sprint** *(expr_1, ..., expr_n)*  
[Function]  
Evaluates and displays its arguments one after the other 'on a line' starting at the leftmost position. The expressions are printed with a space character right next to the number, and it disregards line length. *newline()* might be used for line breaking.

Example: Sequential printing with *sprint*. Creating a new line with *newline()*.

```lisp
(%i1) for n:0 thru 19 do sprint(fib(n))$
   0 1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 1597 2584 4181
(%i2) for n:0 thru 22 do (sprint(fib(n)),
```
if mod(n,10) = 9 then newline() )$
0 1 1 2 3 5 8 13 21 34
55 89 144 233 377 610 987 1597 2584 4181
6765 10946 17711

writebyte (byte, stream) [Function]
Writes byte to stream which must be a binary output stream. writebyte returns byte.

Example: Write some bytes to a binary file output stream. In this example all bytes correspond to printable characters and are printed by printfile. The bytes remain in the stream until flush_output or close have been called.

(%i1) ibase: obase: 16.$
(%i2) bytes: map(cint, charlist("GNU/Linux"));
   (%o2) [47, 4E, 55, 2F, 4C, 69, 6E, 75, 78]
(%i3) out: openw_binary("test.bin");
   (%o3) <output stream test.bin>
(%i4) for i thru 3 do writebyte(bytes[i], out);
   done
(%i5) printfile("test.bin")$
(%i6) flength(out);
   (%o6) 0
(%i7) flush_output(out);
   (%o7) true
(%i8) flength(out);
   (%o8) 3
(%i9) printfile("test.bin")$
GNU
(%i10) for b in rest(bytes,3) do writebyte(b, out);
   done
(%i10b) close(out);
   (%o10b) true
(%i10c) printfile("test.bin")$
GNU/Linux

91.3 Characters
Characters are strings of length 1.

adjust_external_format () [Function]
Prints information about the current external format of the Lisp reader and in case the external format encoding differs from the encoding of the application which runs Maxima adjust_external_format tries to adjust the encoding or prints some help or instruction. adjust_external_format returns true when the external format has been changed and false otherwise.
Functions like \[\texttt{cint}\], \[\texttt{unicode}\], \[\texttt{octets\_to\_string}\], \[\texttt{string\_to\_octets}\], and \[\texttt{string\_to\_octets}\], need UTF-8 as the external format of the Lisp reader to work properly over the full range of Unicode characters.

Examples (Maxima on Windows, March 2016): Using \texttt{adjust\_external\_format} when the default external format is not equal to the encoding provided by the application.

1. Command line Maxima

In case a terminal session is preferred it is recommended to use Maxima compiled with SBCL. Here Unicode support is provided by default and calls to \texttt{adjust\_external\_format} are unnecessary.

If Maxima is compiled with CLISP or GCL it is recommended to change the terminal encoding from CP850 to CP1252. \texttt{adjust\_external\_format} prints some help.

CCL reads UTF-8 while the terminal input is CP850 by default. CP1252 is not supported by CCL. \texttt{adjust\_external\_format} prints instructions for changing the terminal encoding and external format both to iso-8859-1.

2. wxMaxima

In wxMaxima SBCL reads CP1252 by default but the input from the application is UTF-8 encoded. Adjustment is needed.

Calling \texttt{adjust\_external\_format} and restarting Maxima permanently changes the default external format to UTF-8.

\begin{verbatim}
(%i1) adjust_external_format();
The line
(setf sb-impl::*default-external-format* :utf-8)
has been appended to the init file
C:/Users/Username/.sbclrc
Please restart Maxima to set the external format to UTF-8.
(%i1) false

Restarting Maxima.

(%i1) adjust_external_format();
The external format is currently UTF-8
and has not been changed.
(%i1) false
\end{verbatim}

**alphacharp**   \(\text{(char)}\)

| Function |

Returns true if char is an alphabetic character.

To identify a non-US-ASCII character as an alphabetic character the underlying Lisp must provide full Unicode support. E.g. a German umlaut is detected as an alphabetic character with SBCL in GNU/Linux but not with GCL. (In Windows Maxima, when compiled with SBCL, must be set to UTF-8. See \[\texttt{adjust\_external\_format}\], page 1196, for more.)

Example: Examination of non-US-ASCII characters.

The underlying Lisp (SBCL, GNU/Linux) is able to convert the typed character into a Lisp character and to examine.

\begin{verbatim}
(%i1) alphacharp("ü");
\end{verbatim}
In GCL this is not possible. An error break occurs.

(%i1) alphacharp("u");  
(%o1) true

(%i2) alphacharp("ü");  

package stringproc: ü cannot be converted into a Lisp character. -- an error.

alphanumericp (char)  
Returns true if char is an alphabetic character or a digit (only corresponding US-ASCII characters are regarded as digits).

Note: See remarks on [alphacharp], page 1197.

ascii (int)  
Returns the US-ASCII character corresponding to the integer int which has to be less than 128.

See [unicode], page 1200, for converting code points larger than 127.

Examples:

(%i1) for n from 0 thru 127 do (  
ch: ascii(n),  
if alphacharp(ch) then sprint(ch),  
if n = 96 then newline() )$  
A B C D E F G H I J K L M N O P Q R S T U V W X Y Z  
a b c d e f g h i j k l m n o p q r s t u v w x y z

cequal (char_1, char_2)  
Returns true if char_1 and char_2 are the same character.

cequalignore (char_1, char_2)  
Like cequal but ignores case which is only possible for non-US-ASCII characters when the underlying Lisp is able to recognize a character as an alphabetic character.

See remarks on [alphacharp], page 1197.

cgreaterp (char_1, char_2)  
Returns true if the code point of char_1 is greater than the code point of char_2.

cgreaterpignore (char_1, char_2)  
Like cgreaterp but ignores case which is only possible for non-US-ASCII characters when the underlying Lisp is able to recognize a character as an alphabetic character.

See remarks on [alphacharp], page 1197.

charp (obj)  
Returns true if obj is a Maxima-character. See introduction for example.

cint (char)  
Returns the Unicode code point of char which must be a Maxima character, i.e. a string of length 1.
Examples: The hexadecimal code point of some characters (Maxima with SBCL on GNU/Linux).

\[(%i1)\) \text{obase: 16.}\$ 
\[(%i2)\) \text{map(cint, ["$","£","e"])}; 
\[(%o2)\] \[24, 0A3, 20AC\]

Warning: It is not possible to enter characters corresponding to code points larger than 16 bit in wxMaxima with SBCL on Windows when the external format has not been set to UTF-8. See [adjust_external_format], page 1196.

CMUCL doesn't process these characters as one character. \text{cint} then returns \text{false}. Converting a character to a code point via UTF-8-octets may serve as a workaround:
\text{utf8_to_unicode(string_to_octets(character))};

See [utf8_to_unicode], page 1201, [string_to_octets], page 1211.

\text{clessp} (\text{char}_1, \text{char}_2)
\begin{itemize}
  \item [Function] \text{clessp} \text{returns} \text{true} \text{if} \text{the code point of} \text{char}_1 \text{is less than the code point of} \text{char}_2.
\end{itemize}

\text{clesspignore} (\text{char}_1, \text{char}_2)
\begin{itemize}
  \item [Function] \text{clesspignore} \text{is like} \text{clessp} \text{but ignores case which is only possible for non-US-ASCII characters when the underlying Lisp is able to recognize a character as an alphabetic character. See remarks on} [alphacharp], page 1197.
\end{itemize}

\text{constituent} (\text{char})
\begin{itemize}
  \item [Function] \text{constituent} \text{returns} \text{true} \text{if} \text{char} \text{is a graphic character but not a space character. A graphic character is a character one can see, plus the space character. (constituent is defined by Paul Graham. See Paul Graham, ANSI Common Lisp, 1996, page 67.)}
\end{itemize}

\text{digitcharp} (\text{char})
\begin{itemize}
  \item [Function] \text{digitcharp} \text{returns} \text{true} \text{if} \text{char} \text{is a digit where only the corresponding US-ASCII-character is regarded as a digit.}
\end{itemize}

\text{lowercasep} (\text{char})
\begin{itemize}
  \item [Function] \text{lowercasep} \text{returns} \text{true} \text{if} \text{char} \text{is a lowercase character.}
  \item [Note] \text{See remarks on} [alphacharp], page 1197.
\end{itemize}

\text{newline}
\begin{itemize}
  \item [Variable] \text{newline} \text{is the newline character (ASCII-character 10).}
\end{itemize}

\text{space}
\begin{itemize}
  \item [Variable] \text{space} \text{is the space character.}
\end{itemize}

\text{tab}
\begin{itemize}
  \item [Variable] \text{tab} \text{is the tab character.}
\end{itemize}
**unicode (arg)**

Returns the character defined by arg which might be a Unicode code point or a name string if the underlying Lisp provides full Unicode support.

Example: Characters defined by hexadecimal code points (Maxima with SBCL on GNU/Linux).

```maxima
(%i1) ibase: 16.$
(%i2) map(unicode, [24, 0A3, 20AC]);
(%o2) [$, £, €]
```

Warning: In wxMaxima with SBCL on Windows it is not possible to convert code points larger than 16 bit to characters when the external format has not been set to UTF-8. See [adjust_external_format], page 1196, for more information.

CMUCL doesn’t process code points larger than 16 bit. In these cases `unicode` returns `false`. Converting a code point to a character via UTF-8 octets may serve as a workaround:

```maxima
octets_to_string(unicode_to_utf8(code_point));
```

See [octets_to_string], page 1210, [unicode_to_utf8], page 1200.

In case the underlying Lisp provides full Unicode support the character might be specified by its name. The following is possible in ECL, CLISP and SBCL, where in SBCL on Windows the external format has to be set to UTF-8. `unicode(name)` is supported by CMUCL too but again limited to 16 bit characters.

The string argument to `unicode` is basically the same string returned by `printf` using the "@c" specifier. But as shown below the prefix "#" must be omitted. Underlines might be replaced by spaces and uppercase letters by lowercase ones.

Example (continued): Characters defined by names (Maxima with SBCL on GNU/Linux).

```maxima
(%i3) printf(false, "-@c", unicode(0DF));
(%o3) #\LATIN_SMALL_LETTER_SHARP_S
(%i4) unicode("LATIN_SMALL_LETTER_SHARP_S");
(%o4) ß
(%i5) unicode("Latin small letter sharp s");
(%o5) ß
```

**unicode_to_utf8 (code_point)**

Returns a list containing the UTF-8 code corresponding to the Unicode code point.

Examples: Converting Unicode code points to UTF-8 and vice versa.

```maxima
(%i1) ibase: obase: 16.$
(%i2) map(cint, ["$","£","€"]);
(%o2) [24, 0A3, 20AC]
(%i3) map(unicode_to_utf8, %);
(%o3) [[24],[0C2, 0A3],[0E2, 82, 0AC]]
(%i4) map(utf8_to_unicode, %);
(%o4) [24, 0A3, 20AC]
```

**uppercasep (char)**

Returns `true` if char is an uppercase character.

Note: See remarks on [alphacharp], page 1197.
**us_ascii_only**  
[Variable]  
This option variable affects Maxima when the character encoding provided by the application which runs Maxima is UTF-8 but the external format of the Lisp reader is not equal to UTF-8.  
On GNU/Linux this is true when Maxima is built with GCL and on Windows in wxMaxima with GCL- and SBCL-builds. With SBCL it is recommended to change the external format to UTF-8. Setting `us_ascii_only` is unnecessary then. See [adjust_external_format], page 1196, for details.  
`us_ascii_only` is false by default. Maxima itself then (i.e. in the above described situation) parses the UTF-8 encoding.  
When `us_ascii_only` is set to true it is assumed that all strings used as arguments to string processing functions do not contain Non-US-ASCII characters. Given that promise, Maxima avoids parsing UTF-8 and strings can be processed more efficiently.

**utf8_to_unicode (list)**  
[Function]  
Returns a Unicode code point corresponding to the list which must contain the UTF-8 encoding of a single character.  
Examples: See [unicode_to_utf8], page 1200.

### 91.4 String Processing

Position indices in strings are 1-indexed like in Maxima lists. See example in [charat], page 1201.

**charat (string, n)**  
[Function]  
Returns the n-th character of string. The first character in string is returned with n = 1.  
\[
\begin{align*}
    (%i1) \text{charat}("Lisp",1); \\
    (%o1) L \\
    (%i2) \text{charlist}("Lisp")[1]; \\
    (%o2) L
\end{align*}
\]

**charlist (string)**  
[Function]  
Returns the list of all characters in string.  
\[
\begin{align*}
    (%i1) \text{charlist}("Lisp"); \\
    (%o1) [L, i, s, p]
\end{align*}
\]

**eval_string (str)**  
[Function]  
Parse the string str as a Maxima expression and evaluate it. The string str may or may not have a terminator (dollar sign $ or semicolon ;). Only the first expression is parsed and evaluated, if there is more than one. Complain if str is not a string.  
Examples:  
\[
\begin{align*}
    (%i1) \text{eval_string} \("\text{foo: 42; bar: foo}^2 + \text{baz}\") ; \\
    (%o1) 42 \\
    (%i2) \text{eval_string} \("\text{(foo: 42, bar: foo}^2 + \text{baz)}\") ; \\
    (%o2) \text{baz + 1764}
\end{align*}
\]

See also [parse_string], page 1202, and [eval_string_lisp], page 27.
parse_string (str)  [Function]
Parse the string str as a Maxima expression (do not evaluate it). The string str may
or may not have a terminator (dollar sign $ or semicolon ;). Only the first expression
is parsed, if there is more than one.
Complain if str is not a string.
Examples:

(%i1) parse_string ("foo: 42; bar: foo^2 + baz");
(%o1) foo : 42
(%i2) parse_string ("(foo: 42, bar: foo^2 + baz)");
     2
(%o2) (foo : 42, bar : foo + baz)

See also [eval_string], page 1201.

scopy (string)  [Function]
Returns a copy of string as a new string.

sdowncase  [Function]
sdowncase (string)
sdowncase (string, start)
sdowncase (string, start, end)
Like [supcase], page 1206, but uppercase characters are converted to lowercase.

sequal (string_1, string_2)  [Function]
Returns true if string_1 and string_2 contain the same sequence of characters.

sequalignore (string_1, string_2)  [Function]
Like sequal but ignores case which is only possible for non-US-ASCII characters
when the underlying Lisp is able to recognize a character as an alphabetic character.
See remarks on [alphacharp], page 1197.

sexplode (string)  [Function]
sexplode is an alias for function charlist.

simpplode  [Function]
simpplode (list)
simpplode (list, delim)
simpplode takes a list of expressions and concatenates them into a string. If no
delimiter delim is specified, simpplode uses no delimiter. delim can be any string.
See also concat, sconcat, string and printf.
Examples:

(%i1) simpplode(["xx","3","":",expand((x+y)\^3)]);
    \^3
(%o1) xx[3]:y 3+3*x*y +3*x +y +x
(%i2) simpplode( sexplode("stars")," * ");
    s * t * a * r * s
(%i3) simpplode( ["One","more","coffee."]," " );
(%o3) One more coffee.
**sinsert (seq, string, pos)**  
Returns a string that is a concatenation of `substring(string, 1, pos-1)`, the string `seq` and `substring(string, pos)`. Note that the first character in `string` is in position 1.

Examples:

```lisp
(%i1) s: "A submarine."$
(%i2) concat( substring(s,1,3),"yellow ",substring(s,3) );
(%o2) A yellow submarine.
(%i3) sinsert("hollow ",s,3);
(%o3) A hollow submarine.
```

**sinvertcase**  
`sinvertcase(string)`  
`sinvertcase(string, start)`  
`sinvertcase(string, start, end)`  
Returns `string` except that each character from position `start` to `end` is inverted. If `end` is not given, all characters from `start` to the end of `string` are replaced.

Examples:

```lisp
(%i1) sinvertcase("sInvertCase");
(%o1) SiNVERTcASE
```

**slength (string)**  
Returns the number of characters in `string`.

**smake (num, char)**  
Returns a new string with a number of `num` characters `char`.

Example:

```lisp
(%i1) smake(3,"w");
(%o1) www
```

**smismatch**  
`smismatch(string_1, string_2)`  
`smismatch(string_1, string_2, test)`  
Returns the position of the first character of `string_1` at which `string_1` and `string_2` differ or false. Default test function for matching is `sequal`. If `smismatch` should ignore case, use `sequalignore` as test.

Example:

```lisp
(%i1) smismatch("seven","seventh");
(%o1) 6
```

**split**  
`split(string)`  
`split(string, delim)`  
`split(string, delim, multiple)`  
Returns the list of all tokens in `string`. Each token is an unparsed string. `split` uses `delim` as delimiter. If `delim` is not given, the space character is the default delimiter. `multiple` is a boolean variable with `true` by default. Multiple delimiters are read as
one. This is useful if tabs are saved as multiple space characters. If \texttt{multiple} is set to \texttt{false}, each delimiter is noted.

Examples:

\begin{verbatim}
(%i1) split("1.2 2.3 3.4 4.5");
(%o1) [1.2, 2.3, 3.4, 4.5]
(%i2) split("first;;third;fourth",";",false);
(%o2) [first, , third, fourth]
\end{verbatim}

\textbf{\texttt{sposition}}(char, \texttt{string}) \hfill [Function]

Returns the position of the first character in \texttt{string} which matches \texttt{char}. The first character in \texttt{string} is in position 1. For matching characters ignoring case see \texttt{[ssearch]}, page 1204.

\textbf{\texttt{sremove}} \hfill [Function]

\begin{verbatim}
sremove (seq, \texttt{string})
sremove (seq, \texttt{string}, \texttt{test})
sremove (seq, \texttt{string}, \texttt{test}, \texttt{start})
sremove (seq, \texttt{string}, \texttt{test}, \texttt{start}, \texttt{end})
\end{verbatim}

Returns a string like \texttt{string} but without all substrings matching \texttt{seq}. Default test function for matching is \texttt{sequal}. If \texttt{sremove} should ignore case while searching for \texttt{seq}, use \texttt{sequalignore} as test. Use \texttt{start} and \texttt{end} to limit searching. Note that the first character in \texttt{string} is in position 1.

Examples:

\begin{verbatim}
(%i1) sremove("n't","I don't like coffee.");
(%o1) I do like coffee.
(%i2) sremove("DO ",%,'sequalignore);
(%o2) I like coffee.
\end{verbatim}

\textbf{\texttt{sremovefirst}} \hfill [Function]

\begin{verbatim}
sremovefirst (seq, \texttt{string})
sremovefirst (seq, \texttt{string}, \texttt{test})
sremovefirst (seq, \texttt{string}, \texttt{test}, \texttt{start})
sremovefirst (seq, \texttt{string}, \texttt{test}, \texttt{start}, \texttt{end})
\end{verbatim}

Like \texttt{sremove} except that only the first substring that matches \texttt{seq} is removed.

\textbf{\texttt{sreverse}} \hfill [Function]

 Returns a string with all the characters of \texttt{string} in reverse order.

See also \texttt{reverse}.

\textbf{\texttt{ssearch}} \hfill [Function]

\begin{verbatim}
ssearch (seq, \texttt{string})
ssearch (seq, \texttt{string}, \texttt{test})
ssearch (seq, \texttt{string}, \texttt{test}, \texttt{start})
ssearch (seq, \texttt{string}, \texttt{test}, \texttt{start}, \texttt{end})
\end{verbatim}

Returns the position of the first substring of \texttt{string} that matches the string \texttt{seq}. Default test function for matching is \texttt{sequal}. If \texttt{ssearch} should ignore case, use \texttt{sequalignore} as test. Use \texttt{start} and \texttt{end} to limit searching. Note that the first character in \texttt{string} is in position 1.
Example:

```
(%i1) ssearch("~s","~{~S ~}~%","sequalignore");
(%o1) 4
```

**ssort**

```
ssort (string)
```

Returns a string that contains all characters from `string` in an order such there are no two successive characters `c` and `d` such that `test (c, d)` is false and `test (d, c)` is true. Default test function for sorting is `clessp`. The set of test functions is `{clessp, clesspignore, cgreaterp, cgreaterpignore, cequal, cequalignore}`.

Examples:

```
(%i1) ssort("I don't like Mondays.");
(%o1) 'IMaddeikInnoosty
(%i2) ssort("I don't like Mondays.", 'cgreaterpignore);
(%o2) ytsoonnMlkIiedda.'
```

**ssubst**

```
ssubst (new, old, string)
ssubst (new, old, string, test)
ssubst (new, old, string, test, start)
ssubst (new, old, string, test, start, end)
```

Returns a string like `string` except that all substrings matching `old` are replaced by `new`. `old` and `new` need not to be of the same length. Default test function for matching is `sequal`. If `ssubst` should ignore case while searching for `old`, use `sequalignore` as test. Use `start` and `end` to limit searching. Note that the first character in `string` is in position 1.

Examples:

```
(%i1) ssubst("like","hate","I hate Thai food. I hate green tea.");
(%o1) I like Thai food. I hate green tea.
(%i2) ssubst("Indian","thai", %, 'sequalignore, 8, 12);
(%o2) I like Indian food. I like green tea.
```

**ssubstfirst**

```
ssubstfirst (new, old, string)
ssubstfirst (new, old, string, test)
ssubstfirst (new, old, string, test, start)
ssubstfirst (new, old, string, test, start, end)
```

Like `ssubst` except that only the first substring that matches `old` is replaced.

**strim**

```
strim (seq, string)
```

Returns a string like `string`, but with all characters that appear in `seq` removed from both ends.

Examples:

```
(%i1) "/* comment */$
(%i2) strim(" /*", %);
(%o2) comment
```
\(\text{(\%i3)}\) \text{slen\texttt{g}(\%)}; \\
\(\text{(\%o3)}\) \hspace{1cm} 7

\text{striml} (\text{\texttt{seq}}, \text{\texttt{string}}) \quad \text{[Function]} \\
Like \text{strim} except that only the left end of \text{\texttt{string}} is trimmed.

\text{strimr} (\text{\texttt{seq}}, \text{\texttt{string}}) \quad \text{[Function]} \\
Like \text{strim} except that only the right end of \text{\texttt{string}} is trimmed.

\text{stringp} (\text{\texttt{obj}}) \quad \text{[Function]} \\
Returns \text{true} if \text{\texttt{obj}} is a string. See introduction for example.

\text{substring} \quad \text{[Function]} \\
\text{substring} (\text{\texttt{string}}, \text{\texttt{start}}) \\
\text{substring} (\text{\texttt{string}}, \text{\texttt{start}}, \text{\texttt{end}}) \\
Returns the substring of \text{\texttt{string}} beginning at position \text{\texttt{start}} and ending at position \text{\texttt{end}}. The character at position \text{\texttt{end}} is not included. If \text{\texttt{end}} is not given, the substring contains the rest of the string. Note that the first character in \text{\texttt{string}} is in position 1.

Examples:
\(\text{(\%i1)}\) \text{substring("substring",}4\)); \\
\(\text{(\%o1)}\) \hspace{1cm} \text{string} \\
\(\text{(\%i2)}\) \text{substring(\%,}4\},6\)); \\
\(\text{(\%o2)}\) \hspace{1cm} \text{in}

\text{supcase} \quad \text{[Function]} \\
\text{supcase} (\text{\texttt{string}}) \\
\text{supcase} (\text{\texttt{string}}, \text{\texttt{start}}) \\
\text{supcase} (\text{\texttt{string}}, \text{\texttt{start}}, \text{\texttt{end}}) \\
Returns \text{\texttt{string}} except that lowercase characters from position \text{\texttt{start}} to \text{\texttt{end}} are replaced by the corresponding uppercase ones. If \text{\texttt{end}} is not given, all lowercase characters from \text{\texttt{start}} to the end of \text{\texttt{string}} are replaced.

Example:
\(\text{(\%i1)}\) \text{supcase("english",}1\},2\)); \\
\(\text{(\%o1)}\) \hspace{1cm} \text{English}

\text{tokens} \quad \text{[Function]} \\
\text{tokens} (\text{\texttt{string}}) \\
\text{tokens} (\text{\texttt{string}}, \text{\texttt{test}}) \\
Returns a list of tokens, which have been extracted from \text{\texttt{string}}. The tokens are substrings whose characters satisfy a certain test function. If \text{\texttt{test}} is not given, \text{\texttt{constituent}} is used as the default test. \{\text{\texttt{constituent}}, \text{\texttt{alphacharp}}, \text{\texttt{digitcharp}}, \text{\texttt{lowercasep}}, \text{\texttt{uppercasep}}, \text{\texttt{charp}}, \text{\texttt{characterp}}, \text{\texttt{alphanumericp}}\} is the set of test functions. (The Lisp-version of \text{\texttt{tokens}} is written by Paul Graham. ANSI Common Lisp, 1996, page 67.)

Examples:
\(\text{(\%i1)}\) \text{tokens("24 October 2005")}; \\
\(\text{(\%o1)}\) \hspace{1cm} \{24, October, 2005\}
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(%i2) tokens("05-10-24", 'digitcharp);
(%o2) [05, 10, 24]
(%i3) map(parse_string, %);
(%o3) [5, 10, 24]

91.5 Octets and Utilities for Cryptography

base64 (arg) [Function]
Returns the base64-representation of arg as a string. The argument arg may be a string, a non-negative integer or a list of octets.
Examples:
(%i1) base64: base64("foo bar baz");
(%o1) Zm9vIGJhciBiYXo=
(%i2) string: base64_decode(base64);
(%o2) foo bar baz
(%i3) obase: 16.$
(%i4) integer: base64_decode(base64, 'number);
(%o4) 666f6f206261722062617a
(%i5) octets: base64_decode(base64, 'list);
(%o5) [66, 6F, 6F, 20, 62, 61, 72, 20, 62, 61, 7A]
(%i6) ibase: 16.$
(%i7) base64(octets);
(%o7) Zm9vIGJhciBiYXo=

Note that if arg contains umlauts (resp. octets larger than 127) the resulting base64-string is platform dependent. However the decoded string will be equal to the original.

base64_decode [Function]

base64_decode (base64-string)
base64_decode (base64-string, return-type)

By default base64_decode decodes the base64-string back to the original string.
The optional argument return-type allows base64_decode to alternatively return the corresponding number or list of octets. return-type may be string, number or list.
Example: See [base64], page 1207.

crc24sum [Function]

crc24sum (octets)
crc24sum (octets, return-type)

By default crc24sum returns the CRC24 checksum of an octet-list as a string.
The optional argument return-type allows crc24sum to alternatively return the corresponding number or list of octets. return-type may be string, number or list.
Example:

-----BEGIN PGP SIGNATURE-----
Version: GnuPG v2.0.22 (GNU/Linux)

iQEcBAEBAgAGBQJvdCTzAAoJEG/1Mgf2DWAqCSYH/AhVFwhu1D89C3/QFcgyvZTM
wnOyZBUURJAL/cT+1ngkLEpp3hEbbREcugWp+Tm6aw3R4CdJ7G3FLxExBH/5KnDHi
(%i1) ibase : obase : 16.$
(%i2) sig64 : sconcat(
"iQEcBAEBAgAGBQJVdCTzAAoJEG/1Mgf2DWAqCSYH/AbVFwhu1D89C3/QFcgVvZTM",
"wn0YzBUURJAL/cT+IngkLEpp3hEbREcugWp+Tm6aw3R4CdJ7G3FLxExBH/5KnDHi",
"rBQu+I7+3ySk2hpryQ6Wx5J9uZSa4YmsNteR8up0zGkaulJeWkS4pjiRM+auWVe",
"vajlKZCIK52P080DG7Q2dpshh4fgTeNwqCuBiHz73t8g1IaLdhDN6EzJvGizam",
"/spqT/sTo6sw8yDOJjvU+Qvn6/mSMjC/YxjhrMaQt9EMrRlAz4ukBF5uG1S7mXOH",
"WdiwkSPZ3gnIBhM9SuC076gLWZUNs6NqTeE3UzMrJDAFhH3jYk1T7mysCvdtIkms=")
(%o2) $WmeC$

md5sum

Returns the MD5 checksum of a string, non-negative integer, list of octets, or binary
(not character) input stream. A file for which an input stream is opened may be an
ordinary text file; it is the stream which needs to be binary, not the file itself.
When the argument is an input stream, md5sum reads the entire content of the stream,
but does not close the stream.

The default return value is a string containing 32 hex characters. The optional argument
return-type allows md5sum to alternatively return the corresponding number or
list of octets. return-type may be string, number or list.

Note that in case arg contains German umlauts or other non-ASCII characters (resp.
ocets larger than 127) the MD5 checksum is platform dependent.

Examples:

(%,i1) ibase : obase : 16.$
(%,i2) msg : "foo bar baz"$
(%,i3) string : md5sum(msg);
(%,i4) integer : md5sum(msg, 'number);
(%,i5) octets : md5sum(msg, 'list);
(%,i6) sdowncase( printf(false, "~{~2,'0x~^:~}", octets) );

The argument may be a binary input stream.

(%,i1) S: openr_binary (file_search ("md5.lisp"));
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(%o1) #<INPUT BUFFERED FILE-STREAM (UNSIGNED-BYTE 8)
/home/robert/maxima/maxima-code/share/stringproc/md5.lisp>
(%i2) md5sum (S);
(%o2) 31a512ed53daf5b99495c9d05559355f
(%i3) close (S);
(%o3) true

mgf1_sha1

mgf1_sha1 (seed, len)

mgf1_sha1 (seed, len, return-type)

Returns a pseudo random number of variable length. By default the returned value
is a number with a length of len octets.

The optional argument return-type allows mgf1_sha1 to alternatively return the corre-

csponding list of len octets. return-type may be number or list.

The computation of the returned value is described in RFC 3447, appendix B.2.1

MGF1. SHA1 is used as hash function, i.e. the randomness of the computed number
relies on the randomness of SHA1 hashes.

Example:

(%i1) ibase: obase: 16.$
(%i2) number: mgf1_sha1(4711., 8);
(%o2) 0e0252e5a2a42fea1
(%i3) octets: mgf1_sha1(4711., 8, 'list);
(%o3) [0E0,25,2E,5A,2A,42,0FE,0A1]

number_to_octets (number)

Returns an octet-representation of number as a list of octets. The number must be
a non-negative integer.

Example:

(%i1) ibase : obase : 16.$
(%i2) octets: [0ca,0fe,0ba,0be]$  
(%i3) number: octets_to_number(octets);
(%o3) 0cafebabef
(%i4) number_to_octets(number);
(%o4) [0CA, 0FE, 0BA, 0BE]

octets_to_number (octets)

Returns a number by concatenating the octets in the list of octets.

Example: See [number_to_octets], page 1209.

octets_to_oid (octets)

Computes an object identifier (OID) from the list of octets.

Example: RSA encryption OID

(%i1) ibase : obase : 16.$
(%i2) oid: octets_to_oid([2A,86,48,86,0F7,0D,1,1,1]);
(%o2) 1.2.840.113549.1.1.1
(%i3) oid_to_octets(oid);
(%o3) [2A, 86, 48, 86, 0F7, 0D, 1, 1, 1]
octets_to_string

octets_to_string (octets)
octets_to_string (octets, encoding)

Decodes the list of octets into a string according to current system defaults. When decoding octets corresponding to Non-US-ASCII characters the result depends on the platform, application and underlying Lisp.

Example: Using system defaults (Maxima compiled with GCL, which uses no format definition and simply passes through the UTF-8-octets encoded by the GNU/Linux terminal).

(%i1) octets: string_to_octets("abc");
(%o1) [61, 62, 63]
(%i2) octets_to_string(octets);
(%o2) abc
(%i3) ibase: obase: 16.$
(%i4) unicode(20AC);
(%o4) €
(%i5) octets: string_to_octets(%);
(%o5) [0E2, 82, 0AC]
(%i6) octets_to_string(octets);
(%o6) €
(%i7) utf8_to_unicode(octets);
(%o7) 20AC

In case the external format of the Lisp reader is equal to UTF-8 the optional argument encoding allows to set the encoding for the octet to string conversion. If necessary see [adjust_external_format], page 1196, for changing the external format.

Some names of supported encodings (see corresponding Lisp manual for more):
CCL, CLISP, SBCL: utf-8, ucs-2be, ucs-4be, iso-8859-1, cp1252, cp850
CMUCL: utf-8, utf-16-be, utf-32-be, iso8859-1, cp1252
ECL: utf-8, ucs-2be, ucs-4be, iso-8859-1, windows-cp1252, dos-cp850

Example (continued): Using the optional encoding argument (Maxima compiled with SBCL, GNU/Linux terminal).

(%i8) string_to_octets("€", "ucs-2be");
(%o8) [20, 0AC]

oid_to_octets (oid-string)  [Function]

Converts an object identifier (OID) to a list of octets.

Example: See [octets_to_oid], page 1209.

sha1sum

sha1sum (arg)
sha1sum (arg, return-type)

Returns the SHA1 fingerprint of a string, a non-negative integer or a list of octets. The default return value is a string containing 40 hex characters.

The optional argument return-type allows sha1sum to alternatively return the corresponding number or list of octets. return-type may be string, number or list.
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Example:

```lisp
(%i1) ibase: obase: 16.$
(%i2) msg: "foo bar baz"$
(%i3) string: sha1sum(msg);
(%o3) c7567e8b39e2428e38bf9c9226ac68de4c67dc39
(%i4) integer: sha1sum(msg, 'number);
(%o4) 0c7567e8b39e2428e38bf9c9226ac68de4c67dc39
(%i5) octets: sha1sum(msg, 'list);
(%o5) [0C7,56,7E,8B,39,0E2,42,8E,38,0BF,9C,92,26,0AC,68,0DE,4C,67,0DC,39]
(%i6) sdowncase( printf(false, "~{~2,'0x~^:~}", octets) );
```

Note that in case `arg` contains German umlauts or other non-ASCII characters (resp. octets larger than 127) the SHA1 fingerprint is platform dependent.

`sha256sum` [Function]

```lisp
sha256sum (arg)
sha256sum (arg, return-type)
```

Returns the SHA256 fingerprint of a string, a non-negative integer or a list of octets. The default return value is a string containing 64 hex characters.

The optional argument `return-type` allows `sha256sum` to alternatively return the corresponding number or list of octets (see `[sha1sum], page 1210`).

Example:

```lisp
(%i1) string: sha256sum("foo bar baz");
(%o1) dbd318c1c462ae872f41109a4df3048871a03ded0fe0e757ced57dad6f2d7
```

Note that in case `arg` contains German umlauts or other non-ASCII characters (resp. octets larger than 127) the SHA256 fingerprint is platform dependent.

`string_to_octets` [Function]

```lisp
string_to_octets (string)
string_to_octets (string, encoding)
```

Encodes a `string` into a list of octets according to current system defaults. When encoding strings containing Non-US-ASCII characters the result depends on the platform, application and underlying Lisp.

In case the external format of the Lisp reader is equal to UTF-8 the optional argument `encoding` allows to set the encoding for the string to octet conversion. If necessary see `[adjust_external_format], page 1196, for changing the external format.

See `[octets_to_string], page 1210, for examples and some more information.
92 to_poly_solve

92.1 Functions and Variables for to_poly_solve

The packages to_poly and to_poly_solve are experimental; the specifications of the functions in these packages might change or the some of the functions in these packages might be merged into other Maxima functions.

Barton Willis (Professor of Mathematics, University of Nebraska at Kearney) wrote the to_poly and to_poly_solve packages and the English language user documentation for these packages.

%and

The operator %and is a simplifying nonshort-circuited logical conjunction. Maxima simplifies an %and expression to either true, false, or a logically equivalent, but simplified, expression. The operator %and is associative, commutative, and idempotent. Thus when %and returns a noun form, the arguments of %and form a non-redundant sorted list; for example

```
(%i1) a %and (a %and b);  
   (%o1) a %and b
```

If one argument to a conjunction is the explicit the negation of another argument, %and returns false:

```
(%i2) a %and (not a);  
   (%o2) false
```

If any member of the conjunction is false, the conjunction simplifies to false even if other members are manifestly non-boolean; for example

```
(%i3) 42 %and false;  
   (%o3) false
```

Any argument of an %and expression that is an inequation (that is, an inequality or equation), is simplified using the Fourier elimination package. The Fourier elimination simplifier has a pre-processor that converts some, but not all, nonlinear inequations into linear inequations; for example the Fourier elimination code simplifies \( \text{abs}(x) + 1 > 0 \) to true, so

```
(%i4) (x < 1) %and (abs(x) + 1 > 0);  
   (%o4) x < 1
```

Notes

- The option variable prederror does not alter the simplification %and expressions.
- To avoid operator precedence errors, compound expressions involving the operators %and, %or, and not should be fully parenthesized.
- The Maxima operators and and or are both short-circuited. Thus and isn’t associative or commutative.

Limitations The conjunction %and simplifies inequations locally, not globally. This means that conjunctions such as

```
(%i5) (x < 1) %and (x > 1);
```
do not simplify to false. Also, the Fourier elimination code ignores the fact database;

Finally, nonlinear inequations that aren’t easily converted into an equivalent linear inequation aren’t simplified.

There is no support for distributing \texttt{%and} over \texttt{%or}; neither is there support for distributing a logical negation over \texttt{%and}.

To use \texttt{load("to_poly_solve")}

Related functions \texttt{%or}, \texttt{%if}, \texttt{and}, \texttt{or}, \texttt{not}

Status The operator \texttt{%and} is experimental; the specifications of this function might change and its functionality might be merged into other Maxima functions.

\texttt{%if (bool, a, b)}

The operator \texttt{%if} is a simplifying conditional. The conditional \texttt{bool} should be boolean-valued. When the conditional is true, return the second argument; when the conditional is false, return the third; in all other cases, return a noun form.

Maxima inequations (either an inequality or an equality) are not boolean-valued; for example, Maxima does not simplify \(5 < 6\) to true, and it does not simplify \(5 = 6\) to false; however, in the context of a conditional to an \texttt{%if} statement, Maxima automatically attempts to determine the truth value of an inequation. Examples:

\begin{verbatim}
(%i1) f : %if(x # 1, 2, 8);
(%o1) %if(x - 1 # 0, 2, 8)
(%i2) [subst(x = -1,f), subst(x=1,f)];
(%o2) [2, 8]
\end{verbatim}

If the conditional involves an inequation, Maxima simplifies it using the Fourier elimination package.

Notes

- If the conditional is manifestly non-boolean, Maxima returns a noun form:

\begin{verbatim}
(%i3) %if(42,1,2);
(%o3) %if(42, 1, 2)
\end{verbatim}

- The Maxima operator \texttt{if} is nary, the operator \texttt{%if} isn’t nary.

Limitations The Fourier elimination code only simplifies nonlinear inequations that are readily convertible to an equivalent linear inequation.

To use: \texttt{load("to_poly_solve")}

Status: The operator \texttt{%if} is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

\texttt{%or}

The operator \texttt{%or} is a simplifying nonshort-circuited logical disjunction. Maxima simplifies an \texttt{%or} expression to either true, false, or a logically equivalent, but simplified,
expression. The operator \%or is associative, commutative, and idempotent. Thus when \%or returns a noun form, the arguments of \%or form a non-redundant sorted list; for example

\begin{verbatim}
(%i1) a \%or (a \%or b);
(%o1) a \%or b
\end{verbatim}

If one member of the disjunction is the explicit the negation of another member, \%or returns true:

\begin{verbatim}
(%i2) a \%or (not a);
(%o2) true
\end{verbatim}

If any member of the disjunction is true, the disjunction simplifies to true even if other members of the disjunction are manifestly non-boolean; for example

\begin{verbatim}
(%i3) 42 \%or true;
(%o3) true
\end{verbatim}

Any argument of an \%or expression that is an inequation (that is, an inequality or equation), is simplified using the Fourier elimination package. The Fourier elimination code simplifies \(\text{abs}(x) + 1 > 0\) to true, so we have

\begin{verbatim}
(%i4) (x < 1) \%or (abs(x) + 1 > 0);
(%o4) true
\end{verbatim}

Notes

- The option variable \texttt{prederror} does \textit{not} alter the simplification of \%or expressions.
- You should parenthesize compound expressions involving the operators \%and, \%or, and not; the binding powers of these operators might not match your expectations.
- The Maxima operators \texttt{and} and \texttt{or} are both short-circuited. Thus \texttt{or} isn’t associative or commutative.

Limitations

The conjunction \%or simplifies inequations \textit{locally, not globally}. This means that conjunctions such as

\begin{verbatim}
(%i1) (x < 1) \%or (x >= 1);
(%o1) (x > 1) \%or (x >= 1)
\end{verbatim}

do \textit{not} simplify to true. Further, the Fourier elimination code ignores the fact database;

\begin{verbatim}
(%i2) assume(x > 5);
(%o2) [x > 5]
(%i3) (x > 1) \%and (x > 2);
(%o3) (x > 1) \%and (x > 2)
\end{verbatim}

Finally, nonlinear inequations that aren’t easily converted into an equivalent linear inequation aren’t simplified.

The algorithm that looks for terms that cannot both be false is weak; also there is no support for distributing \%or over \%and; neither is there support for distributing a logical negation over \%or.

To use \texttt{load(“to_poly_solve“)}
Related functions %or, %if, and, or, not

Status The operator %or is experimental; the specifications of this function might change and its functionality might be merged into other Maxima functions.

complex_number_p (x) [Function]
The predicate complex_number_p returns true if its argument is either \( a + \%i \cdot b \), \( a \cdot \%i \cdot b \), or \( \%i \), where \( a \) and \( b \) are either rational or floating point numbers (including big floating point); for all other inputs, complex_number_p returns false; for example

(\(\texttt{i1}\))  \texttt{map('complex_number_p,[2/3, 2 + 1.5 * \%i, \%i]);}
(\(\%\texttt{o1}\))  \[true, true, true\]
(\(\texttt{i2}\))  \texttt{complex_number_p((2+\%i)/(5-\%i));}
(\(\%\texttt{o2}\))  \texttt{false}
(\(\texttt{i3}\))  \texttt{complex_number_p(cos(5 - 2 * \%i));}
(\(\%\texttt{o3}\))  \texttt{false}

Related functions isreal_p

To use load("to_poly_solve")

Status The operator complex_number_p is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

compose_functions (l) [Function]
The function call compose_functions(l) returns a lambda form that is the composition of the functions in the list \( l \). The functions are applied from right to left; for example

(\(\texttt{i1}\))  \texttt{compose_functions([cos, exp]);}
(\(\%\texttt{o1}\))  \texttt{lambda([\%g151], cos(\%e ))}
(\(\%\texttt{i2}\))  \%\texttt{(x)};
(\(\%\texttt{o2}\))  \texttt{cos(\%e )}

When the function list is empty, return the identity function:

(\(\texttt{i3}\))  \texttt{compose_functions([]);}
(\(\%\texttt{o3}\))  \texttt{lambda([\%g152], \%g152)}
(\(\%\texttt{i4}\))  \%\texttt{(x)};
(\(\%\texttt{o4}\))  \texttt{x}

Notes
- When Maxima determines that a list member isn’t a symbol or a lambda form, funmake (not compose_functions) signals an error:

(\(\texttt{i5}\))  \texttt{compose_functions([a < b]);}

\texttt{funmake: first argument must be a symbol, subscripted symbol, string, or lambda expression; found: a < b}
\#0: compose_functions(l=[a < b])(to_poly_solve.mac line 40)
-- an error. To debug this try: debugmode(true);
- To avoid name conflicts, the independent variable is determined by the function new_variable.

(\(\texttt{i6}\))  \texttt{compose_functions([\%g0]);}
(\%o6) \text{lambda}(\text{g154}, \%g0(\text{g154}))
(\%i7) \text{compose\_functions}(\text{g0});
(\%o7) \text{lambda}(\text{g155}, \%g0(\text{g155}))

Although the independent variables are different, Maxima is able to deduce that these lambda forms are semantically equal:

(\%i8) \text{is}\{(\%o6, \%o7)\};
(\%o8) \true

\textbf{To use load}("to\_poly\_solve")

\textbf{Status} The function \texttt{compose\_functions} is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

dfloat (x)

The function \texttt{dfloat} is similar to \texttt{float}, but the function \texttt{dfloat} applies \texttt{rectform} when \texttt{float} fails to evaluate to an IEEE double floating point number; thus

(\%i1) \text{float}(4.5^\text{1} + \%i));
(\%o1) \text{4.5}
(\%i2) \text{dfloat}(4.5^\text{1} + \%i));
(\%o2) 4.48998802962884 \%i + .3000124893895671

\textbf{Notes}

- The rectangular form of an expression might be poorly suited for numerical evaluation—for example, the rectangular form might needlessly involve the difference of floating point numbers (subtractive cancellation).

- The identifier \texttt{float} is both an option variable (default value false) and a function name.

\textbf{Related functions} \texttt{float, bfloat}

\textbf{To use load}("to\_poly\_solve")

\textbf{Status} The function \texttt{dfloat} is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

elem (l, x)

The function \texttt{elem} eliminates the variables in the set or list \texttt{x} from the equations in the set or list \texttt{l}. Each member of \texttt{x} must be a symbol; the members of \texttt{l} can either be equations, or expressions that are assumed to equal zero.

The function \texttt{elem} returns a list of two lists; the first is the list of expressions with the variables eliminated; the second is the list of pivots; thus, the second list is a list of expressions that \texttt{elem} used to eliminate the variables.

Here is an example of eliminating between linear equations:

(\%i1) \text{elem(set(x + y + z = 1, x - y - z = 8, x - z = 1),}
\text{set(x,y))};
(\%o1) [[2 \text{z - 7}, \text{y + 7, z - x + 1}]]

Eliminating \texttt{x} and \texttt{y} yields the single equation \texttt{2 z - 7 = 0}; the equations \texttt{y + 7 = 0} and \texttt{z - z + 1 = 1} were used as pivots. Eliminating all three variables from these equations, triangularizes the linear system:

(\%i2) \text{elem(set(x + y + z = 1, x - y - z = 8, x - z = 1),}
\( \text{set}(x,y,z) \);

\( \%o2 \) \[ [], [2\ z - 7, y + 7, z - x + 1] \]

Of course, the equations needn’t be linear:

\( \%i3 \) \text{elim(set(x^2 - 2 * y^3 = 1, x - y = 5), [x,y]);}

\[ \text{3} \ \text{2} \]

\( \%o3 \) \[ [], [2\ y - y - 10\ y - 24, y - x + 5] \]

The user doesn’t control the order the variables are eliminated. Instead, the algorithm uses a heuristic to \textit{attempt} to choose the best pivot and the best elimination order.

**Notes**

- Unlike the related function \texttt{eliminate}, the function \texttt{elim} does \textit{not} invoke \texttt{solve} when the number of equations equals the number of variables.
- The function \texttt{elim} works by applying resultants; the option variable \texttt{resultant} determines which algorithm Maxima uses. Using \texttt{sqfr}, Maxima factors each resultant and suppresses multiple zeros.
- The \texttt{elim} will triangularize a nonlinear set of polynomial equations; the solution set of the triangularized set \textit{can} be larger than that solution set of the untriangularized set. Thus, the triangularized equations can have \textit{spurious} solutions.

**Related functions** \texttt{elim_allbut, eliminate_using, eliminate}

**Option variables** \texttt{resultant}

To use \texttt{load("to_poly")}

**Status** The function \texttt{elim} is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

\texttt{elim_allbut}(l, x) \hspace{1cm} [\text{Function}]

This function is similar to \texttt{elim}, except that it eliminates all the variables in the list of equations \( l \) \textit{except} for those variables that in in the list \( x \)

\( \%i1 \) \text{elim_allbut([x+y = 1, x - 5*y = 1],[])};

\( \%o1 \) \[ [], [y, y + x - 1] \]

\( \%i2 \) \text{elim_allbut([x+y = 1, x - 5*y = 1],[x])};

\( \%o2 \) \[ [x - 1], [y + x - 1] \]

To use \texttt{load("to_poly")}

**Option variables** \texttt{resultant}

**Related functions** \texttt{elim, eliminate_using, eliminate}

**Status** The function \texttt{elim_allbut} is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

\texttt{eliminate_using}(l, e, x) \hspace{1cm} [\text{Function}]

Using \( e \) as the pivot, eliminate the symbol \( x \) from the list or set of equations in \( l \).

The function \texttt{eliminate_using} returns a set.

\( \%i1 \) \texttt{eq : [x^2 - y^2 - z^3 , x*y - z^2 - 5, x - y + z];}

\( \text{3} \ \text{2} \ \text{2} \ \text{2} \)

\( \%o1 \) \[ [- z - y + x , - z + x y - 5, z - y + x] \]

\( \%i2 \) \texttt{eliminate_using(eq,first(eq),z);}
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\[ %o2 \{ y + (1 - 3 x) y + 3 x y - x - x, \]
\[ 4 3 3 \]
\[ 2 2 4 \]
\[ y - x y + 13 x y - 75 x y + x + 125 \} \]

\( \%i3 \) \text{eliminate\_using(eq, second(eq), z);} \]
\[ 2 2 4 3 3 2 2 4 \]
\[ y - 3 x y + x + 5, y - x y + 13 x y - 75 x y + x + 125 \] \]

\( \%i4 \) \text{eliminate\_using(eq, third(eq), z);} \]
\[ 2 2 3 2 \]
\[ 2 2 3 2 \]
\[ y - 3 x y + x + 5, y + (1 - 3 x) y + 3 x y - x - x \} \]

\textbf{Option variables} \texttt{resultant} \texttt{allbut}

\textbf{Related functions} \texttt{elim, eliminate, elim} \texttt{fourier\_elim}

\texttt{fourier\_elim ([eq1, eq2, \ldots], [var1, var, \ldots])}

Fourier elimination is the analog of Gauss elimination for linear inequations (equations or inequalities). The function call \texttt{fourier\_elim([eq1, eq2, \ldots], [var1, var2, \ldots])} does Fourier elimination on a list of linear inequations [eq1, eq2, \ldots] with respect to the variables [var1, var2, \ldots]; for example

\( \%i1 \) \text{fourier\_elim([y-x < 5, x - y < 7, 10 < y], [x,y]);} \]
\( \%o1 \) \text{[y - 5 < x, x < y + 7, 10 < y]} \]
\( \%i2 \) \text{fourier\_elim([y-x < 5, x - y < 7, 10 < y], [y,x]);} \]
\( \%o2 \) \text{[max(10, x - 7) < y, y < x + 5, 5 < x]} \]

Eliminating first with respect to x and second with respect to y yields lower and upper bounds for x that depend on y, and lower and upper bounds for y that are numbers. Eliminating in the other order gives x dependent lower and upper bounds for y, and numerical lower and upper bounds for x.

When necessary, \texttt{fourier\_elim} returns a \textit{disjunction} of lists of inequations:

\( \%i3 \) \text{fourier\_elim([x # 6], [x]);} \]
\( \%o3 \) \text{[x < 6] or [6 < x]} \]

When the solution set is empty, \texttt{fourier\_elim} returns \texttt{emptyset}, and when the solution set is all reals, \texttt{fourier\_elim} returns \texttt{universalset}; for example

\( \%i4 \) \text{fourier\_elim([x < 1, x > 1], [x]);} \]
\( \%o4 \) \texttt{emptyset} \]
\( \%i5 \) \text{fourier\_elim([minf < x, x < inf], [x]);} \]
\( \%o5 \) \texttt{universalset} \]

For nonlinear inequations, \texttt{fourier\_elim} returns a (somewhat) simplified list of inequations:

\( \%i6 \) \text{fourier\_elim([x^3 - 1 > 0], [x]);} \]
\( 2 \)
\( \%o6 \) \texttt{[1 < x, x + x + 1 > 0] or [x < 1, - (x + x + 1) > 0]} \]
\( \%i7 \) \text{fourier\_elim([cos(x) < 1/2], [x]);}
Instead of a list of inequations, the first argument to `fourier_elim` may be a logical disjunction or conjunction:

```
(%i8) fourier_elim((x + y < 5) and (x - y > 8), [x, y]);
```

```
3
[ y + 8 < x, x < 5 - y, y < - - ]
2
```

```
(%i9) fourier_elim(((x + y < 5) and x < 1) or (x - y > 8), [x, y]);
```

```
[ y + 8 < x ] or [ x < min(1, 5 - y) ]
```

The Fourier elimination code has a preprocessor that converts some nonlinear inequations that involve the absolute value, minimum, and maximum functions into linear in equations. Additionally, the preprocessor handles some expressions that are the product or quotient of linear terms:

```
(%i10) fourier_elim([max(x,y) > 6, x # 8, abs(y-1) > 12], [x, y]);
```

```
[ 6 < x, x < 8, y < - 11 ] or [ 8 < x, y < - 11 ]
```

```
(%i11) fourier_elim([[x+6]/(x-9) <= 6], [x]);
```

```
[ x = 12 ] or [ 12 < x ] or [ x < 9 ]
```

```
(%i12) fourier_elim([x^2 - 1 # 0], [x]);
```

```
[ - 1 < x, x < 1 ] or [ 1 < x ] or [ x < - 1 ]
```

The function `fourier_elim` supports the inequation operators `<`, `<=`, `>`, `>=`, `#`, and `=`.

To use `load("fourier_elim")`

**isreal_p (e)**

The predicate `isreal_p` returns true when Maxima is able to determine that `e` is real-valued on the entire real line; it returns false when Maxima is able to determine that `e` isn’t real-valued on some nonempty subset of the real line; and it returns a noun form for all other cases.

```
(%i1) map('isreal_p, [-1, 0, %i, %pi]);
```

```
[true, true, false, true]
```

Maxima variables are assumed to be real; thus

```
(%i2) isreal_p(x);
```

```
true
```

The function `isreal_p` examines the fact database:

```
(%i3) declare(z,complex)$
```

```
(%i4) isreal_p(z);
```

```
isreal_p(z)
```

**Limitations** Too often, `isreal_p` returns a noun form when it should be able to return false; a simple example: the logarithm function isn’t real-valued on the entire real line, so `isreal_p(log(x))` should return false; however

```
(%i5) isreal_p(log(x));
```
Chapter 92: to_poly_solve

(\%o5) \hspace{1em} \text{isreal}_p(\log(x))

To use load("to_poly_solve")

Related functions \texttt{complex\_number\_p}

Status The function \texttt{isreal\_p} is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

\texttt{new\_variable (type)} \hspace{1em} [\text{Function}]

Return a unique symbol of the form \%[z,n,r,c,g]k, where k is an integer. The allowed values for \texttt{type} are \texttt{integer, natural\_number, real, complex, and general}. (By natural number, we mean the nonnegative integers; thus zero is a natural number. Some, but not all, definitions of natural number \textit{exclude} zero.)

When \texttt{type} isn’t one of the allowed values, \texttt{type} defaults to \texttt{general}. For integers, natural numbers, and complex numbers, Maxima automatically appends this information to the fact database.

(\%i1) map('new\_variable, ['integer, 'natural\_number, 'real, 'complex, 'general]);
(\%o1) [%z144, %n145, %r146, %c147, %g148]
(\%i2) nicedummies(%);
(\%o2) [%z0, %n0, %r0, %c0, %g0]
(\%i3) featurep(%z0, 'integer);
(\%o3) true
(\%i4) featurep(%n0, 'integer);
(\%o4) true
(\%i5) is(%n0 >= 0);
(\%o5) true
(\%i6) featurep(%c0, 'complex);
(\%o6) true

Note Generally, the argument to \texttt{new\_variable} should be quoted. The quote will protect against errors similar to

(\%i7) integer : 12$
(\%i8) new\_variable(integer);
(\%o8) \%g149
(\%i9) new\_variable('integer);
(\%o9) \%z150

Related functions \texttt{nicedummies}

To use load("to_poly\_solve")

Status The function \texttt{new\_variable} is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

\texttt{nicedummies} \hspace{1em} [\text{Function}]

Starting with zero, the function \texttt{nicedummies} re-indexes the variables in an expression that were introduced by \texttt{new\_variable};

(\%i1) new\_variable('integer) + 52 * new\_variable('integer);
(\%o1) 52 \%z136 + \%z135
(%i2) new_variable('integer) - new_variable('integer);  
(%o2) %z137 - %z138  
(%i3) nicedummies(%);  
(%o3) %z0 - %z1

Related functions new_variable

To use load("to_poly_solve")

Status The function nicedummies is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

parg (x) [Function]
The function parg is a simplifying version of the complex argument function carg; thus

(%i1) map('parg,[1,1+%i, %i, -1 + %i, -1]);  
(%o1) [0, ---, ---, -----, %pi]  
   4 2 4

Generally, for a non-constant input, parg returns a noun form; thus

(%i2) parg(x + %i * sqrt(x));  
(%o2) parg(x + %i sqrt(x))

When sign can determine that the input is a positive or negative real number, parg will return a non-noun form for a non-constant input. Here are two examples:

(%i3) parg(abs(x));  
(%o3) 0  
(%i4) parg(-x^2-1);  
(%o4) %pi

Note The sign function mostly ignores the variables that are declared to be complex (declare(x,complex)); for variables that are declared to be complex, the parg can return incorrect values; for example

(%i1) declare(x, complex)$  
(%i2) parg(x^2 + 1);  
(%o2) 0

Related function carg, isreal_p

To use load("to_poly_solve")

Status The function parg is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

real_imagpart_to_conjugate (e) [Function]
The function real_imagpart_to_conjugate replaces all occurrences of realpart and imagpart to algebraically equivalent expressions involving the conjugate.

(%i1) declare(x, complex)$  
(%i2) real_imagpart_to_conjugate(realpart(x) + imagpart(x) = 3);  
   conjugate(x) + x   %i (x - conjugate(x))
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To use load("to_poly_solve")

Status The function real_imagpart_to_conjugate is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

rectform_log_if_constant (e) [Function]
The function rectform_log_if_constant converts all terms of the form log(c) to rectform(log(c)), where c is either a declared constant expression or explicitly declared constant

(%i1) rectform_log_if_constant(log(1-%i) - log(x - %i));

(%o1) - log(x - %i) + ----- - -----
    2  4

(%i2) declare(a,constant, b,constant)$

(%i3) rectform_log_if_constant(log(a + %i*b));

(%o3) ---------- + %i atan2(b, a)
       2

To use load("to_poly_solve")

Status The function rectform_log_if_constant is experimental; the specifications of this function might change and its functionality might be merged into other Maxima functions.

simp_inequality (e) [Function]
The function simp_inequality applies basic simplifications to inequations, returning either a boolean value (true or false) or the original inequation.

The simplification rules used by simp_inequality include some facts about the ranges of the absolute value, power, and exponential functions along with some elementary algebra facts.

For conjunctions or disjunctions of inequations, simp_inequality is applied to each individual inequation, but no effort is made to simplify the entire logical expression. Effectively, simp_inequality creates a new empty context, so database facts are not used to simplify inequations.

load("to_poly_solve") loads this function.

Examples:

(%i2) simp_inequality(1 # 0);
(%o2) true

(%i3) simp_inequality(1 < 0);
(%o3) false

(%i4) simp_inequality(a=a);
(%o4) true

(%i5) simp_inequality(a # a);
(%i5) simp_inequality(a + 1 # a);
(%o5) false
(%i6) simp_inequality(a < a+1);
(%o6) true
(%i7) simp_inequality(abs(x) >= 0);
(%o7) true
(%i8) simp_inequality(exp(x) > 0);
(%o8) true
(%i9) simp_inequality(x^2 >= 0);
(%o9) true
(%i10) simp_inequality(2^x # 0);
(%o10) true
(%i11) simp_inequality(2^(x+1) > 2^x);
(%o11) true
The fact database is not consulted. For example:
(%i13) assume(xx > 0)$
(%i14) simp_inequality(xx > 0);
(%o14) xx>0
And finally, for conjunctions or disjunctions of inequations, each inequation is simplified, but no effort is made to simplify the entire logical expression; for example:
(%i15) simp_inequality((1 > 0) and (x < 0) and (x > 0));
(%o15) x<0 and x>0

standardize_inverse_trig (e)  [Function]
This function applies the identities \(\cot(x) = \frac{\pi}{\tan(1/x)}\), \(\acsc(x) = \frac{\pi}{\sin(1/x)}\), and similarly for \(\asec, \coth, \acsch\) and \(\asech\) to an expression. See Abramowitz and Stegun, Eqs. 4.4.6 through 4.4.8 and 4.6.4 through 4.6.6.
To use load("to_poly_solve")

Status The function standardize_inverse_trig is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

subst_parallel (l, e)  [Function]
When \(l\) is a single equation or a list of equations, substitute the right hand side of each equation for the left hand side. The substitutions are made in parallel; for example
(%i11) load("to_poly_solve")$
(%i12) subst_parallel([x=y,y=x], [x,y]);
(%o12) [y, x]
Compare this to substitutions made serially:
(%i13) subst([x=y,y=x],[x,y]);
(%o13) [x, x]
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The function `subst_parallel` is similar to `sublis` except that `subst_parallel` allows for substitution of nonatoms; for example

```lisp
(%i4) subst_parallel([x^2 = a, y = b], x^2 * y);
(%o4) a b
(%i5) sublis([x^2 = a, y = b], x^2 * y);
sublis: left-hand side of equation must be a symbol; found: x
-- an error. To debug this try: debugmode(true);
```

The substitutions made by `subst_parallel` are literal, not semantic; thus `subst_parallel` does not recognize that \( x \cdot y \) is a subexpression of \( x^2 \cdot y \)

```lisp
(%i6) subst_parallel([x * y = a], x^2 * y);
(%o6) x y
```

The function `subst_parallel` completes all substitutions before simplifications. This allows for substitutions into conditional expressions where errors might occur if the simplifications were made earlier:

```lisp
(%i7) subst_parallel([x = 0], %if(x < 1, 5, log(x)));
(%o7) 5
(%i8) subst([x = 0], %if(x < 1, 5, log(x)));
log: encountered log(0).
-- an error. To debug this try: debugmode(true);
```

Related functions `subst`, `sublis`, `ratsubst`

To use load("to_poly_solve_extra.lisp")

Status The function `subst_parallel` is experimental; the specifications of this function might change and its functionality might be merged into other Maxima functions.

to_poly \( e, l \) [Function]
The function `to_poly` attempts to convert the equation \( e \) into a polynomial system along with inequality constraints; the solutions to the polynomial system that satisfy the constraints are solutions to the equation \( e \). Informally, `to_poly` attempts to polynomialize the equation \( e \); an example might clarify:

```lisp
(%i1) load("to_poly_solve")$
(%i2) to_poly(sqrt(x) = 3, [x]);
2
(%o2) [[%g130 - 3, x = %g130 ],
   \%pi
   [- ---- < parg(%g130), parg(%g130) <= ----], []]
   2
   2
```

The conditions \(-\%pi/2 < \text{parg}(\%g130), \text{parg}(\%g130) <= \%pi/2\) tell us that \%g130 is in the range of the square root function. When this is true, the solution set to \( \sqrt{x} = 3 \) is the same as the solution set to \( \%g130 - 3, x = \%g130^2 \).
To polynomialize trigonometric expressions, it is necessary to introduce a non algebraic substitution; these non algebraic substitutions are returned in the third list returned by \texttt{to_poly}; for example

\begin{verbatim}
(%i3) to_poly(cos(x),[x]);
  2 %i x
(%o3) [\%g131 + 1, [2 \%g131 # 0], [\%g131 = \%e ]]
\end{verbatim}

Constant terms aren’t polynomialized unless the number one is a member of the variable list; for example

\begin{verbatim}
(%i4) to_poly(x = sqrt(5),[x]);
(%o4) [x - sqrt(5)], [], []
(%i5) to_poly(x = sqrt(5),[1,x]);
  2
(%o5) [x - \%g132, 5 = \%g132 ],
        \%pi \%pi
    [- \---- < \text{parg}(\%g132), \text{parg}(\%g132) \leq \----], []
    2  2
\end{verbatim}

To generate a polynomial with $\text{sqrt}(5) + \text{sqrt}(7)$ as one of its roots, use the commands

\begin{verbatim}
(%i6) first(elim_allbut(first(to_poly(x = sqrt(5) + sqrt(7), [1,x])), [x]));
  4  2
(%o6) [x - 24 x + 4]
\end{verbatim}

Related functions \texttt{to_poly_solve}

To use \texttt{load("to_poly")}

\textbf{Status:} The function \texttt{to_poly} is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

\texttt{to_poly_solve (e, l, [options])} \quad \text{[Function]}

The function \texttt{to_poly_solve} tries to solve the equations $e$ for the variables $l$. The equation(s) $e$ can either be a single expression or a set or list of expressions; similarly, $l$ can either be a single symbol or a list of set of symbols. When a member of $e$ isn’t explicitly an equation, for example $x^2 - 1$, the solver assumes that the expression vanishes.

The basic strategy of \texttt{to_poly_solve} is to convert the input into a polynomial form and to call \texttt{algsys} on the polynomial system. Internally \texttt{to_poly_solve} defaults \texttt{algexact} to true. To change the default for \texttt{algexact}, append \texttt{'algexact=false} to the \texttt{to_poly_solve} argument list.

When \texttt{to_poly_solve} is able to determine the solution set, each member of the solution set is a list in a \%union object:

\begin{verbatim}
(%i1) load("to_poly_solve")$

(%i2) to_poly_solve(x*(x-1) = 0, x);
(%o2) \%union([x = 0], [x = 1])
\end{verbatim}

When \texttt{to_poly_solve} is \textit{unable} to determine the solution set, a \%solve nounform is returned (in this case, a warning is printed)

\begin{verbatim}
(%i3) to_poly_solve(x^k + 2* x + 1 = 0, x);
\end{verbatim}
Nonalgebraic argument given to 'to_poly'
unable to solve

\( k \)

\(%o3\) \%solve([x + 2 x + 1 = 0], [x])

Substitution into a \%solve nounform can sometimes result in the solution

\(%i4\) \%union([x = - 1])

Especially for trigonometric equations, the solver sometimes needs to introduce an arbitrary integer. These arbitrary integers have the form \%zXXX, where XXX is an integer; for example

\(%i5\) to_poly_solve(sin(x) = 0, x);
\(%o5\) \%union([x = 2 \%pi \%z33 + \%pi], [x = 2 \%pi \%z35])

To re-index these variables to zero, use \texttt{nicedummies}:

\(%i6\) nicedummies(%);
\(%o6\) \%union([x = 2 \%pi \%z0 + \%pi], [x = 2 \%pi \%z1])

Occasionally, the solver introduces an arbitrary complex number of the form \%cXXX or an arbitrary real number of the form \%rXXX. The function \texttt{nicedummies} will re-index these identifiers to zero.

The solution set sometimes involves simplifying versions of various of logical operators including \%and, \%or, or \%if for conjunction, disjunction, and implication, respectively; for example

\(%i7\) sol : to_poly_solve(abs(x) = a, x);
\(%o7\) \%union(%if(isnonnegative_p(a), [x = - a], \%union()),
\%if(isnonnegative_p(a), [x = a], \%union()))

\(%i8\) subst(a = 42, sol);
\(%o8\) \%union([x = - 42], [x = 42])

\(%i9\) subst(a = -42, sol);
\(%o9\) \%union()

The empty set is represented by \%union().

The function \texttt{to_poly_solve} is able to solve some, but not all, equations involving rational powers, some nonrational powers, absolute values, trigonometric functions, and minimum and maximum. Also, some it can solve some equations that are solvable in terms of the Lambert W function; some examples:

\(%i10\) load("to_poly_solve")$

\(%i11\) to_poly_solve(set(max(x,y) = 5, x+y = 2), set(x,y));
\(%o11\) \%union([x = - 5, y = 5], [x = 5, y = - 5])

\(%i12\) to_poly_solve(set(sqrt(x) + sqrt(y) = 5, x + y = 10), set(x,y));
\(%o12\) \%union([x = - 5, y = 5], [x = 5, y = - 5])
(%o4) %union([x = - ------------, y = ------------],
         2            2
         3/2      3/2
         [x = ---------, y = - ------------])
         2

(%i5) to_poly_solve(cos(x) * sin(x) = 1/2, x,
    'simpfuncts = ['expand, 'nicedummies]);

(%i6) to_poly_solve(x^(2*a) + x^a + 1, x);

(%i7) to_poly_solve(x * exp(x) = a, x);

For linear inequalities, to_poly_solve automatically does Fourier elimination:

(%i8) to_poly_solve([x + y < 1, x - y >= 8], [x, y]);

Each optional argument to to_poly_solve must be an equation; generally, the order
of these options does not matter.

* simpfuncts = l, where l is a list of functions. Apply the composition of the
  members of l to each solution.
Sometimes additional simplification can revert a simplification; for example

\[(%i3) \text{to_poly_solve}(x^2=1,x);\]
\%(o3) \text{union}([x = -1], [x = 1])
\%(i4) \text{to_poly_solve}(x^2= 1,x, 'simpfuncs = [polarform]);
\%(i5) \text{union}([x = 1], [x = %e])

Maxima doesn't try to check that each member of the function list \(l\) is purely a simplification; thus

\%(i5) \text{to_poly_solve}(x^2 = %i,x, 'simpfuncs = [lambda([s],s^2)]);
\%(o5) \text{union}([x = %i])

To convert each solution to a double float, use \text{simpfunc = ['dfloat]}:

\%(i6) \text{to_poly_solve}(x^3 +x + 1 = 0,x,
'simpfuncs = ['dfloat]), algexact : true;
\%(o6) \text{union}([x = -0.6823278038280178],
[x = 0.3411639019140089 - 1.16154139997251 %i],
[x = 1.16154139997251 %i + 0.3411639019140089])

• \text{use_grobner = true} With this option, the function \text{poly_reduced_grobner} is
applied to the equations before attempting their solution. Primarily, this option
provides a workaround for weakness in the function \text{algsys}. Here is an example
of such a workaround:

\%(i7) \text{to_poly_solve}([x^2+y^2=2^2,(x-1)^2+(y-1)^2=2^2],[x,y],
'use_grobner = true);
\%(o7) \text{union}([x = -\frac{\sqrt{7}}{2}, y = -\frac{\sqrt{7}}{2}])
\%(i8) \text{to_poly_solve}([x^2+y^2=2^2,(x-1)^2+(y-1)^2=2^2],[x,y]);
\%(o8) \text{union}()

• \text{maxdepth = k}, where \(k\) is a positive integer. This function controls the maximum
recursion depth for the solver. The default value for \text{maxdepth} is five. When the
recursions depth is exceeded, the solver signals an error:

\%(i9) \text{to_poly_solve}(\cos(x) = x,x, 'maxdepth = 2);
\(\text{Unable to solve}\)
\%(o9) \text{solve}([\cos(x) = x], [x], \text{maxdepth = 2})

• \text{parameters = 1}, where \(1\) is a list of symbols. The solver attempts to return a
solution that is valid for all members of the list \(l\); for example:

\%(i10) \text{to_poly_solve}(a * x = x, x);
\%(o10) \text{union}([x = 0])
\%(i11) \text{to_poly_solve}(a * x = x, x, 'parameters = [a]);
In (%o2), the solver introduced a dummy variable; to re-index the these dummy variables, use the function `nicedummies`:

(%i12) nicedummies(%);
(%o12) %union(%if(a - 1 = 0, [x = %c0], %union()),
      %if(a - 1 # 0, [x = 0], %union()))

The `to_poly_solve` uses data stored in the hashed array `one_to_one_reduce` to solve equations of the form \( f(a) = f(b) \). The assignment `one_to_one_reduce['f,'f] : lambda([a,b], a=b)` tells `to_poly_solve` that the solution set of \( f(a) = f(b) \) equals the solution set of \( a = b \); for example

(%i13) one_to_one_reduce['f,'f] : lambda([a,b], a=b)$

(%i14) to_poly_solve(f(x^2-1) = f(0),x);
(%o14) %union([x = - 1], [x = 1])

More generally, the assignment `one_to_one_reduce['f,'g] : lambda([a,b], w(a, b) = 0)` tells `to_poly_solve` that the solution set of \( f(a) = f(b) \) equals the solution set of \( w(a, b) = 0 \); for example

(%i15) one_to_one_reduce['f,'g] : lambda([a,b], a = 1 + b/2)$

(%i16) to_poly_solve(f(x) - g(x),x);
(%o16) %union([x = 2])

Additionally, the function `to_poly_solve` uses data stored in the hashed array `function_inverse` to solve equations of the form \( f(a) = b \). The assignment `function_inverse['f] : lambda([s], g(s))` informs `to_poly_solve` that the solution set to \( f(x) = b \) equals the solution set to \( x = g(b) \); two examples:

(%i17) function_inverse['Q] : lambda([s], P(s))$

(%i18) to_poly_solve(Q(x-1) = 2009,x);
(%o18) %union([x = P(2009) + 1])

(%i19) function_inverse['G] : lambda([s], s+new_variable(integer));
(%o19) lambda([s], s + new_variable(integer))

(%i20) to_poly_solve(G(x - a) = b,x);
(%o20) %union([x = b + a + %z125])

Notes

- The solve variables needn’t be symbols; when `fullratsubst` is able to appropriately make substitutions, the solve variables can be nonsymbols:

(%i11) to_poly_solve([x^2 + y^2 + x * y = 5, x * y = 8],
[x^2 + y^2, x * y]);
(%o11) %union([x y = 8, y + x = - 3])

- For equations that involve complex conjugates, the solver automatically appends the conjugate equations; for example
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(%i1) declare(x, complex)$

(%i2) to_poly_solve(x + (5 + %i) * conjugate(x) = 1, x);
       %i + 21
(%o2) %union([x = - -------------])
               25 %i - 125

(%i3) declare(y, complex)$

(%i4) to_poly_solve(set(conjugate(x) - y = 42 + %i, x + conjugate(y) = 0), set(x, y));
       %i - 42 %i + 42
(%o4) %union([x = - ---------, y = - --------])
             2      2

• For an equation that involves the absolute value function, the to_poly_solve
  consults the fact database to decide if the argument to the absolute value is
  complex valued. When

  (%i1) to_poly_solve(abs(x) = 6, x);
       %i
(%o1) %union([x = - 6], [x = 6])

(%i2) declare(z, complex)$

(%i3) to_poly_solve(abs(z) = 6, z);
       %i
(%o3) %union(%if((%c11 # 0) %and (%c11 conjugate(%c11) - 36 = 0), [z = %c11], %union( )))

This is the only situation that the solver consults the fact database. If a solve
variable is declared to be an integer; for example, to_poly_solve ignores this
declaration.

Relevant option variables algexact, resultant, algebraic

Related functions to_poly

To use load(“to_poly_solve”)    

Status: The function to_poly_solve is experimental; its specifications might change
and its functionality might be merged into other Maxima functions.

%union (soln_1, soln_2, soln_3, ...)
%union ()

%union(soln_1, soln_2, soln_3, ...) represents the union of its arguments, each
of which represents a solution set, as determined by to_poly_solve. %union() rep-
resents the empty set.

In many cases, a solution is a list of equations [x = ..., y = ..., z = ...] where x,
y, and z are one or more unknowns. In such cases, to_poly_solve returns a %union
expression containing one or more such lists.

The solution set sometimes involves simplifying versions of various of logical operators
including %and, %or, or %if for conjunction, disjunction, and implication, respectively.

Examples:
%union(...) represents the union of its arguments, each of which represents a solution
set, as determined by to_poly_solve. In many cases, a solution is a list of equations.
(%i1) load ("to_poly_solve")$
(%i2) to_poly_solve ([sqrt(x^2 - y^2), x + y], [x, y]);
(%o2) %union([x = 0, y = 0], [x = %c13, y = - %c13])

%union() represents the empty set.

(%i1) load ("to_poly_solve")$
(%i2) to_poly_solve (abs(x) = -1, x);
(%o2) %union()

The solution set sometimes involves simplifying versions of various of logical operators.

(%i1) load ("to_poly_solve")$
(%i2) sol : to_poly_solve (abs(x) = a, x);
(%o2) %union(%if(isnonnegative_p(a), [x = - a], %union()),
       %if(isnonnegative_p(a), [x = a], %union())))
(%i3) subst (a = 42, sol);
(%o3) %union([x = - 42], [x = 42])
(%i4) subst (a = -42, sol);
(%o4) %union()
93 unit

93.1 Introduction to Units

The *unit* package enables the user to convert between arbitrary units and work with dimensions in equations. The functioning of this package is radically different from the original Maxima units package - whereas the original was a basic list of definitions, this package uses rulesets to allow the user to chose, on a per dimension basis, what unit final answers should be rendered in. It will separate units instead of internmixing them in the display, allowing the user to readily identify the units associated with a particular answer. It will allow a user to simplify an expression to its fundamental Base Units, as well as providing fine control over simplifying to derived units. Dimensional analysis is possible, and a variety of tools are available to manage conversion and simplification options. In addition to customizable automatic conversion, *units* also provides a traditional manual conversion option.

Note - when unit conversions are inexact Maxima will make approximations resulting in fractions. This is a consequence of the techniques used to simplify units. The messages warning of this type of substitution are disabled by default in the case of units (normally they are on) since this situation occurs frequently and the warnings clutter the output. (The existing state of ratprint is restored after unit conversions, so user changes to that setting will be preserved otherwise.) If the user needs this information for units, they can set *unitverbose:on* to reactivate the printing of warnings from the unit conversion process.

*unit* is included in Maxima in the share/contrib/unit directory. It obeys normal Maxima package loading conventions:

```
(%i1) load("unit")$
```

```
* Units version 0.50 *
* Definitions based on the NIST Reference on *
* Constants, Units, and Uncertainty *
* Conversion factors from various sources including *
* NIST and the GNU units package *

Redefining necessary functions...
WARNING: DEFUN/DEFMACRO: redefining function TOLEVEL-MACSYMA-EVAL ...
WARNING: DEFUN/DEFMACRO: redefining function MSETCHK ...
WARNING: DEFUN/DEFMACRO: redefining function KILL1 ...
WARNING: DEFUN/DEFMACRO: redefining function NFORMAT ...
Initializing unit arrays...
Done.
```

The WARNING messages are expected and not a cause for concern - they indicate the *unit* package is redefining functions already defined in Maxima proper. This is necessary in order to properly handle units. The user should be aware that if other changes have been made to these functions by other packages those changes will be overwritten by this loading process.

The *unit.mac* file also loads a lisp file *unit-functions.lisp* which contains the lisp functions needed for the package.
Clifford Yapp is the primary author. He has received valuable assistance from Barton Willis of the University of Nebraska at Kearney (UNK), Robert Dodier, and other intrepid folk of the Maxima mailing list.

There are probably lots of bugs. Let me know. \texttt{float} and \texttt{numer} don’t do what is expected.

TODO : dimension functionality, handling of temperature, showabbr and friends. Show examples with addition of quantities containing units.

\section*{93.2 Functions and Variables for Units}

\texttt{setunits (list)} \hfill \textbf{[Function]}

By default, the \texttt{unit} package does not use any derived dimensions, but will convert all units to the seven fundamental dimensions using MKS units.

\begin{verbatim}
(%i2) N;
   kg m
   ----
   2
   s

(%i3) dyn;
   1 kg m
   (------) (----)
   100000  2
   s

(%i4) g;
   1
   (----) (kg)
   1000

(%i5) centigram*inch/minutes^2;
   127 kg m
   (-------------) (----)
   1800000000000  2
   s

(%i6) setunits([centigram,in\chinch,minute]);
   done

(%i7) N;
   1800000000000 %in\ chg
   (------------) (----)
   127  2
   %min

(%i8) dyn;
   18000000 %in\ chg
   (------------) (----)
   127  2
   %min
\end{verbatim}
The setting of units is quite flexible. For example, if we want to get back to kilograms, meters, and seconds as defaults for those dimensions we can do:

\[
\text{(i11) setunits([kg, m, s]);}
\]
\[
\text{(o11) done}
\]
\[
\text{(i12) centigram\cdot\text{inch}/\text{minutes}^2;}
\]
\[
\text{(o12) \frac{127}{18000000000000} \frac{\text{kg\cdot m}}{\text{s}}}
\]

Derived units are also handled by this command:

\[
\text{(i17) setunits(N);}
\]
\[
\text{(o17) done}
\]
\[
\text{(i18) N;}
\]
\[
\text{(o18) N}
\]
\[
\text{(i19) dyn;}
\]
\[
\text{(o19) \frac{1}{100000} \frac{\text{N}}{\text{s}}}
\]
\[
\text{(i20) kg\cdot m/s^2;}
\]
\[
\text{(o20) N}
\]
\[
\text{(i21) centigram\cdot\text{inch}/\text{minutes}^2;}
\]
\[
\text{(o21) \frac{127}{18000000000000} \frac{\text{N}}{\text{s}}}
\]

Notice that the unit package recognized the non MKS combination of mass, length, and inverse time squared as a force, and converted it to Newtons. This is how Maxima works in general. If, for example, we prefer dyne to Newtons, we simply do the following:

\[
\text{(i22) setunits(dyn);}
\]
\[
\text{(o22) done}
\]
\[
\text{(i23) kg\cdot m/s^2;}
\]
\[
\text{(o23) \frac{100000}{100000} \text{(dyn)}}
\]
\[
\text{(i24) centigram\cdot\text{inch}/\text{minutes}^2;}
\]
\[
\text{(o24) \frac{127}{18000000000000} \text{(dyn)}}
\]

To discontinue simplifying to any force, we use the uforget command:

\[
\text{(i26) uforget(dyn);}
\]
\[
\text{(o26) false}
\]
This would have worked equally well with `uforget(N)` or `uforget(%force)`. See also `uforget`. To use this function write first `load("unit")`.

**uforget (list)**

By default, the `unit` package converts all units to the seven fundamental dimensions using MKS units. This behavior can be changed with the `setunits` command. After that, the user can restore the default behavior for a particular dimension by means of the `uforget` command:

```context
(\texttt{(\%i13)} setunits([centigram,inch,minute]));
(\texttt{(\%o13)} done)
(\texttt{(\%i14)} centigram*inch/minutes^2;)
(\texttt{(\%o14)} \frac{\text{cg}}{\min})
(\texttt{(\%i15)} uforget([cg,\text{\%in,\%min]}));
(\texttt{(\%o15)} [false, false, false])
(\texttt{(\%i16)} centigram*inch/minutes^2;)
(\texttt{(\%o16)} \frac{\text{cg}}{\min})
```

`uforget` operates on dimensions, not units, so any unit of a particular dimension will work. The dimension itself is also a legal argument.

See also `setunits`. To use this function write first `load("unit")`.

**convert (expr, list)**

When resetting the global environment is overkill, there is the `convert` command, which allows one time conversions. It can accept either a single argument or a list of units to use in conversion. When a convert operation is done, the normal global evaluation system is bypassed, in order to avoid the desired result being converted again. As a consequence, for inexact calculations "rat" warnings will be visible if the global environment controlling this behavior (`ratprint`) is true. This is also useful for spot-checking the accuracy of a global conversion. Another feature is `convert` will allow a user to do Base Dimension conversions even if the global environment is set to simplify to a Derived Dimension.
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(%i2) kg*m/s^2;

(%o2) \frac{kg \cdot m}{2 \cdot s}

(%i3) convert(kg*m/s^2, [g, km, s]);

(%o3) \frac{g \cdot km}{2 \cdot s}

(%i4) convert(kg*m/s^2, [g, inch, minute]);

'rat' replaced 39.37007874015748 by 5000/127 = 39.37007874015748

(%o4) \frac{\frac{18000000000}{127}}{2 \cdot \frac{s}{\text{min}}}

(%i5) convert(kg*m/s^2, [N]);

(%o5) \text{N}

(%i6) convert(kg*m^2/s^2, [N]);

(%o6) \text{m N}

(%i7) setunits([N, J]);

(%o7) \text{done}

(%i8) convert(kg*m^2/s^2, [N]);

(%o8) \text{m N}

(%i9) convert(kg*m^2/s^2, [N, inch]);

'rat' replaced 39.37007874015748 by 5000/127 = 39.37007874015748

(%o9) \frac{\frac{5000}{127}}{2 \cdot \frac{\text{in}}{\text{N}}}

(%i10) convert(kg*m^2/s^2, [J]);

(%o10) \text{J}

(%i11) kg*m^2/s^2;

(%o11) \text{J}

(%i12) setunits([g, inch, s]);

(%o12) \text{done}

(%i13) kg*m/s^2;

(%o13) \text{N}

(%i14) uforget(N);

(%o14) \text{false}

(%i15) kg*m/s^2;

(%o15) \frac{5000000}{127} \cdot \frac{\text{g}}{\text{in}}
(%i16) convert(kg*m/s^2, [g, inch, s]);

'rat' replaced 39.37007874015748 by 5000/127 = 39.37007874015748

(%o16) (-------) (-----)
      127   2

s

See also setunits and uforget. To use this function write first load("unit").

usersetunits

[Optional variable]

Default value: none

If a user wishes to have a default unit behavior other than that described, they can make use of maxima-init.mac and the usersetunits variable. The unit package will check on startup to see if this variable has been assigned a list. If it has, it will use setunits on that list and take the units from that list to be defaults. uforget will revert to the behavior defined by usersetunits over its own defaults. For example, if we have a maxima-init.mac file containing:

usersetunits : [N,J];

we would see the following behavior:

(%i1) load("unit")$

*******************************************************************
* Units version 0.50 *
* Definitions based on the NIST Reference on *
* Constants, Units, and Uncertainty *
* Conversion factors from various sources including *
* NIST and the GNU units package *
*******************************************************************

Redefining necessary functions...
WARNING: DEFUN/DEFMACRO: redefining function
    TOPLEVEL-MACSYMA-EVAL ...
WARNING: DEFUN/DEFMACRO: redefining function MSETCHK ...
WARNING: DEFUN/DEFMACRO: redefining function KILL1 ...
WARNING: DEFUN/DEFMACRO: redefining function NFORMAT ...
Initializing unit arrays...
Done.
User defaults found...
User defaults initialized.

(%i12) kg*m/s^2;
(%o12) N
(%i13) kg*m^2/s^2;
(%o13) J
(%i14) kg*m^-3/s^2;
(%o14) J m
(%i15) kg*m*km/s^2;
(%o15) (1000) (J)
Without \textit{usersetunits}, the initial inputs would have been converted to MKS, and \textit{uforget} would have resulted in a return to MKS rules. Instead, the user preferences are respected in both cases. Notice these can still be overridden if desired. To completely eliminate this simplification - i.e. to have the user defaults reset to factory defaults - the \texttt{dontusedimension} command can be used. \textit{uforget} can restore user settings again, but only if \texttt{usedimension} frees it for use. Alternately, \texttt{kill(usersetunits)} will completely remove all knowledge of the user defaults from the session. Here are some examples of how these various options work.

\begin{verbatim}
(%i6) setunits([dyn,eV]);
               done
(%i7) kg*m/s^2;
       (100000) (dyn)
(%i8) kg*m^-2/s^-2;
        (6241509596477042688) (eV)
(%i9) kg*m^-3/s^-2;
          (6241509596477042688) (eV m)
(%i10) kg*m*km/s^-2;
            (6241509596477042688000) (eV)
(%i11) uforget([dyn,eV]);
        [false, false]
(%i12) kg*m/s^2;
         N
(%i13) kg*m^-2/s^-2;
        J
(%i14) kg*m^-3/s^-2;
       J m
(%i15) kg*m*km/s^-2;
      (1000) (J)
\end{verbatim}
dontusedimension(J);
(%o11)  [%energy, %force]
kg*m/s^2;
(%o12)  kg m
-----
2
s

kg*m^2/s^2;
(%o13)  kg m
-----
2
s

setunits([dyn,eV]);
(%o14)  done
kg*m/s^2;
(%o15)  kg m
-----
2
s

kg*m^2/s^2;
(%o16)  kg m
-----
2
s

uforget([dyn,eV]);
(%o17)  [false, false]
kg*m/s^2;
(%o18)  kg m
-----
2
s

kg*m^2/s^2;
(%o19)  kg m
-----
2
s

usedimension(N);
Done. To have Maxima simplify to this dimension, use setunits([unit]) to select a unit.
(%o20)  true
usedimension(J);
(%o21)  true
Unfortunately this wide variety of options is a little confusing at first, but once the user grows used to them they should find they have very full control over their working environment.

**metricexpandall (x)**

Rebuilds global unit lists automatically creating all desired metric units.  

- **x** is a numerical argument which is used to specify how many metric prefixes the user wishes defined. The arguments are as follows, with each higher number defining all lower numbers' units:
0 - none. Only base units
1 - kilo, centi, milli
(default) 2 - giga, mega, kilo, hecto, deka, deci, centi, milli,
      micro, nano
3 - peta, tera, giga, mega, kilo, hecto, deka, deci,
      centi, milli, micro, nano, pico, femto
4 - all

Normally, Maxima will not define the full expansion since this results in a very large
number of units, but metricexpandall can be used to rebuild the list in a more or
less complete fashion. The relevant variable in the unit.mac file is %unitexpand.

%unitexpand [Variable]
Default value: 2
This is the value supplied to metricexpandall during the initial loading of unit.
94 wrstcse

94.1 Introduction to wrstcse

wrstcse is a naive go at interval arithmetics is powerful enough to perform worst case calculations that appear in engineering by applying all combinations of tolerances to all parameters.

This approach isn’t guaranteed to find the exact combination of parameters that results in the worst-case. But it avoids the problems that make a true interval arithmetics affected by the halting problem as an equation can have an infinite number of local minima and maxima and it might be impossible to algorithmically determine which one is the global one.

Tolerances are applied to parameters by providing the parameter with a tol[n] that wrstcse will vary between -1 and 1. Using the same n for two parameters will make both parameters tolerate in the same way.

load ("wrstcse") loads this package.

94.2 Functions and Variables for wrstcse

\textbf{wc\_typicalvalues (expression, [num])} \hspace{1cm} \textbf{[Function]}

Returns what happens if all tolerances (that are represented by tol [n] that can vary from 0 to 1) happen to be 0.

Example:

\texttt{(\%i1) load("wrstcse")$}
\texttt{(\%i2) vals: [}
\texttt{ \hspace{1cm} R_1= 1000.0*(1+tol[1]*.01),}
\texttt{ \hspace{1cm} R_2= 2000.0*(1+tol[2]*.01) }
\texttt{ ];}
\texttt{(\%o2) [R_1 = 1000.0 (0.01 tol + 1),}
\texttt{ \hspace{1cm} 1}
\texttt{ \hspace{1cm} R_2 = 2000.0 (0.01 tol + 1)]}
\texttt{(\%i3) divider:U_Out=U_In*R_1/(R_1+R_2);}
\texttt{ \hspace{1cm} R_1 U_In}
\texttt{(\%o3) \hspace{1cm} U_Out = \hspace{1cm} ---------}
\texttt{ \hspace{1cm} R_2 + R_1}
\texttt{(\%i4) wc\_typicalvalues(vals);}
\texttt{(\%o4) \hspace{1cm} [R_1 = 1000.0, R_2 = 2000.0]}
\texttt{(\%i5) wc\_typicalvalues(subst(vals,divider));}
\texttt{(\%o5) \hspace{1cm} U_Out = 0.3333333333333333 U_In}

\textbf{wc\_inputvalueranges (expression, [num])} \hspace{1cm} \textbf{[Function]}

Convenience function: Displays a list which parameter can vary between which values.

Example:

\texttt{(\%i1) load("wrstcse")$}
(%i2) vals: [
   R_1= 1000.0*(1+tol[1]*.01),
   R_2= 2000.0*(1+tol[2]*.01)
];
(%o2) [R_1 = 1000.0 (0.01 tol + 1),
      R_2 = 2000.0 (0.01 tol + 1)]

(%i3) wc_inputvalueranges(vals);
   [ R_1 min = 990.0 typ = 1000.0 max = 1010.0 ]
(%o3) [ ]
   [ R_2 min = 1980.0 typ = 2000.0 max = 2020.0 ]

Function wc_systematic (expression, [num])
Systematically introduces num values per parameter into expression and returns a list of the result. If no num is given, num defaults to 3.

See also wc_montecarlo.

Example:

(%i1) load("wrstcse")$
(%i2) vals: [
   R_1= 1000.0*(1+tol[1]*.01),
   R_2= 2000.0*(1+tol[2]*.01)
];
(%o2) [R_1 = 1000.0 (0.01 tol + 1),
      R_2 = 2000.0 (0.01 tol + 1)]

(%i3) divider: U_Out=U_In*(R_1)/(R_1+R_2);
   R_1 U_In
(%o3) U_Out = --------
      R_2 + R_1

(%i4) wc_systematic(subst(vals,rhs(divider)));
(%o4) [0.3333333333333334 U_In, 0.3311036789297659 U_In,
      0.3289036544850498 U_In, 0.3355704697986577 U_In,
      0.3333333333333333 U_In, 0.3311258278145696 U_In,
      0.3377926421404682 U_In, 0.3355481727574751 U_In,
      0.3333333333333333 U_In]

Function wc_montecarlo (expression, num)
Introduces num random values per parameter into expression and returns a list of the result.

See also wc_systematic.

Example:

(%i1) load("wrstcse")$
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(%i2) vals: [
    R_1= 1000.0*(1+tol[1]*.01),
    R_2= 2000.0*(1+tol[2]*.01)
];
(%o2) [R_1 = 1000.0 (0.01 tol + 1),
    1
    R_2 = 2000.0 (0.01 tol + 1)]

(%i3) divider: U_Out=U_In*(R_1)/(R_1+R_2);
(%o3) U_Out = ---------
    R_2 + R_1

(%i4) wc_montecarlo(subst(vals,rhs(divider)),10);
(%o4) [0.3365488313167528 U_In, 0.3339089445851889 U_In,
    0.31465142884122 U_In, 0.3447359711624277 U_In,
    0.3294005710066001 U_In, 0.3330897542463686 U_In,
    0.3397591863729343 U_In, 0.3227030530673181 U_In,
    0.3385512773502185 U_In, 0.314764470912582 U_In]

wc_mintypmax (expr, [n])
[Function]
Prints the minimum, maximum and typical value of expr. If n is positive, n values
for each parameter will be tried systematically. If n is negative, -n random values
are used instead. If no n is given, 3 is assumed.

Example:
(%i1) load("wrstcse")$
(%i2) ratprint:false$
(%i3) vals: [
    R_1= 1000.0*(1+tol[1]*.01),
    R_2= 1000.0*(1+tol[2]*.01)
];
(%o3) [R_1 = 1000.0 (0.01 tol + 1),
    1
    R_2 = 1000.0 (0.01 tol + 1)]

(%i4) assume(U_In>0);
(%o4) [U_In > 0]
(%i5) divider:U_Out=U_In*R_1/(R_1+R_2);
(%o5) U_Out = ---------
    R_2 + R_1

(%i6) lhs(divider)=wc_mintypmax(subst(vals,rhs(divider)));
(%o6) U_Out = [min = 0.495 U_In, typ = 0.5 U_In,
    max = 0.505 U_In]

wc_tolappend (list)
[Function]
Appends two list of parameters with tolerances renumbering the tolerances of both
lists so they don’t coincide.
Example:

(%i1) load("wrstcse")$
(%i2) val_a: [
    R_1= 1000.0*(1+tol[1]*.01),
    R_2= 1000.0*(1+tol[2]*.01)
];
(%o2) [R_1 = 1000.0 (0.01 tol + 1), 1
    R_2 = 1000.0 (0.01 tol + 1)]

(%i3) val_b: [
    R_3= 1000.0*(1+tol[1]*.01),
    R_4= 1000.0*(1+tol[2]*.01)
];
(%o3) [R_3 = 1000.0 (0.01 tol + 1), 1
    R_4 = 1000.0 (0.01 tol + 1)]

(%i4) wc_tolappend(val_a,val_b);
(%o4) [R_1 = 1000.0 (0.01 tol + 1), 2
    R_2 = 1000.0 (0.01 tol + 1), R_3 = 1000.0 (0.01 tol + 1),
    R_4 = 1000.0 (0.01 tol + 1)]

wc_mintypmax2tol (tolname, minval, typval, maxval) [Function]
Generates a parameter that uses the tolerance tolname that tolerates between the
given values.

Example:

(%i1) load("wrstcse")$
(%i2) V_F: U_Diode=wc_mintypmax2tol(tol[1],.5,.75,.82);
(%o2) U_Diode = (- 0.09000000000000002 tol ) + 0.16 tol + 0.75

(%i3) lhs(V_F)=wc_mintypmax(rhs(V_F));
(%o3) U_Diode = [min = 0.5, typ = 0.75, max = 0.8199999999999998]
95 zeilberger

95.1 Introduction to zeilberger

zeilberger is an implementation of Zeilberger’s algorithm for definite hypergeometric summation, and also Gosper’s algorithm for indefinite hypergeometric summation.

zeilberger makes use of the "filtering" optimization method developed by Axel Riese.

zeilberger was developed by Fabrizio Caruso.

load ("zeilberger") loads this package.

95.1.1 The indefinite summation problem

zeilberger implements Gosper’s algorithm for indefinite hypergeometric summation. Given a hypergeometric term $F_k$ in $k$ we want to find its hypergeometric anti-difference, that is, a hypergeometric term $f_k$ such that

$$F_k = f_{k+1} - f_k.$$

95.1.2 The definite summation problem

zeilberger implements Zeilberger’s algorithm for definite hypergeometric summation. Given a proper hypergeometric term (in $n$ and $k$) $F_{n,k}$ and a positive integer $d$ we want to find a $d$-th order linear recurrence with polynomial coefficients (in $n$) for $F_{n,k}$ and a rational function $R$ in $n$ and $k$ such that

$$a_0 F_{n,k} + \ldots + a_d F_{n+d}, \; k = \Delta_k (R (n, k) F_{n,k}),$$

where $\Delta_k$ is the $k$-forward difference operator, i.e., $\Delta_k (t_k) \equiv t_{k+1} - t_k$.

95.1.3 Verbosity levels

There are also verbose versions of the commands which are called by adding one of the following prefixes:

Summary Just a summary at the end is shown
Verbose Some information in the intermediate steps
VeryVerbose More information
Extra Even more information including information on the linear system in Zeilberger’s algorithm

For example:
95.2 Functions and Variables for zeilberger

**AntiDifference** $(F_k, k)$

Returns the hypergeometric anti-difference of $F_k$, if it exists.
Otherwise **AntiDifference** returns `no_hyp_antidifference`.

**Gosper** $(F_k, k)$

Returns the rational certificate $R(k)$ for $F_k$, that is, a rational function such that
$F_k = R(k+1) F_{k+1} - R(k) F_k$, if it exists. Otherwise, **Gosper** returns `no_hyp_sol`.

**GosperSum** $(F_k, k, a, b)$

Returns the summation of $F_k$ from $k = a$ to $k = b$ if $F_k$ has a hypergeometric
anti-difference. Otherwise, **GosperSum** returns `nongosper_summable`.

Examples:

```maxima
(%i1) load ("zeilberger")$
(%i2) GosperSum ((-1)^k*k / (4*k^2 - 1), k, 1, n);
Dependent equations eliminated: (1)

\[
3 n + 1
\]
\[
(n + -) (- 1)
\]
\[
2 1
\]
\[
- \frac{2}{2} - \frac{4}{4}
\]
\[
2 (4 (n + 1) - 1)
\]

(%o2)

(%i3) GosperSum (1 / (4*k^2 - 1), k, 1, n);

\[
3
\]
\[
- n -
\]
\[
2 1
\]
\[
- \frac{2}{2} - \frac{2}{4}
\]
\[
4 (n + 1) - 1
\]

(%o3)

(%i4) GosperSum (x^k, k, 1, n);

\[
x x
\]
\[
n + 1
\]
\[
- \frac{x}{x - 1} - \frac{x}{x - 1}
\]

(%o4)

(%i5) GosperSum ((-1)^k*a! / (k!*(a - k)!), k, 1, n);

\[
a! (n + 1) (- 1)
\]
\[
\frac{n + 1}{a! (a - 1)! (n + 1)!}
\]
\[
\frac{a}{a! (a - 1)!}
\]

(%o5)

(%i6) GosperSum (k*k!, k, 1, n);
Dependent equations eliminated: (1)

\[
(n + 1)! - 1
\]

(%o6)

(%i7) GosperSum (((k + 1)*k! / (k + 1)!), k, 1, n);

\[
(n + 1) (n + 2) (n + 1)!
\]

(%o7)
```
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(%i8) GosperSum (1 / (a - k)!*k!), k, 1, n);
(%o8) NON_GOSPER_SUMMABLE

parGosper \(F_{n,k}, k, n, d\)

Attempts to find a \(d\)-th order recurrence for \(F_{n,k}\).

The algorithm yields a sequence \([s_1, s_2, ..., s_m]\) of solutions. Each solution has the form

\[R(n,k), [a_0, a_1, ..., a_d]\].

parGosper returns \([]\) if it fails to find a recurrence.

Zeilberger \(F_{n,k}, k, n\)

Attempts to compute the indefinite hypergeometric summation of \(F_{n,k}\).

Zeilberger first invokes Gosper, and if that fails to find a solution, then invokes parGosper with order 1, 2, 3, ..., up to \(\text{MAX}_\text{ORD}\). If Zeilberger finds a solution before reaching \(\text{MAX}_\text{ORD}\), it stops and returns the solution.

The algorithm yields a sequence \([s_1, s_2, ..., s_m]\) of solutions. Each solution has the form

\[R(n,k), [a_0, a_1, ..., a_d]\].

Zeilberger returns \([]\) if it fails to find a solution.

Zeilberger invokes Gosper only if Gosper_in_Zeilberger is true.

95.3 General global variables

\textbf{MAX}_\text{ORD}

\[\text{Global variable}\]

Default value: 5

\text{MAX}_\text{ORD} is the maximum recurrence order attempted by Zeilberger.

\textbf{simplified_output}

\[\text{Global variable}\]

Default value: false

When \text{simplified_output} is true, functions in the \text{zeilberger} package attempt further simplification of the solution.

\textbf{linear_solver}

\[\text{Global variable}\]

Default value: linsolve

\text{linear_solver} names the solver which is used to solve the system of equations in Zeilberger’s algorithm.

\textbf{warnings}

\[\text{Global variable}\]

Default value: true

When \text{warnings} is true, functions in the \text{zeilberger} package print warning messages during execution.
Gosper_in_Zeilberger  [Global variable]
  Default value: true
  When Gosper_in_Zeilberger is true, the Zeilberger function calls Gosper before calling parGosper. Otherwise, Zeilberger goes immediately to parGosper.

trivial_solutions  [Global variable]
  Default value: true
  When trivial_solutions is true, Zeilberger returns solutions which have certificate equal to zero, or all coefficients equal to zero.

95.4 Variables related to the modular test

mod_test  [Global variable]
  Default value: false
  When mod_test is true, parGosper executes a modular test for discarding systems with no solutions.

modular_linear_solver  [Global variable]
  Default value: linsolve
  modular_linear_solver names the linear solver used by the modular test in parGosper.

ev_point  [Global variable]
  Default value: big_primes[10]
  ev_point is the value at which the variable $n$ is evaluated when executing the modular test in parGosper.

mod_big_prime  [Global variable]
  Default value: big_primes[1]
  mod_big_prime is the modulus used by the modular test in parGosper.

mod_threshold  [Global variable]
  Default value: 4
  mod_threshold is the greatest order for which the modular test in parGosper is attempted.
96 Error and warning messages

This chapter provides detailed information about the meaning of some error messages or on how to recover from errors.

96.1 Error messages

96.1.1 apply: no such "list" element

One common cause for this error message is that square brackets operator ([ ]) was used trying to access a list element that whose element number was < 1 or > length(list).

96.1.2 argument must be a non-atomic expression

This normally means that a list, a set or something else that consists of more than one element was expected. One possible cause for this error message is a construct of the following type:

```lisp
(%i1) l:[1,2,3];
(%o1) [1, 2, 3]
(%i2) append(l,4);
append: argument must be a non-atomic expression; found 4
-- an error. To debug this try: debugmode(true);
```

The correct way to append variables or numbers to a list is to wrap them in a single-element list first:

```lisp
(%i1) l:[1,2,3];
(%o1) [1, 2, 3]
(%i2) append(l,[4]);
(%o2) [1, 2, 3, 4]
```

96.1.3 assignment: cannot assign to <function name>

Maxima supports several assignment operators. When trying to define a function := has to be used.

96.1.4 expt: undefined: 0 to a negative exponent.

This message notifies about a classical division by zero error.

96.1.5 incorrect syntax: , is not a prefix operator

This might be caused by a command starting with a comma (,) or by one comma being directly followed by another one.

96.1.6 incorrect syntax: Illegal use of delimiter )

Common reasons for this error appearing are a closing parenthesis without an opening one or a closing parenthesis directly preceded by a comma.
96.1.7 loadfile: failed to load <filename>

This error message normally indicates that the file exists, but can not be read. If the file is present and readable there is another possible for this error message: Maxima can compile packages to native binary files in order to make them run faster. If after compiling the file something in the system has changed in a way that makes it incompatible with the binary the file cannot be loaded any more. Maxima normally puts binary files it creates from its own packages in a folder named binary within the folder whose name it is printed after typing:

```
(%i1) maxima_userdir;
(%o1) /home/gunter/.maxima
```

If this directory is missing maxima will recreate it again as soon as it has to compile a package.

96.1.8 makelist: second argument must evaluate to a number

makelist expects the second argument to be the name of the variable whose value is to be stepped. This time instead of the name of a still-undefined variable maxima has found something else, possibly a list or the name of a list.

96.1.9 Only symbols can be bound

The most probable cause for this error is that there was an attempt to either use a number or a variable whose numerical value is known as a loop counter.

96.1.10 operators of arguments must all be the same

One possible reason for this error message to appear is a try to use append in order to add an equation to a list:

```
(%i1) l:[a=1,b=2,c=3];
(%o1) [a = 1, b = 2, c = 3]
(%i2) append(l,d=5);
append: operators of arguments must all be the same.
-- an error. To debug this try: debugmode(true);
```

In order to add an equation to a list it has to be wrapped in a single-element list first:

```
(%i1) l:[a=1,b=2,c=3];
(%o1) [a = 1, b = 2, c = 3]
(%i2) append(l,[d=5]);
(%o2) [a = 1, b = 2, c = 3, d = 5]
```

96.1.11 Out of memory

Lisp typically handles several types of memory containing at least one stack and a heap that contains user objects. To avoid running out of memory several approaches might be useful:

- If possible, the best solution normally is to use an algorithm that is more memory-efficient.
- Compiling a function might drastically reduce the amount of memory it needs.
- Arrays of a fixed type might be more memory-efficient than lists.
• If maxima is run by sbcl sbcl’s memory limit might be set to a value that is too low to solve the current problem. In this case the command-line option \texttt{--dynamic-space-size <n>} allows to tell sbcl to reserve \texttt{n} megabytes for the heap. It is to note, though, that sbcl has to handle several distinct types of memory and therefore might be able to only reserve about half of the available physical memory. Also note that 32-bit processes might only be able to access 2GB of physical memory.

96.1.12 \textbf{part: fell off the end}

\texttt{part()} was used to access the \texttt{n} element of something that has less than \texttt{n} items.

96.1.13 \textbf{undefined variable (draw or plot)}

A function could not be plotted since it still contained a variable maxima doesn’t know the value of.

In order to find out which variable this could be it is sometimes helpful to temporarily replace the name of the drawing command (\texttt{draw2d, plot2d} or similar) by a random name (for example \texttt{ddraw2d}) that doesn’t coincide with the name of an existing function to make maxima print out what parameters the drawing command sees.

\begin{verbatim}
(%i1) load("draw")$
(%i2) f(x):=sin(omega*t);
(%o2) f(x) := sin(omega t)
(%i3) draw2d(
    explicit(
        f(x),
        x,1,10)
  );
draw2d (explicit): non defined variable
    -- an error. To debug this try: debugmode(true);
(%i4) ddraw2d(
    explicit(
        f(x),
        x,1,10)
  );
(%o4) ddraw2d(explicit(sin(omega t), x, 1, 10))
\end{verbatim}

96.1.14 \textbf{VTK is not installed, which is required for Scene}

This might either mean that VTK is actually not installed - or cannot be found by maxima - or that Maxima has no write access to the temporary directory whose name is output if the following maxima command is entered:

\begin{verbatim}
(%i1) maxima_tempdir;
(%o1) /tmp
\end{verbatim}

Note: The \texttt{scene()} command requires VTK with TCL/TK bindings.

96.2 \textbf{Warning messages}
96.2.1 Encountered undefined variable \(<x>\) in translation

A function was compiled but the type of the variable \(x\) was not known. This means that the compiled command contains additional code that makes it retain all the flexibility maxima provides in respect to this variable. If \(x\) isn’t meant as a variable name but just a named option to a command prepending the named option by a single quote (‘) should resolve this issue.

96.2.2 Rat: replaced \(<x>\) by \(<y> = <z>\)

Floating-point numbers provide a maximum number of digits that is typically high, but still limited. Good examples that this limitation might be too low even for harmless-looking examples include Wilkinson’s Polynomial, The Rump polynomial and the fact that an exact 1/10 cannot be expressed as a binary floating-point number. In places where the floating-point error might add up or hinder terms from cancelling each other out maxima therefore by default replaces them with exact fractions. See also ratprint, ratepsilon, bftorat, fpprintprec and rationalize.
97 Command-line options

97.1 Command line options

The following command line options are available for Maxima:

- `f file, --batch=file`
  
  Process file in noninteractive mode.

- `--batch-lisp=file`
  
  Process Lisp file file in noninteractive mode.

- `--batch-string=string`
  
  Process string in noninteractive mode.

- `d, --directories`
  
  Display Maxima directory information.

- `--disable-readline`
  
  Disable readline support.

- `g, --enable-lisp-debugger`
  
  Enable Lisp debugger.

- `h, --help`
  
  Display a brief usage summary.

- `--init=string`
  
  Load the Maxima and Lisp initialization files `string.mac` and `string.lisp` at startup.

- `--init-mac=file`
  
  Load the Maxima initialization file file at startup.

- `--init-lisp=file`
  
  Load the Lisp initialization file file at startup.

- `l lisp, --lisp=lisp`
  
  Use Lisp implementation lisp. Use `--list-avail` to see the list of possible values.

- `--list-avail`
  
  List the available Lisp and Maxima versions.

- `p lisp_file, --preload-lisp=lisp_file`
  
  Preload `lisp_file`.

- `q, --quiet`
  
  Suppress Maxima start-up message.

- `r string, --run-string=string`
  
  Process string in interactive mode.

- `s port, --server=port`
  
  Connect Maxima to server on port. Note that this does not create a Maxima server; Maxima is the client.
-u version, --use-version=version
   Launch Maxima version version. Use --list-avail to see the list of possible values.

--userdir=directory
   Use directory for user directory (default is %USERPROFILE%/maxima for Windows, $HOME/.maxima for others)

-v, --verbose
   Print extra information from the Maxima wrapper script.

--version
   Print the (default) installed version.

--very-quiet
   Suppress expression labels and the Maxima start-up message.

-X Lisp options, --lisp-options=Lisp options
   Options to be given to the underlying Lisp.

--no-init, --norc
   Do not load the init file(s) on startup.
Appendix A  Function and Variable Index

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